Hacettepe Journal of Mathematics and Statistics
Volume 43 (4) (2014), 603 – 611

# On property $(gw1)$

M. H. M. Rashid*

**Abstract**

In this note we introduce and study the property $(gw1)$, which extend property $(gw)$ introduced by Amouch and Berkani in [7]. We investigate the property $(gw1)$ in connection with Weyl type theorems, and establish for abounded linear operator defined on a Banach space the sufficient and necessary conditions for which property $(gw1)$ holds. We also study the property $(gw1)$ for operators satisfying the single valued extension property (SVEP). Classes of operators are considered as illustrating examples.

Received 30/11/2010 : Accepted 22/04/2013

*2000 AMS Classification:* 47A10; 47A11; 47A53

**Keywords:** Generalized Weyl's theorem, Generalized $a$-Weyl's theorem, Property $(gw1)$, Property $(gw)$, Polaroid operators, Perturbation Theory

## 1. Introduction

Throughout this paper, $\mathbf{B}(\mathcal{X})$ denote the algebra of all bounded linear operators acting on a Banach space $\mathcal{X}$. For $T \in \mathbf{B}(\mathcal{X})$, let $T^*$, $\ker(T)$, $\mathcal{R}(T)$, $\sigma(T)$, $\sigma_p(T)$ and $\sigma_a(T)$ denote respectively the adjoint, the null space, the range, the spectrum, the point spectrum and the approximate point spectrum of $T$. Let $\alpha(T)$ and $\beta(T)$ be the nullity and the deficiency of $T$ defined by

$$\alpha(T) := dim\ker(T) \qquad \text{and} \qquad \beta(T) := codim\mathcal{R}(T).$$

If the range $\mathcal{R}(T)$ of $T$ is closed and $\alpha(T) < \infty$ (resp. $\beta(T) < \infty$), then $T$ is called an upper semi-Fredholm (resp. a lower semi-Fredholm ) operator. In the sequel $SF_+(\mathcal{X})$ (resp. $SF_-(\mathcal{X})$) will denote the set of all upper (resp. lower ) semi-Fredholm operators. If $T \in \mathbf{B}(\mathcal{X})$ is either upper or lower semi- Fredholm, then $T$ is called a semi-Fredholm operator, and the index of $T$ is defined by

$$ind(T) = \alpha(T) - \beta(T).$$

If both $\alpha(T)$ and $\beta(T)$ are finite, then $T$ is a Fredholm operator. An operator $T$ is called Weyl if it is Fredholm of index zero.
Let $a := asc(T)$ be the ascent of an operator $T$ ; i.e., the smallest nonnegative integer p such that $\ker(T^a) = \ker(T^{a+1})$. If such integer does not exist we put $asc(T) = \infty$ . Analogously, let $d := dsc(T)$ be descent of an operator $T$ ; i.e., the smallest nonnegative integer $d$ such that $\mathcal{R}(T^d) = \mathcal{R}(T^{d+1})$, and if such integer does not exist we put $d(T) = \infty$. It is well known that if $asc(T)$ and $dsc(T)$ are both finite then $asc(T) = dsc(T)$ [23, Proposition 38.3]. Moreover, $0 < asc(T - \lambda I) = dsc(T - \lambda I) < \infty$ precisely when $\lambda$ is a pole of the resolvent of $T$ , see Heuser [23, Proposition 50.2].

*Dept. of Mathematics & Statistics, Faculty of Science, Mu'tah University, Al-karak, Jordan
E-mail: malik_okasha@yahoo.com

An operator $T \in \mathbf{B}(\mathcal{X})$ is called Browder if it is Fredholm "of finite ascent and descent". The Weyl spectrum of $T$ is defined by $\sigma_W(T) := \{\lambda \in \mathbb{C} : T - \lambda I \quad \text{is not Weyl}\}$. For $T \in \mathbf{B}(\mathcal{X})$, let $SF_+^-(\mathcal{X}) := \{T \in SF_+(\mathcal{X}) : ind(T) \leq 0\}$. Then the upper Weyl spectrum of $T$ is defined by $\sigma_{SF_+^-}(T) := \left\{\lambda \in \mathbb{C} : T - \lambda I \notin SF_+^-(\mathcal{X})\right\}$. Let $\Delta(T) = \sigma(T) \setminus \sigma_W(T)$ and $\Delta_a(T) = \sigma_a(T) \setminus \sigma_{SF_+^-}(T)$. Following Coburn [17], we say that Weyl's theorem holds for $T \in \mathbf{B}(\mathcal{X})$ (in symbols, $T \in \mathcal{W}$) if $\Delta(T) = E^0(T)$, where $E^0(T) = \{\lambda \in iso\sigma(T) : 0 < \alpha(T - \lambda I) < \infty\}$ and that Browder's theorem holds for $T$ (in symbols, $T \in \mathcal{B}$) if $\sigma_b(T) = \sigma_W(T)$, where

$$\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda I \quad \text{is not Browder}\}.$$

Here and elsewhere in this paper, for $K \subset \mathbb{C}$, $isoK$ is the set of isolated points of $K$.

According to Rakočević [28], an operator $T \in \mathbf{B}(\mathcal{X})$ is said to satisfy a-Weyl's theorem (in symbols, $T \in a\mathcal{W}$) if $\Delta_a(T) = E_a^0(T)$, where

$$E_a^0(T) = \{\lambda \in iso\sigma_a(T) : 0 < \alpha(T - \lambda I) < \infty\}.$$

It is known [28] that an operator satisfying a- Weyl's theorem satisfies Weyl's theorem, but the converse does not hold in general.

