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A NOTE ON HARMONIC MAPS OF STATISTICAL MANIFOLDS

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Abstract. We show an existence and uniqueness result for a class of maps from a flat statistical manifold into a Riemannian manifold in a given homotopy class when the target Riemannian manifold is of negative sectional curvature under a global topological non-triviality condition. We also show that due to dualistic structure of the domain manifold the result is still valid in dual coordinates.

1. Introduction

This paper is based on the application of the results of Jost and Şimşir, [1, 2]. Moreover, their deep relationship to the statistical manifolds is deduced. Furthermore, the system of affine harmonic map equations is extended to dual flat coordinates with inverse Kähler affine metric.

The concept of affine harmonic maps, geometry of statistical manifolds and its close relationship to affine differential geometry and information geometry is summarized in preliminaries. Then, the following section is devoted to the results on harmonic maps of statistical manifolds.

2. Preliminaries

In Riemannian geometry higher dimensional generalizations of geodesics are harmonic maps. They are the critical points of an energy integral that involves the metric. Therefore, they have a variational structure which depends on the Levi-Cività connection of the underlying Riemannian metric. An affine manifold, however, naturally possesses a different connection, a flat affine connection that has nothing to do with the Levi-Cività connection of the auxiliary Riemannian metric. In particular, that Riemannian metric need not to be flat. Thus, harmonic maps are not naturally defined on such manifolds. Affine harmonic maps, as introduced and studied in [1, 2], are determined by the affine connection, and the resulting

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equations do not satisfy a variational principle. The absence of a variational structure makes the analysis more difficult. Therefore, an additional global non-triviality condition to guarantee the existence of an affine harmonic map in a given homotopy class is needed. As in the case of ordinary harmonic maps, non-positive curvature of the target manifold is also required.

A statistical manifold is simply a Riemannian manifold (M, g) together with two torsion free connections ∇ and ∇^* that satisfy a duality relation with respect to the Riemannian metric g. Indeed, for all vector fields X, Y on M, $Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla^*_X Z)$. One may immediately observe that if $\nabla = \nabla^*$ the geometry reduces to the Riemannian one. There is a close relationship between the concept of a statistical manifold and affine differential geometry. One may refer to the works of Lauritzen, Kurose and Noguchi [8, 7, 9] for a detailed study of statistical manifolds.

An affine manifold is a differential manifold whose coordinate changes are affine transformations which leads to existence of a torsion-free connection with vanishing curvature. Once we have such a structure we may introduce a two tensor $g_{ij} = \frac{\partial^2 F}{\partial x^i \partial x^j} dx^i \otimes dx^j$ where F is a strictly convex function. Thus, g is symmetric and positive definite, that is to say, a Riemannian metric on M. Such structures are first introduced by Cheng and Yau, [6]. One may recover mutually flat connections from this structure. Conversely, given mutually flat connections one may obtain local potential functions. The details can be found in Jost and Şimşir, [1, 2]. If g is a Kähler affine metric and D is the flat affine connection of the affine manifold M, the triple (M, D, g) is called a Kähler affine manifold. Throughout this text Riemannian manifolds satisfying a duality relationship with respect to the metric are called statistical manifolds referring their relationship with information geometry. However, in case dual connections are flat the concept of Kähler affine manifolds is more transparent.

On the other hand, the set of probability distributions constitute a statistical model as a manifold. This leads to the concept of information geometry which can be described as applying the techniques of differential geometry to statistics, [4]. By means of this model, the relationship between the geometric structure of the manifold and statistical estimation can be analyzed. One may approach information geometry either from the point of view of dually flat connections or the Fisher information metric. Dually flat connections are investigated by Chentsov and Amari [5, 4] as a basis of information geometry.

2.1. **Statistical manifolds.**

Definition 1. *For a torsion-free affine connection ∇ and a Riemannian metric g on a manifold M, the triple (M, ∇, g) is called a statistical Riemannian manifold if it admits another torsion-free connection ∇^* satisfying*

$$Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla^*_X Z)$$

for arbitrary X, Y and Z in $\mathfrak{X}(M)$, where $\mathfrak{X}(M)$ is the set of all tangent vector fields on M.

