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Common fixed point theorems in cone Banach type spaces

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Abstract

In this paper, we give some generalized theorems on points of coincidence and common fixed points for two weakly compatible mappings on a cone Banach type space.

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1. Introduction

In 1980, Rzepecki [15] provide a generalization of metric spaces. He defined a metric d_E on a set X by $d_E : X \times X \to S$, where E is a Banach space and S is a normal cone in E with partial order $\preceq$, and he generalized the fixed point theorems of Maia type. In 1987, Lin [9] introduced the notion of K-metric spaces and considered some results of Khan and Imdad [7] in K-metric spaces. In 2007, Huang and Zhang [8] introduced cone metric spaces and defined some properties of convergence of sequences and completeness in cone metric spaces, also they proved a fixed point theorem of cone metric spaces. Beginning around the year 2007, the fixed point theorems in cone metric spaces have been extensively proved by a number of authors and there are many interesting results concerning these theorems (see [1]–[3], [5], [11]–[14]).

In this paper, we propose the notion of cone Banach type spaces and prove the generalization of some known results on points of coincidence and the generalization of some common fixed point theorems for two weakly compatible mappings in cone Banach type spaces.

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2. preliminaries

2.1. Definition. [11] Let E be a real Banach space with norm $\|.\|$ and P be a subset of E. P is called a cone if and only if the following conditions are satisfied:

(P1) P is closed, nonempty and $P \neq \{0\}$;
(P1) $a, b \geq 0$ and $x, y \in P \Rightarrow ax + by \in P$;
(P3) $x \in P$ and $-x \in P \Rightarrow x = 0$.

Let $P \subset E$ be a cone, we define a partial ordering $\preceq$ on E with respect to P by $x \preceq y$ if and only if $y - x \in P$. we write $x \prec y$ whenever $x \preceq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in intP$(interior of P). The cone $P \subset E$ is called normal if there is a positive real number k such that for all $x, y \in E$,

$$0 \preceq x \preceq y \Rightarrow \|x\| \leq k\|y\|.$$

The least positive number satisfying the above inequality is called the normal constant of P. It is clear that $k \geq 1$. Rezapour and Hamlbarani [14] proved that existence of an ordered Banach space E with cone P which is not normal but with $intP \neq \emptyset$.

Throughout this paper, we assume that E is a real Banach space and P is a cone suth that $intP \neq \emptyset$.

2.2. Definition. [11]. Let X be a nonempty set. A function $d : X \times X \to E$ is said to be a cone b-metric function on X with the constant $K \geq 1$ if the following conditions are satisfied:

(1) $0 \preceq d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \preceq K\Big(d(x, y) + d(y, z)\Big)$ for all $x, y, z \in X$;

then the pair (X, d) is called the cone b-metric space (or cone metric type space (in brief $CMTS$)).

2.3. Definition. [5] Let X be a vector space over $\mathbb{R}$. Suppose the mapping $\|.\|_P : X \to E$ satisfies:

(i) $\|x\|_P \succ 0$ for all $x \in X$;
(ii) $\|x\|_P = 0$ if and only if $x = 0$;
(iii) $\|x + y\|_P \preceq \|x\|_P + \|y\|_P$ for all $x, y \in X$;
(iv) $\|kx\|_P = |k|\|x\|_P$ for all $x \in X$ and all $k \in \mathbb{R}$;

then $\|.\|_P$ is called cone norm on X and the pair $(X, \|.\|_P)$ is called a cone normed space (in brief CNS). Note that each CNS is cone metric space (in brief CMS). Indeed, $d(x, y) = \|x - y\|_P$.

Similar to the definition of $CMTS$, we give the following definition:

2.4. Definition. Let X be a vector space over $\mathbb{R}$. Suppose the mapping $\|.\|_P : X \to E$ satisfies:

(i) $\|x\|_P \succeq 0$ for all $x \in X$;
(ii) $\|x\|_P = 0$ if and only if $x = 0$;
(iii) $\|x + y\|_P \preceq K\Big(\|x\|_P + \|y\|_P\Big)$ for all $x, y \in X$ and for constant $K \geq 1$ (triangle - type inequality);
(iv) $\|rx\|_P = |r|\|x\|_P$ for all $x \in X$ and all $r \in \mathbb{R}$;

then the pair $(X, \|.\|_P)$ is called a *cone normed type space* (in brief $CNTS$).