For $T \in \mathbf{B}(\mathcal{X})$ and a nonnegative integer $n$ define $T_n$ to be the restriction of $T$ to $\mathcal{R}(T^n)$ viewed as a map from $\mathcal{R}(T^n)$ into $\mathcal{R}(T^n)$ ( in particular $T_0 = T$). If for some integer n the range space $\mathcal{R}(T^n)$ is closed and $T_n$ is an upper (resp. a lower) semi-Fredholm operator, then $T$ is called an upper (resp. a lower) semi- B-Fredholm operator. In this case the index of $T$ is defined as the index of the semi-B-Fredholm operator $T_n$, see [8]. Moreover, if $T_n$ is a Fredholm operator, then $T$ is called a B-Fredholm operator. A semi-B-Fredholm operator is an upper or a lower semi-B-Fredholm operator. An operator $T \in \mathbf{B}(\mathcal{X})$ is said to be a B-Weyl operator if it is a B-Fredholm operator of index zero. The B-Weyl spectrum $\sigma_{BW}(T)$ of T is defined by

$$\sigma_{BW}(T) := \{\lambda \in \mathbb{C} : T - \lambda I \quad \text{is not a B-Weyl operator}\}.$$

Given $T \in \mathbf{B}(\mathcal{X})$, we say that the generalized Weyl's theorem holds for $T$ (and we write $T \in g\mathcal{W}$) if

$$\sigma(T) \setminus \sigma_{BW}(T) = E(T),$$

where $E(T)$ is the set of all isolated eigenvalues of $T$, and that the generalized Browder's theorem holds for $T$ (in symbols, $T \in g\mathcal{B}$) if

$$\sigma(T) \setminus \sigma_{BW}(T) = \pi(T),$$

where $\pi(T)$ is the set of all poles of $T$, see [11, Definition 2.13]. It is known [11, 21] that

$$g\mathcal{W} \subseteq g\mathcal{B} \cap \mathcal{W} \quad \text{and that} \quad g\mathcal{B} \cup \mathcal{W} \subseteq \mathcal{B}.$$

Moreover, given $T \in g\mathcal{B}$, it is clear that $T \in g\mathcal{W}$ if and only if $E(T) = \pi(T)$. Generalized Weyl's theorem has been studied in [6, 12, 9, 10, 11, 19] and the references therein.

Let $SBF_+(\mathcal{X})$ be the class of all upper semi-B-Fredholm operators,

$$SBF_+^-(\mathcal{X}) = \{T \in SBF_+(\mathcal{X}) : ind(T) \leq 0\}.$$

The upper B-Weyl spectrum of $T$

$$\sigma_{SBF_+^-}(T) := \left\{\lambda \in \mathbb{C} : T - \lambda I \notin SBF_+^-(\mathcal{X})\right\}.$$

We say that $T$ obeys generalized a-Weyl's theorem (in symbols, $T \in ga\mathcal{W}$), if

$$\sigma_{SBF_+^-}(T) = \sigma_a(T) \setminus E_a(T);$$

where $E_a(T)$ is the set of all eigenvalues of $T$ which are isolated in $\sigma_a(T)$ ( [11, Definition 2.13]). Generalized a-Weyl's theorem has been studied in [11, 13, 14].

## 2. Results

We will say that $T \in \mathbf{B}(\mathcal{X})$ has the single-valued extension property at $\lambda_0$, (SVEP for short) if for every open neighborhood $U$ of $\lambda_0$, the only analytic function $f : U \to X$ which satisfies the equation: $(T - \lambda I)f(\lambda) = 0$, for all $\lambda \in U$ is the function $f \equiv 0$. $T \in \mathbf{B}(\mathcal{X})$ is said to have the SVEP if $T$ has the SVEP at every point $\lambda \in \mathbb{C}$(see [26]).

**2.1. Remark.** For $T \in \mathbf{B}(\mathcal{X})$ , let $\Delta^g(T) = \sigma(T) \setminus \sigma_{BW}(T)$ and $\Delta^g_a(T) = \sigma_a(T) \setminus \sigma_{SBF^-_+}(T)$. If $T^*$ has the SVEP, then it is known [24, page 35] that $\sigma(T) = \sigma_a(T)$ and from [5, Theorem 2.9] we have $\sigma_{SBF^-_+}(T) = \sigma_{BW}(T)$. Thus $E^a(T) = E(T)$ and $\Delta^g_a(T) = \Delta^g(T)$.

**2.2. Definition.** ( [28]) A bounded linear operator $T \in \mathbf{B}(\mathcal{X})$ is said to satisfy property $(w)$ if

$$\Delta^g(T) = \sigma_a(T) \setminus \sigma_{SF^-_+}(T) = E^0(T).$$

**2.3. Definition.** ( [7]) A bounded linear operator $T \in \mathbf{B}(\mathcal{X})$ is said to satisfy property $(gw)$ if

$$\Delta^g_a(T) = E(T).$$

**2.4. Definition.** ( [16]) A bounded linear operator $T \in \mathbf{B}(\mathcal{X})$ is said to satisfy property $(w1)$ if

$$\Delta^g(T) = \sigma_a(T) \setminus \sigma_{SF^-_+}(T) \subseteq E^0(T).$$

**2.5. Definition.** A bounded linear operator $T \in \mathbf{B}(\mathcal{X})$ is said to satisfy property $(gw1)$ if

$$\Delta^g_a(T) \subseteq E(T).$$

**2.6. Theorem.** Let $T \in \mathbf{B}(\mathcal{X})$. If property $(gw1)$ holds for $T$, then property $(w1)$ holds for $T$.

*Proof.* Assume that $T$ satisfies property $(gw1)$ and let $\lambda \in \Delta_a(T)$. Since $\sigma_{SBF^-_+}(T) \subseteq \sigma_{SF^-_+}(T)$, then $\lambda \in \Delta^g_a(T) \subseteq E(T)$. As $\alpha(T - \lambda I) < \infty$, then $\lambda \in E^0(T)$ and $\Delta^g(T) \subseteq E^0(T)$. □

**2.7. Theorem.** Let $T \in \mathbf{B}(\mathcal{X})$. Then

(1) property $(gw)$ holds for $T$ if and only if $T$ satisfies property $(gw1)$ and $E(T) = \pi(T)$.
(2) property $(gw)$ holds for $T$ if and only if $T$ satisfies property $(gw1)$ and $\sigma_{SBF^-_+}(T) \cap E(T) = \emptyset$.