∇ and ∇^* are called dual affine connections with respect to g and the triple (g, ∇, ∇^*) is called a dualistic structure on M.

A flat connection ∇ and a Riemannian metric g on a differentiable manifold M is a Kähler affine structure if and only if it satisfies the Codazzi equation $D_X g(Y,Z) = D_Y g(X,Z)$, [10]. This is automatically satisfied for statistical manifolds due to their definition. Note that Shima and Japanese school calls Kähler affine structures as Hessian structures since they are Hessian of a convex potential function.

Consider $\mathbb{R}^n$ with its standard affine coordinate system $\{x_1, ..., x_n\}$ and let D be the canonical flat affine connection, i.e, $D_{\frac{\partial}{\partial x^i}} \dfrac{\partial}{\partial x^j} = 0$. Let $\Omega \subset \mathbb{R}^n$ be a domain and let φ be a strictly convex function on Ω. With the Kähler affine metric $g = \dfrac{\partial^2 \varphi}{\partial x^i \partial x^j} dx^i dx^j$, the triple (Ω, D, g) is a Kähler affine manifold. This triple is a flat statistical manifold. Conversely a flat statistical manifold is locally isometric to (Ω, D, g) as required.

2.2. **Affine harmonic maps.**

Definition 2. *Let M be an affine manifold and N be a Riemannian manifold. A map $f : M \longrightarrow N$ that satisfies*

$$g^{ij}\left(\frac{\partial^2 f^\alpha}{\partial x^i \partial x^j} + \Gamma^\alpha_{\beta\delta} \frac{\partial f^\beta}{\partial x^i} \frac{\partial f^\delta}{\partial x^j}\right) = 0$$

is called an affine harmonic map [1, 2].

Note that this is an elliptic, semi-linear system of partial differential equations and the metric g is any Riemannian metric on the affine manifold (M, D). Then, we have the following result:

Theorem 1 (Jost - Şimşir, [2])**.** *Let M be a compact affine manifold, N a compact Riemannian manifold of nonpositive sectional curvature. Let $g : M \to N$ be continuous, and suppose g is not homotopic to a map $g_0 : M \to N$ for which there is a nontrivial parallel section of $g_0^{-1}TN$. Then g is homotopic to an affine harmonic map $f : M \to N$.*

Affine harmonic maps are determined by the affine structure hence they lack variational structure which make their analysis harder. In a weaker case of the above theorem we replace affine manifold by a Kähler affine manifold in which case the Kähler affine metric hence the affine differential operator $L := g^{ij} \dfrac{\partial^2}{\partial x^i \partial x^j}$ becomes invariant under affine transformations, [1]. Furthermore, one may recover the dual flat connections from the metric.

3. Results on harmonic maps of statistical manifolds

In this chapter, some results on harmonic maps of statistical manifolds is obtained under the light of Theorem 1 described in 2.2.

Theorem 2. *Let (M, D, g) be a n-dimensional compact flat statistical manifold and N a compact Riemannian manifold of nonpositive sectional curvature. Let $g : M \to N$ be continuous, and suppose g is not homotopic to a map $g_0 : M \to N$ for which there is a nontrivial parallel section of $g_0^{-1}TN$. Then g is homotopic to an affine harmonic map $f : M \to N$ and this map is unique in its homotopy class.*

Proof. From the definition of statistical manifold the Codazzi equation

$$Xg(Y, Z) = g(D_X Y, Z) + g(Y, D_X^* Z)$$

is satisfied for the flat connection D where D^* is the dual flat connection. Since D is flat it defines a flat structure and $(x^i)_{1 \leq i \leq n}$ is an affine coordinate system for D. Hence, this yields a Kähler affine structure. Therefore, a flat statistical manifold is a Kähler affine manifold. As a consequence, proof of existence part follows from the Theorem 1. Using the argument of Al'ber [3], one can also show that the affine harmonic map is unique in its homotopy class under the assumptions of the above theorem. In fact, here, we also need the global condition. For the details one may refer to [1, 2]. □