Note that each $CNTS$ is $CMTS$. Indeed, $d(x, y) = \|x - y\|_P$.

2.5. Example. *Let $C_b(X) = \{f : X \to \mathbb{C} : \sup_{x\in X} |f(x)| < \infty\}$. Define $\|.\|_P : C_b(X) \to \mathbb{R}$ by*

$$\|f\|_P = \sqrt[3]{\sup_{x\in X} |f(x)|^3}.$$

Then $\|.\|_P$ satisfies the following properties:

(i) $\|f\|_P > 0$ *for all* $f \in C_b(X)$;
(ii) $\|f\|_P = 0$ *if and only if* $f = 0$;
(iii) $\|f + g\|_P \leq \sqrt[3]{4}\Big(\|f\|_P + \|g\|_P\Big)$ *for all* $f, g \in C_b(X)$;
(iv) $\|rf\|_P = |r|\|f\|_P$ *for all* $f \in C_b(X)$ *and all* $r \in \mathbb{R}$.

2.6. Definition. Let $(X, ||.||_P, K)$ be a $CNTS$, let $\{x_n\}$ be a sequence in X and $x \in X$. Then

(i) $\{x_n\}$ converges to x whenever for every $c \in E$ with $0 \ll c$ there is a natural number N, such that $\|x_n - x\|_P \ll c$ for all $n > N$. It is denoted by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$;
(ii) $\{x_n\}$ is a Cauchy sequence whenever for every $c \in E$ with $0 \ll c$ there is a natural number N, such that $\|x_n - x_m\|_P \ll c$ for all $n, m > N$;
(iii) $(X, \|.\|_P, K)$ is a complete cone normed type space if every Cauchy sequence is convergent. Complete cone normed type spaces will be called *cone Banach type spaces.*

2.7. Lemma. *Let $(X, \|.\|_P, K)$ be a $CNTS$, P be a normal cone with normal constant M, and $\{x_n\}$ be a sequence in X. Then,*

(i) *the sequence $\{x_n\}$ converges to x if and only if $\|x_n - x\|_P \to 0$, as $n \to \infty$;*
(ii) *the sequence $\{x_n\}$ is Cauchy if and only if $\|x_n - x_m\|_P \to 0$ as $n, m \to \infty$;*
(iii) *if the sequence $\{x_n\}$ converges to x and the sequence $\{y_n\}$ converges to y, then $\|x_n - y_n\|_P \to \|x - y\|_P$.*

Proof. The proof is similar to proof of Lemmas 1-5 of [8], by taking $d(x, y) = \|x-y\|_P$. □

From now on, we assume that P is a normal cone with $intP \neq \emptyset$.

2.8. Lemma. *Let $\{y_n\}$ be a sequence in a cone Banach type space $(X, \|.\|_P, K)$ such that*

$$d(y_n, y_{n+1}) \leq \lambda d(y_{n-1}, y_n),$$

for some $0 < \lambda < 1/K$ and all $n \in \mathbb{N}$, where $d(x, y) = \|x - y\|_P$. Then $\{y_n\}$ is a Cauchy sequence in $(X, \|.\|_P, K)$.

2.9. Definition. Let S and T be two self-mappings on a cone metric type space (X, d). A point $z \in X$ is called a coincidence point of S and T if $Sz = Tz$, and it is called a common fixed point of S and T if $Sz = z = Tz$. Moreover, a pair of self-mappings (S, T) is called weakly compatible on X if they commute at their coincidence points, i.e.,

$$z \in X, \quad Sz = Tz \Rightarrow STz = TSz.$$

2.10. Theorem. *Let C be a subset of a cone Banach type space $(X, \|.\|_P, K)$ and $d : X \times X \to E$ be such that $d(x, y) = \|x - y\|_P$. Suppose that $F, T : C \longrightarrow C$ are two mappings such that $TC \subset FC$ and FC is closed and convex. If there exists some constant $1 - \frac{1}{K} < \frac{r}{2} < 1$ such that*

$$d(Fy, Ty) + rd(Fx, Fy) \preceq d(Fx, Tx), \tag{2.1}$$

for all $x, y \in C$, then F and T have at least one point of coincidence. Moreover, if F and T are weakly compatible, then F and T have a unique common fixed point.