*Proof.* (1). Suppose that $T$ has property $(gw)$, then property $(gw1)$ holds for $T$. Let $\lambda \in E(T)$, then $\lambda \in \Delta^g_a(T)$, thus $T - \lambda I \in SBF^-_+(\mathcal{X})$. Since $\lambda \in iso\sigma(T)$, we know that $T - \lambda I \in g\mathcal{B}$ and hence $\lambda \in \pi(T)$. Conversely, suppose $T$ satisfies property $(gw1)$ and $E(T) = \pi(T)$. Let $\lambda \in E(T)$, which means that $\lambda \in \Delta^g_a(T)$, thus property $(gw)$ holds for $T$.
(2). Suppose that $T$ has property $(gw)$ and this implies that property $(gw1)$ holds for $T$, and $\sigma_{SBF^-_+}(T) \cap E(T) = \emptyset$. For the converse, if $\lambda \in E(T)$, $\lambda \notin \sigma_{SBF^-_+}(T)$ since $\sigma_{SBF^-_+}(T) \cap E(T) = \emptyset$. Then $\lambda \in \Delta^g_a(T)$, hence $\Delta^g_a(T) = E(T)$. □

The following example shows that property $(gw1)$ does not implies property $(gw)$ in general.

**2.8. Example.** Let $S \in \mathbf{B}(\mathcal{X})$ be any quasi-nilpotent operator acting on an infinite dimensional Banach space $\mathcal{X}$ such that $\mathcal{R}(S^n)$ is non-closed for all$n$. Let $T = 0 \oplus S$ defined on the Banach space $\mathcal{X} \oplus \mathcal{X}$. Since $\mathcal{R}(T^n) = \mathcal{R}(S^n)$ is non-closed for all $n$, then $T$ is not a semi-B-Fredholm operator, so $\sigma_{SBF_+^-}(T) = \{0\}$. Since $\sigma_a(T) = \{0\}$ and $E(T) = \{0\}$, then $T$ does not satisfies property $(gw)$. But $T$ satisfies property $(gw1)$, since $\Delta_a^g(T) = \emptyset \subseteq E(T)$.

The following example shows that property $(gw1)$ does not implies that $\sigma_{SBF_+^-}(T) \cap E(T) = \emptyset$.

**2.9. Example.** Let $\mathcal{X} = \ell^p$, let $T_1, T_2 \in \mathbf{B}(\mathcal{X})$ be given by

$$T(x_1, x_2, \cdots) := \left(0, \frac{1}{2}x_1, \frac{1}{3}x_2, \frac{1}{4}x_3, \cdots\right) \quad \text{and} \quad T_2 := 0,$$

and let

$$T := \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} \in \underline{(}\mathcal{X} \oplus \mathcal{X}).$$

Then

$$\sigma(T) = \sigma_a(T) = \sigma_{BW}(T) = \sigma_{SBF_+^-}(T) = E(T) = \{0\}$$

and

$$\Delta_a^g(T) = \emptyset.$$

Therefore, property $(gw1)$ holds for $T$ but $\sigma_{SBF_+^-}(T) \cap E(T) = \{0\}$.

**2.10. Theorem.** Let $T \in \mathbf{B}(\mathcal{X})$. Then the following statements are equivalent:

(a) Property $(gw1)$ holds for $T$;
(b) $\sigma_{SBF_+^-}(T) = \sigma_a(T) \cap \sigma_{BW}(T)$;
(c) $\sigma_a(T) = \sigma_{SBF_+^-}(T) \cup E(T)$;
(d) $\Delta_a^g(T) \subseteq \pi(T)$.

*Proof.* $(a) \Leftrightarrow (b)$. Suppose $T$ has property $(gw1)$. Clearly, $\sigma_{SBF_+^-}(T) \subseteq \sigma_{BW}(T) \cap \sigma_a(T)$. We only need to prove that $\sigma_{SBF_+^-}(T) \supseteq \sigma_{BW}(T) \cap \sigma_a(T)$. Let $\lambda \notin \sigma_{SBF_+^-}(T)$, then $T - \lambda I \in SBF_+^-(\mathcal{X})$, thus $T - \lambda I$ is semi-B-Fredholm and $ind(T - \lambda I) \leq 0$ or $\lambda \in \Delta_a^g(T)$. Since $T$ has property $(gw1)$, we know that if $\lambda \in \Delta_a^g(T)$, $T - \lambda I \in g\mathcal{B}$, which means that $\lambda \notin \sigma_a(T) \cap \sigma_{BW}(T)$. Conversely, let $\lambda \in \Delta_a^g(T)$, since $\sigma_{SBF_+^-}(T) = \sigma_a(T) \cap \sigma_{BW}(T)$, it follows that $T - \lambda I \in g\mathcal{B}$, hence $\lambda \in E(T)$, which means that property $(gw1)$ holds for $T$.
$(a) \Leftrightarrow (c)$. Suppose $T$ satisfies property $(gw1)$. $\sigma_a(T) \supseteq \sigma_{SBF_+^-}(T) \cup E(T)$ is clear. Let $\lambda \notin \sigma_{SBF_+^-}(T) \cup E(T)$, then $T - \lambda I \in SBF_+^-(\mathcal{X})$ and $ind(T - \lambda I) \leq 0$. If $\alpha(T - \lambda I) = 0$, then $\lambda \notin \sigma_a(T)$; if $\alpha(T - \lambda I) > 0$, then $\lambda \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$, since $T$ satisfies property $(gw1)$, it follows that $\lambda \in E(T)$. It is in contradiction to the fact that $\lambda \notin E(T) \cup \sigma_{SBF_+^-}(T)$. Thus $\sigma_a(T) = \sigma_{SBF_+^-}(T) \cup E(T)$. For the converse, if $\sigma_a(T) = \sigma_{SBF_+^-}(T) \cup E(T)$, then $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) \subseteq E(T)$, which means that property $(gw1)$ holds for $T$.
$(a) \Leftrightarrow (d)$. Suppose $T$ has property $gw1$. Let $\lambda \in \Delta_a^g(T)$, then $\lambda \in E(T)$, since $T - \lambda I$ is upper semi-B-Fredholm and $\lambda \in iso\sigma(T)$, we know that $T - \lambda I \in g\mathcal{B}$, hence $\lambda \in \pi(T)$. Conversely, using the fact that $\pi(T) \subseteq E(T)$, if $\Delta_a^g(T) \subseteq \pi(T)$, then $T$ has property $(gw1)$. $\square$