From Theorem 1 we immediately arrive at the following corollaries:

Corollary 1. *Let M be a compact flat statistical manifold, N a compact Riemannian manifold of negative sectional curvature. Let $g : M \to N$ be continuous, and suppose g is not homotopic to a map onto a closed geodesic of N. Then g is homotopic to an affine harmonic map.*

Corollary 2. *Let M be a compact flat statistical affine manifold, N a compact Riemannian manifold of nonpositive sectional curvature. Let $g : M \to N$ be smooth and satisfy $e(g^*TN) \neq 0$, where e is the Euler class. Then g is homotopic to an affine harmonic map.*

Proof. The two corollaries follow from Theorem 1 because their assumptions imply that g cannot be homotopic to a map $g_0 : M \to N$ for which there is a nontrivial parallel section of $g_0^{-1}TN$. For the first corollary, if the tangent space of $g_0(M)$ has a parallel section $g_0(M)$ itself ought to be a flat subspace in a negatively curved target space N and this kind of subspaces are one dimensional hence homotopic to closed geodesics of N. For the second corollary, remember that a vector with a parallel section has vanishing Euler class. □

Let (M, D, g) be a flat statistical manifold. In this case, M becomes a Kähler affine manifold. Hence, metric g in local coordinates is of the form $g_{ij} = \dfrac{\partial^2 \varphi}{\partial x^i \partial x^j}$. Denoting the Christoffel symbols of the flat connection D by Γ^i_{jk} and lowering the index i by the metric g, we obtain $\Gamma_{ijk} = g_{il}\Gamma^l_{jk}$. If the Levi-Cività connection of the metric g is denoted by $\widehat{\nabla}$ then

$$\Gamma_{ijk} = \widehat{\Gamma}_{ijk} - \frac{1}{2}\partial_i\partial_j\partial_k\varphi. \tag{3.1}$$

where

$$\widehat{\Gamma}_{ijk} = \langle \widehat{\nabla}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\rangle. \tag{3.2}$$

From (3.1) and (3.2),

$$\widehat{\Gamma}_{ijk} = \frac{1}{2}\partial_i\partial_j\partial_k\varphi,$$

and $\Gamma_{ijk} + \Gamma^*_{ijk} = 2\widehat{\Gamma}_{ijk}$ where D^* is the dual flat connection of D and its Christoffel symbols in x-coordinates are $\Gamma^*_{ijk} = \partial_i\partial_j\partial_k\varphi$. Note that since D^* is not flat in x-coordinates, the theorem and its corollaries are not valid when we replace (M, D, g) by (M, D^*, g) in x-coordinates.

The dual affine coordinates x^* can be obtained as follows;

$$\begin{aligned} x^*_j &= \partial_j\varphi \\ g_{ij} &= \partial_i x^*_j. \end{aligned} \tag{3.3}$$

The corresponding local potential function is obtained through the following Legendre transformation.

$$\begin{aligned} \phi(x^*) &= \max_x(x^i x^*_i - \varphi(x)), \quad \varphi(x) + \phi(x^*) - x\cdot x^* = 0, \\ x^j &= \partial^j\phi(x^*), \quad g^{ij} = \frac{\partial x^j}{\partial x^*_i} = \partial^i\partial^j\phi(x^*). \end{aligned} \tag{3.4}$$

This is how we obtain dual flat connection through the metric. These kind of dually flat structures are investigated by Chensov [5] and Amari [4] and information geometry has founded. Conversely, if we have dually flat structure local potentials of the Kähler affine structure shall be obtained.