In the following, let $H(T)$ be the class of all complex-valued functions which are analytic on a neighborhood of $\sigma(T)$ and are not constant on any component of $\sigma(T)$.

**2.11. Theorem.** Let $T \in \mathbf{B}(\mathcal{X})$. Suppose that property $(gw1)$ holds for $T$. Then the following statements are equivalent:

(i) For any $f \in H(T)$, property $(gw1)$ holds for $f(T)$;

(ii) For any $f \in H(T)$, $f(\sigma_{SBF_+^-}(T)) = \sigma_{SBF_+^-}(f(T))$, and if $\sigma_a(T) \neq \sigma_{SBF_+^-}(T)$, then $\sigma(T) = \sigma_a(T)$;

(ii) For each pair $\lambda, \mu \in \mathbb{C} \setminus \sigma_{SBF_+^-}(T)$, $ind(T-\lambda I)ind(T-\mu I) \geq 0$, and if $\sigma_a(T) \neq \sigma_{SBF_+^-}(T)$, then $\sigma(T) = \sigma_a(T)$.

*Proof.* $(i) \Rightarrow (ii)$. $\sigma_{SBF_+^-}(f(T)) \subseteq f(\sigma_{SBF_+^-}(T))$ is clear. We need to prove $\sigma_{SBF_+^-}(f(T)) \supseteq f(\sigma_{SBF_+^-}(T))$. Let $\mu_0 \notin \sigma_{SBF_+^-}(f(T))$, then $f(T) - \mu_0 I \in SBF_+^-(\mathcal{X})$. Let

$$f(T) - \mu_0 I = (T - \lambda_1 I)^{n_1}(T - \lambda_2 I)^{n_2} \cdots (T - \lambda_k I)^{n_k} g(T),$$

where $\lambda_i \neq \lambda_j$ and $g(T)$ is invertible. Thus $T - \lambda_j I$ is upper semi-B-Fredholm operator and $\mu_0 \notin \sigma_a(f(T))$ or $\mu_0 \in \Delta_a^g(f(T))$. If $\mu_0 \notin \sigma_a(f(T))$, then $f(T) - \mu_0 I$ is bounded from below, which means that each $T - \lambda_j I$ is bounded from below. Then $\mu_0 \notin f(\sigma_{SBF_+^-}(T))$. If $\mu_0 \in \Delta_a^g(f(T))$, since property $(gw1)$ holds for $f(T)$, we know that $f(T) - \mu_0 I \in g\mathcal{B}$. Hence $T - \lambda_j I \in g\mathcal{B}$ and $\lambda_j \notin \sigma_{SBF_+^-}(T)$. Then $\mu_0 \notin f(\sigma_{SBF_+^-}(T))$. Next we will prove if $\sigma_a(T) \neq \sigma_{SBF_+^-}(T)$, then $\sigma(T) = \sigma_a(T)$. Let $\lambda_0 \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$. Then $T - \lambda_0 I \in g\mathcal{B}$ because property $(gw1)$ holds for $T$. For any $\mu_0 \notin \sigma_a(T)$, $\alpha(T - \mu_0 I) = 0$. Let $f(T) = (T - \mu_0 I)(T - \lambda_0 I)$, then $0 \in \sigma_a(f(T)) \setminus \sigma_{SBF_+^-}(f(T))$. Since $f(T)$ has property $(gw1)$, we know that $f(T) \in g\mathcal{B}$. This implies that $f(T) - \mu_0 I \in g\mathcal{B}$. The fact $\alpha(T - \mu_0 I) = 0$ tell us that $T - \mu_0 I$ is invertible, which means that $\mu_0 \notin \sigma(T)$. Hence $\sigma(T) = \sigma_a(T)$.

$(ii) \Rightarrow (i)$. Let $\mu_0 \in \Delta_a^g(f(T))$, then $f(T) - \mu_0 I \in SBF_+^-(\mathcal{X})$ and $\alpha(f(T) - \mu_0 I) > 0$. Let

$$f(T) - \mu_0 I = (T - \lambda_1 I)^{n_1}(T - \lambda_2 I)^{n_2} \cdots (T - \lambda_k I)^{n_k} g(T),$$

where $\lambda_i \neq \lambda_j$ and $g(T)$ is invertible. Since $f(\sigma_{SBF_+^-}(T)) = \sigma_{SBF_+^-}(f(T))$ and $\mu_0 \notin \sigma_{SBF_+^-}(f(T))$, it follows that $\lambda_j \notin \sigma_{SBF_+^-}(T)$. Then $T - \lambda_j I \in SBF_+^-(\mathcal{X})$. Let $\alpha(T - \lambda_j I = 0)$ if $1 \leq j \leq i$ and $\alpha(T - \lambda_j I > 0)$ if $i < j \leq k$. Then $T - \lambda_j I$ is bounded from below if $1 \leq j \leq i$. Using the fact $\sigma(T) = \sigma_a(T)$ we know that $T - \lambda_j I$ is invertible. If $i < j \leq k$, then $\lambda_j \notin \Delta_a^g(T)$, since $T$ has property $(gw1)$, $T - \lambda_j I \in g\mathcal{B}$. Thus $f(T) - \mu_0 I \in g\mathcal{B}$ and $\mu_0 \in E(f(T))$. Hence property $(gw1)$ holds for $f(T)$.

$(i) \Rightarrow (iii)$. Suppose that there exist $\lambda, \mu \in \sigma_{SBF_+^-}(T)$ such that $ind(T-\lambda I)ind(T-\mu I) < 0$. Let $ind(T - \lambda I) = k > 0$, then $T - \lambda I$ is B-Fredholm. If $ind(T - \mu I) = -t < 0$, $t \neq \infty$, then let $f(T) = (T - \lambda I)^t (T - \mu I)^k$; if $ind(T - I) = \infty$, then let $f(T) = (T - \lambda I)(T - \mu I)$. Thus $0 \in \Delta_a^g(f(T))$, since $f(T)$ has property $(gw1)$, we know that $f(T) \in g\mathcal{B}$. Thus $T - \lambda I \in g\mathcal{B}$ and $T - \mu I \in g\mathcal{B}$. It is in contradiction to the fact that $ind(T - \lambda I) > 0$. Hence for each pair $\lambda, \mu \in \mathbb{C} \setminus \sigma_{SBF_+^-}(T)$, $ind(T - \lambda I)ind(T - \mu I) \geq 0$.

$(iii) \Rightarrow (i)$. Let $\mu_0 \notin \Delta_a^g(f(T))$, then $f(T) - \mu_0 \in SBF_+^-(\mathcal{X})$. Let

$$f(T) - \mu_0 I = (T - \lambda_1 I)^{n_1}(T - \lambda_2 I)^{n_2} \cdots (T - \lambda_k I)^{n_k} g(T),$$

where $\lambda_i \neq \lambda_j$ and $g(T)$ is invertible. Then for any $\lambda_j$, $T - \lambda_j I$ is upper semi-B-Fredholm and

$$\sum_{j=1}^{k} ind(T - \lambda_j)^{n_j} \leq 0.$$

By condition $ind(T-\lambda_j I)\leq 0$, we get that $T-\lambda_j I\in SBF_+^-(\mathcal{X})$. Since the rest of the proof is similar to the proof of $(ii)\Rightarrow(i)$ we omit it, and hence property $(gw1)$ holds for $f(T)$. $\square$

**2.12. Theorem.** Let $T\in\mathbf{B}(\mathcal{X})$. Then the following statements are equivalent:

(I) $T$ satisfies property $(gw1)$ and $E(T)=\pi(T)$

(II) T satisfies generalized Weyl's theorem and $ind(T-\lambda I)=0$ for all $\lambda\in\Delta_a^g(T)$.

*Proof.* $(I)\Rightarrow(II)$. Suppose that $T$ satisfies property $(gw)$ and $E(T)=\pi(T)$. Let $\lambda\in\Delta^g(T)$. Since $\sigma_{SBF_+^-}(T)\subseteq\sigma_{BW}(T)$, then $\lambda\notin\sigma_{SBF_+^-}(T)$. If $\alpha(T-\lambda I)=0$, as $\lambda\notin\sigma_{BW}(T)$, then $T-\lambda I$ will be invertible. But this is impossible since $\lambda\in\sigma(T)$. Hence $\alpha(T-\lambda I)>0$ and $\lambda\in\sigma_a(T)$. As $T$ satisfies property $(gw1)$, then $\lambda\in E(T)$. This implies that $\Delta^g(T)\subseteq E(T)$. To show the opposite inclusion, let $\lambda\in E(T)$ be arbitrary. Since $T$ satisfies property $(gw1)$, and $E(T)=\pi(T)$, then $\lambda\notin\sigma_{SBF_+^-}(T)$ and hence $ind(T-\lambda I)\leq 0$. On the other hand, as $\lambda\in E(T)$, then $\lambda\in iso\sigma(T)$, and hence $T^*$ has the SVEP at $\lambda$. By [3], we have $ind(T-\lambda I\geq 0$. So $ind(T-\lambda I)=0$, and $\lambda\notin\sigma_{BW}(T)$. Hence $\sigma(T)\setminus\sigma_{BW}(T)=E(T)$ and $ind(T-\lambda I)=0$ for all $\lambda\in\Delta_a^g(T)$.
$(II)\Rightarrow(I)$. Conversely, assume that $T$ satisfies generalized Weyl's theorem and $ind(T-\lambda I)=0$ for all $\lambda\in\Delta_a^g(T)$. If $\lambda\in\Delta_a^g(T)$, then $T-\lambda I$ is a semi-B-Fredholm operator such that $ind(T-\lambda I)=0$. Hence $T-\lambda I$ is a B-Weyl operator. Since $T$ satisfies generalized Weyl's theorem, then $\lambda\in E(T)$ and hence $\Delta_a^g(T)\subseteq E(T)$. To show $E(T)=\pi(T)$, let $\lambda\in E(T)$, then $T-\lambda I$ is a B-Weyl operator and since $\lambda\in\sigma(T)$, then $\alpha(T-\lambda I)>0$. Thus $\lambda\in\pi(T)$. Consequently $T$ satisfies property $(gw1)$ and $E(T)=\pi(T)$. $\square$

The following example shows that generalized a-Weyl's theorem and generalized Weyl's theorem does not imply property $(gw1)$.