Let D and D^* be dually flat connections and Let $\{x^1, \ldots, x^n\}$ be the affine coordinate system that is obtained from D. In this case, the vector fields $\partial_i = \dfrac{\partial}{\partial x^i}$ are parallel. Then, the vector fields ∂^j can be defined as follows:

$$g(\partial_i, \partial^j) = \delta^j_i =$$

Observe that $Vg(\partial_i, \partial^j) = g(D_V\partial_i, \partial^j) + g(\partial_i, D^*_V\partial^j)$ for every vector field V. Since ∂_i is parallel for D, so is ∂^j for D^*. As D^* is torsion-free, for all j, k we get $[\partial^j, \partial^k] = 0$. The affine coordinates x^*_j of D^* is obtained from $\partial^j = \dfrac{\partial}{\partial x^*_j}$. The position of the indices has a specific importance since it shows the transformation behaviour under coordinate transformations. For instance, when we transform x-coordinates ∂_i transforms contravariantly whereas ∂^j transforms covariantly. In particular, $\partial^j = (\partial^j x^i)\partial_i$ and $\partial_i = (\partial_i x_j^*)\partial^j$ gives the transformation rule between x and x^*-coordinates. Moreover, since $g(\partial_i, \partial^j) = \delta_i^j$

$$g_{ij} := g(\partial_i, \partial_j) = \frac{\partial x_j^*}{\partial x^i}$$
$$g^{ij} := g(\partial^i, \partial^j) = \frac{\partial x^i}{\partial x_j^*}.$$

We would like to find local potential functions $\varphi(x)$ and $\phi(x^*)$ that satisfy $x_i^* = \partial_i \varphi(x),\ x^i = \partial^i \phi(x^*)$. The first equation can be solved locally iff $\partial_i x_j^* = \partial_j x_i^*$. This is equivalent to the condition that g is symmetric and $g_{ij} = \partial_i \partial_j \varphi$. Therefore, φ is a strictly convex function. From the duality define, $\phi := x^i x_i^* - \varphi$ to get

$$\partial^i \phi = x^i + \frac{\partial x^j}{\partial x_i^*} x_j^* - \frac{\partial x^j}{\partial x_i^*}\frac{\partial \varphi}{\partial x^j} = x^i.$$

Since φ and ϕ are strictly convex functions, they relate to each other by a Legendre transform,

$$\phi(x^*) = \max_x (x^i - x_i^* - \varphi(x))$$
$$\varphi(x) = \max_{x^*} (x^i - x_i^* - \phi(x^*)).$$

Moreover,

$$\widehat{\Gamma}^{ijk} = -\widehat{\Gamma}_{ijk} = -\frac{1}{2}\partial_i \partial_j \partial_k \varphi,$$

is the Christoffel symbols of the Levi-Cività connection of the metric g^{ij} in x-coordinates and the Christoffel symbols of D^* in x^*-coordinates

$$\Gamma^{ijk} = \widehat{\Gamma}^{ijk} - \frac{1}{2}\partial_i \partial_j \partial_k \varphi = -\Gamma^*_{ijk}$$

becomes zero. Note that these formulas are only valid where φ and ϕ are locally defined. Hence, in $(x_i^*)_{1\le i\le n}$ flat affine coordinates together with the inverse Kähler affine metric g^{-1}, the triple (M, D^*, g^{-1}) becomes a flat statistical manifold. In this dual flat affine coordinate system (x^*), affine harmonic map equation reduces to

$$g_{ij}\left(\partial^i \partial^j f^\alpha + \Gamma^\alpha_{\beta\delta}\partial^i f^\beta \partial^j f^\delta\right) = 0.$$

and hence the following theorem holds:

Theorem 3. *Let (M, D^*, g^{-1}) be a n-dimensional compact flat statistical manifold and N a compact Riemannian manifold of nonpositive sectional curvature. Let $g : M \to N$ be continuous, and suppose g is not homotopic to a map $g_0 : M \to N$ for which there is a nontrivial parallel section of $g_0^{-1}TN$. Then g is homotopic to an affine harmonic map $f : M \to N$ and this map is unique in its homotopy class.*

Note that, the corollaries of Theorem 2 is also valid for Theorem 3. Working with the dual flat connection, hence, dual flat affine coordinates forces us to use not the metric g itself but its inverse so that the definition of affine harmonic map system makes sense.

It should be possible and of interest in information geometry to construct examples of affine harmonic maps for specific families of probability distributions considered as flat statistical manifolds equipped with the Fisher information metric.

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