**2.13. Example.** Let $\mathcal{X}=\ell^2(\mathbb{N})$ and $S\in\mathbf{B}(\mathcal{X})$ be the unilateral right shift and let $V$ defined by

$$V(x_1,x_2,\cdots)=(0,x_2,x_3,\cdots),\quad (x_n)\in\ell^2(\mathbb{N}).$$

If $T=S\oplus V$, then $\sigma(T)=\mathbf{D}(0,1)$ the closed unit disc in $\mathbb{C}$, $iso\sigma(T)=\emptyset$ and $\sigma_a(T)=\mathbf{C}(0,1)\cup\{0\}$, where $\mathbf{C}(0,1)$ is unit circle of $\mathbb{C}$. This implies that

$$\sigma_{SBF_+^-}(T)=\mathbf{C}(0,1)\quad\text{and}\quad\Delta_a^g(T)=\{0\}\,.$$

Moreover we have $E(T)=\emptyset$ and $E_a(T)=\{0\}$. Hence $T$ satisfies generalized a- Weyl's theorem and so $T$ satisfies generalized Weyl's theorem. But $T$ does not satisfy property $(gw1)$.

**2.14. Theorem.** Let $T\in\mathbf{B}(\mathcal{X})$. If $T^*$ has the SVEP, then the following statements are equivalent:

(1) Property $(gw)$ holds for $T$ ;

(2) generalized Weyl's theorem holds for $T$ ;

(3) generalized a-Weyl's theorem holds for $T$;

(4) Property $(gw1)$ holds for $T$ and $E(T)=\pi(T)$

*Proof.* Suppose that $T^*$ has the SVEP, then as it had been already mentioned by Remark 2.1, we have

$$\sigma_a(T)=\sigma(T),\sigma_{SBF_+^-}(T)=\sigma_{BW}(T),E_a(T)=E(T)$$

and $\Delta_a^g(T)=\Delta^g(T)$. The equivalence between $(1),(2)$, and $(3)$ follows from [7, Theorem 2.8]. Since $(1)\Leftrightarrow(4)$ has already been proved, the proof is complete. $\square$

**2.15. Theorem.** Let $T\in\mathbf{B}(\mathcal{X})$. If $T$ has SVEP then $\sigma_{SBF_+^-}(T^*)=\sigma_{BW}(T)$.

*Proof.* The inclusion $\sigma_{SBF_+^-}(T^*) \subseteq \sigma_{BW}(T)$ holds for every $T \in \mathbf{B}(\mathcal{X})$. Suppose that $\lambda \notin \sigma_{SBF_+^-}(T^*)$. Then $T^* - \lambda I^* \in SBF_+(\mathcal{X})$ with $ind(T^* - \lambda I^*) \leq 0$. By duality $T - \lambda I$ is lower semi-B-Fredholm and the SVEP of $T$ entails that $asc(T - \lambda I) < \infty$. By [1, Theorem 3.4] we have $ind(T - \lambda I) \leq 0$, thus $ind(T^* - \lambda I^*) = -ind(T - \lambda I) \geq 0$. Therefore, $ind(T^* - \lambda I^*) = ind(T - \lambda I) = 0$, and, again by [1, Theorem 3.4], we have $dsc(T - \lambda I) < \infty$, thus $\lambda \notin \sigma_{BW}(T)$. □

A bounded operator $T \in \mathbf{B}(\mathcal{X})$ is said to be polaroid if $iso\sigma(T) = \emptyset$ or every isolated point of $\sigma(T)$ is a pole of the resolvent of $T$ (see [4]).

**2.16. Theorem.** Suppose that $T \in \mathbf{B}(\mathcal{X})$. Then the following statements hold:

(i) If $T$ is polaroid and $T^*$ has SVEP then property $(gw)$ holds for $T$.
(ii) If $T$ is polaroid and $T$ has SVEP then property $(gw)$ holds for $T^*$.

*Proof.* (i). Note that by Remark 2.1 we have $\sigma(T) = \sigma_a(T)$. Suppose first that $iso\sigma(T) = \emptyset$. Then $E(T) = \emptyset$. We show that also $\Delta_a^g(T)$ is empty. By Remark 2.1 we have $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \sigma(T) \setminus \sigma_{BW}(T)$ and the last set is empty, since $\sigma(T)$ has no isolated points. Therefore, $T$ satisfies property $(gw)$. Consider the other case, $iso\sigma(T) \neq \emptyset$. Suppose that $\lambda \in E(T)$. Then $\lambda$ is isolated in $\sigma(T)$ and hence, by the polaroid condition, $\lambda$ is a pole of the resolvent of $T$, i.e. $asc(T - \lambda I) = dsc(T - \lambda I) < \infty$. By assumption $\alpha(T - \lambda I) < \infty$, so by [1, Theorem 3.1] $\beta(T - \lambda I) < \infty$,and hence $T - \lambda I$ is a Fredholm operator. Therefore, by 2.1, $\lambda \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \sigma(T) \setminus \sigma_{BW}(T)$.. Conversely, if $\lambda \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \sigma(T) \setminus \sigma_{BW}(T)$ then $\lambda$ is an isolated point of $\sigma(T)$ . Clearly, $0 < \alpha(T - \lambda I)$, so $\lambda \in E(T)$ and hence $T$ satisfies property $(gw)$.
(ii). First note that since $T$ has SVEP then $\sigma(T^*) = \sigma(T) = \sigma(T^*)$, see Corollary 2.45 of [1]. Suppose first that $iso\sigma(T) = \sigma(T^*) = \emptyset$. Then $E(T^*) = \emptyset$. By Theorem 2.15 we have $\sigma_a(T^*) \setminus \sigma_{SBF_+^-}(T^*) = \sigma(T) \setminus \sigma_{BW}(T) = \emptyset$, so $T^*$ satisfies property $(gw)$. Suppose that $iso\sigma(T) \neq \emptyset$ and let $\lambda \in E(T^*)$. Then $\lambda$ is an isolated point of $\sigma(T^*) = \sigma(T)$ , hence a pole of the resolvent of $T^*$, since $T^*$ is polaroid by Theorem 2.5 of [4]. By assumption $\alpha(T^* - \lambda I)^p$ and since the ascent and the descent of $T^* - \lambda I^*$ are both finite it then follows by Theorem 3.1 of [1] that $\beta(T - \lambda I) = \alpha(T - \lambda I) < \infty$, so $T^* - \lambda I^* \in g\mathcal{B}$ and hence also $T - \lambda I \in g\mathcal{B}$. Therefore, $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$ and by Theorem 2.15and Remark 2.1 it then follows that $\lambda \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$. Conversely, if $\lambda \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \sigma(T) \setminus \sigma_{BW}(T)$, then $\lambda$ is an isolated point of the spectrum of $\sigma(T) = \sigma(T^*)$, $T - \lambda I \in g\mathcal{B}$, or equivalently $T^* - \lambda I^* \in g\mathcal{B}$. Since $\alpha(T^* - \lambda I^*) = \beta(T^* - \lambda I^*)$ we then have $\alpha(T^* - \lambda I^*) > 0$ (otherwise $\lambda \notin \sigma(T^*)$). Clearly, $\alpha(T^* - \lambda I^*) > 0$, since by assumption $T^* - \lambda I^* \in SF_+^-(\mathcal{X}^*)$ , so that $\lambda \in E(T^*)$. Thus $T^*$ satisfies property $(gw)$. □

**2.17. Corollary.** Suppose that $T \in \mathbf{B}(\mathcal{X})$. Then the following statements hold:

(i) If $T$ is polaroid and $T^*$ has SVEP then property $(gw1)$ holds for $T$.
(ii) If $T$ is polaroid and $T$ has SVEP then property $(gw1)$ holds for $T^*$.

The following example shows that in the statements $(i)$ of Theorem 2.16 and Corollary 2.17 the assumption that $T^*$ has SVEP cannot be replaced by the assumption that $T$ has SVEP.

**2.18. Example.** Denote by $S$ the unilateral right shift on $\ell^2(\mathbb{N})$ and define

$$V(x_1, x_2, \cdots) = (0, x_2, x_3, \cdots) \quad \text{for all } (x_n) \in \ell^2(\mathbb{N}).$$

Clearly, $V$ is a quasi-nilpotent operator. Let $T = S \oplus V$. We have $\sigma(T) = \mathbf{D}, D$ the closed unit disc of $\mathbb{C}$, so $iso\sigma(T) = E(T) = \emptyset$ and hence $T$ is polaroid. Moreover,

$\sigma_a(T) = \partial\mathbf{D} \cup \{0\}$. Since $\sigma_a(T)$ does not cluster at $\lambda$ $T$ has SVEP at 0, as well as at the points $\lambda \notin \sigma_a(T)$. Since $T$ has SVEP at all points $\partial\sigma(T)$ it then follows that $T$ has SVEP. Finally, $\sigma_{SBF_+^-}(T) = \partial\sigma(T)$ so $\Delta_a^g(T) = \{0\} \neq E(T) = \emptyset$, thus $T$ does not satisfy property $(gw1)$ and hence does not satisfy property $(gw)$.

Analogously, in the statements $(ii)$ of Theorem 2.16 and Corollary 2.17 the assumption that $T$ has SVEP cannot be replaced by the assumption that $T^*$ has SVEP.

**2.19. Example.** Let us consider the left shift $L \in (\ell^2(\mathbb{N}))$, and let $V^*$ be the adjoint of the quasi-nilpotent operator $V$ defined in Example 2.18. We have $L^* = R$, $R$ the unilateral right shift. If we define $W := L \oplus V^*$ then, as observed in Example 2.18 $W^* = R \oplus V$ has SVEP. From Example 2.18 we also know that $\sigma(W) = \overline{\sigma(W^*)} = \mathbf{D}$, so $iso\sigma(W^*) = E(W^*) = \emptyset$ and hence $W$ is polaroid. Moreover, $\sigma_a(W^*) = \partial\mathbf{D} \cup \{0\}$. Finally, $\sigma_{SBF_+^-}(W^*) = \partial\sigma(W^*)$ so $\Delta_a^g(W^*) = \{0\} \neq E(W^*) = \emptyset$, thus $W^*$ does not satisfy property $(gw1)$ and hence does not satisfy property $(gw)$.

**2.20. Example.** Let $\mathcal{H}$ be a Hilbert space, an operator $T$ acting on $\mathcal{H}$ is said to be paranormal if $\|Tx\|^2 \leq \|T^2x\|\,\|x\|$ for all $x \in \mathcal{H}$. Examples of paranormal operators are the $p$-hyponormal or log-hyponormal operators ( [20]). Recall that an operator $T$ is $p$-hyponormal for some $p > 0$, if $(T^*T)^p \geq (TT^*)^p$ and $T$ is said to be log-hyponormal if $T$ is invertible and $\log T^*T \geq \log TT^*$. It follows from [18, Lemma 2.3] that a paranormal operator $T$ is polaroid. Moreover a paranormal operator have the SVEP, see [15, Corollary 2.10]. So if $T$ is paranormal, then $T^*$ satisfies property $(gw1)$.

A bounded operator $T \in \mathbf{B}(\mathcal{X})$ is said to have property $H(p)$ if for all $\lambda \in \mathbb{C}$ there exists a $p := p(\lambda) \in \mathbb{N}$ such that:

$$H_0(T - \lambda I) = \ker(T - \lambda I)^p,$$

where

$$H_0(T - \lambda I) = \left\{x \in \mathcal{X} |\ \lim_{n\to\infty} \|(T - \lambda I)^n x\|^{\frac{1}{n}} = 0\right\}.$$

It is well known that such operators has the SVEP and are polaroid. So if $T^*$ has the property $H(p)$, then $T$ has the property $(gw1)$. Oudghiri [27] observed that every generalized scalar operator and every subscalar operator $T$ (i.e. $T$ is similar to a restriction of a generalized scalar operator to one of its closed invariant subspaces) has property H(p), see [24] for definitions and properties. Consequently, property $H(p)$ is satisfied by $p$-hyponormal operators and log-hyponormal operators [25, Corollary 2], algebraically $w$-hyponormal operators [31], quasi-class $(A, k)$ [29], algebraically $(p, k)$-quasihyponormal [32], algebraically $wF(p, r, q)$ operators with $p, r > 0$ and $q \geq 1$ [30], $M$-hyponormal operators [24, Proposition 2.4.9], and totally paranormal operators [2]. Also totally $*$-paranormal operators have property $H(1)$ [22].

## References


[1] P. Aiena, *Fredholm and local spectral theory with applications to multipliers*, ,Kluwer Acad. Publishers, Dordrecht, 2004.

[2] P. Aiena, F. Villafãne, *Weyl's theorem for some classes of operators*, Integral Equations Operator Theory, **53**, 453-466, 2005.

[3] P. Aiena, *Quasi-Fredholm operators and localized SVEP*, Acta Sci. Math. (Szeged), 73, 251-263, 2007.

[4] Aiena, P., Guillen, J. and Peñna, P., *Property $(w)$ for perturbations of polaroid operators*, Linear Algebra Appl., **428**, 1791-1802, 2008.

[5] P. Aiena and T.L. Miller, *On generalized a-Browder's theorem*, Studia Math. **180** No.3, 285-300, 2007.

[6] M. Amouch, *Generalized a-Weyl's Theorem and the Single-Valued Extension Property*, Extracta Math. **21** No.1, 51-65, 2006.

[7] M. Amouch, M. Berkani, *on the property (gw)*, Mediterr. J. Math., **5**, 371-378, 2008.

[8] M. Berkani, On a class of quasi-Fredholm operators, Integral Equations Operator Theory, **34** No.2, 244–249, 1999.

[9] M. Berkani, *Index of B-Fredholm operators and generalization of a Weyl theorem*, Proc. Amer. Math. Soc., **130**, 1717–1723, 2001.

[10] M. Berkani, *B-Weyl spectrum and poles of the resolvent*, J. Math. Anal. Appl., **272**, 596–603, 2002.

[11] M. Berkani, J. Koliha, *Weyl type theorems for bounded linear operators*, Acta Sci. Math. (Szeged), **69** No.(1-2), 359–376, 2003.

[12] M. Berkani, A. Arroud, *Generalized weyl's theorem and hyponormal operators*, J. Austral. Math. Soc., **76**, 1–12, 2004.

[13] M. Berkani, *On the equivalence of Weyl theorem and generalized Weyl theorem*, Acta Math. Sinica, **272** No.1, 103–110, 2007.

[14] X. H. Cao, *a-Browder's theorem and generalized a-Weyl's theorem*, Acta Math. Sinica, **23** No.5, 951–960, 2007.

[15] N.N. Chourasia and P.B. Ramanujan, *Paranormal operators on Banach spaces*, Bull. Austral. Math. Soc., **21** No. 2, 161-168, 1980.

[16] Chenhui Sun, Xiaohong Cao and Lei Dai, *Property* $(w1)$ *and Weyl type theorem*, J. Math. Anal. Appl., **363**, 1-6, 2010.

[17] L. A. Coburn, *Weyl's theorem for nonnormal operators*, Michigan Math. J., **13**, 285–288, 1966.

[18] R. Curto and Y.M. Han, *Weyl's theorem for algebraically paranormal operators*, Integral Equations Operator Theory, **47** No.3, 307-314, 2003.

[19] B. P. Duggal and S. V. Djordjevic, *Generalized Weyl's theorem for a class of operators satisfying a norm condition II*, Math. Proc. Royal Irish Acad., **104A**, 1–9, 2006.

[20] T. Furuta, M. Ito and T. Yamazaki, *A subclass of paranormal operators including class of log-hyponormal and several related classes*, Sci. Math., **1**, 389-403, 1998.

[21] R. E. Harte, *Invertibility and singularity for bounded linear operators*, Marcel Dekker, New York, 1988.

[22] Y.M. Han, An-Hyun Kim, *A note on ∗-paranormal operators*, Integral Equations Operator Theory, **49**, 435–444, 2004.

[23] H. Heuser, *Functional Analysis*, Dekker, New York, 1982.

[24] K. B. Laursen and M. M. Neumann, *An introduction to local spectral theory*, Oxford, Clarendon, 2000.

[25] C. Lin, Y. Ruan, Z. Yan, p-Hyponormal operators are subscalar, Proc. Amer. Math. Soc., **131** No.9, 2753-2759, 2003.

[26] M. Mbekhta, *Sur la th'eorie spectrale locale et limite de nilpotents*, Proc. Amer. Math. Soc., **3**, 621 - 631, 1990.

[27] M. Oudghiri, *Weyl's and Browder's theorem for operators satisfying the SVEP*, Studia Math., **163**, 85-101, 2004.

[28] V. Rakočević, *Operators obeying a-Weyl's theorem*, Rev. Roumaine Math. Pures Appl., **10**, 915–919, 1986.

[29] M.H.M.Rashid, *Property* $(w)$ *and quasi-class* $(A, k)$ *operators*, Revista De Le Unión Math. Argentina, **52**, 133–142, 2011.

[30] M.H.M.Rashid, *Weyl's theorem for algebraically* $wF(p, r, q)$ *operators with* $p, r > 0$ *and* $q \geq 1$, Ukrainian Math. J., **63** No.8, 1256–1267, 2011.

[31] M.H.M.Rashid and M.S.M.Noorani, *Weyl's type theorems for algebraically* $w$*-hyponormal operators*, Arab. J. Sci. Eng., **35**,103–116, 2010.

[32] M.H.M.Rashid and M.S.M.Noorani, *Weyl's type theorems for algebraically* $(p, k)$*-quasihyponormal operators*, Commun. Korean Math. Soc., **27**, 77–95, 2012.