Proof. Let $x_0 \in C$ be arbitrary. we define a sequence $\{Fx_n\}$ in the following relation:

$$Fx_{n+1} := \frac{Fx_n + Tx_n}{2}, \quad n = 0, 1, 2, \cdots . \tag{2.2}$$

We see that

$$Fx_n - Tx_n = 2\left(Fx_n - \left(\frac{Fx_n + Tx_n}{2} \right) \right) = 2(Fx_n - Fx_{n+1}), \tag{2.3}$$

which implies

$$d(Fx_n, Tx_n) = ||Fx_n - Tx_n||_P = 2||Fx_n - Fx_{n+1}||_P = 2d(Fx_n, Fx_{n+1}), \tag{2.4}$$

for $n = 0, 1, 2, \cdots$. Now, letting $x = x_{n-1}$ and $y = x_n$ in (2.1), using (2.4), we can conclude that

$$2d(Fx_n, Fx_{n+1}) + rd(Fx_{n-1}, Fx_n) \preceq 2d(Fx_{n-1}, Fx_n). \tag{2.5}$$

So

$$d(Fx_n, Fx_{n+1}) \preceq (1 - \frac{r}{2})d(Fx_{n-1}, Fx_n), \tag{2.6}$$

where $1 - \frac{r}{2} < \frac{1}{K}$. Hence by Lemma 2.8, $\{Fx_n\}$ is a Cauchy sequence in FC. Then there exists $z \in C$ such that $Fx_n \to Fz$. Also by (2.2) we can obtain $Tx_n \to Fz$. So by (2.1) we have

$$d(Fz, Tz) \preceq d(Fz, Tz) + rd(Fx_n, Fz) \preceq d(Fx_n, Tx_n). \tag{2.7}$$

Therefore by taking the limit as $n \to \infty$ in (2.7), we obtain $d(Fz, Tz) = 0$, that is, z is a point of coincidence of F and T. Therefore F and T have at least one point of coincidence.

Put $w = Fz = Tz$. If F and T are weakly compatible mappings, then $FTz = TFz$, so $Fw = Tw$.

Now, we show that w is a fixed point of F. Putting $x = w$ and $y = z$ in (2.1), we get

$$d(Fz, Tz) + rd(Fw, Fz) \preceq d(Fw, Tw). \tag{2.8}$$

Hence $d(Fw, Fz) = 0$. That is, $Fw = w$. Therefore $Fw = Tw = w$. So we conclude that $w = Fw = Tw$ is a common fixed point of F and T.
To prove the uniqueness of w, suppose that w_1 is another common fixed point F and T. Replacing x and y by w and w_1 in (2.1), respectively, we get

$$d(Fw_1, Tw_1) + rd(Fw, Fw_1) \preceq d(Fw, Tw). \tag{2.9}$$

Thus,

$$d(w_1, w) \preceq 0.$$

So $w = w_1$. Then w is the unique common fixed point of F and T. □

2.11. Corollary. *Let C be a closed and convex subset of a cone Banach type space $(X, \|.\|_P, K)$ and $d : X \times X \to E$ be such that $d(x, y) = \|x - y\|_P$. Suppose that $T : C \to C$ is a mapping for which there exists some constant $1 - \frac{1}{K} < \frac{r}{2} < 1$ such that*

$$d(y, Ty) + rd(x, y) \preceq d(x, Tx),$$

for all $x, y \in C$. Then T has a unique fixed point.

2.12. Theorem. *Let C be a subset of a cone Banach type space $(X, \|.\|_P, K)$ such that $1 < K \leq 2$. Let $d : X \times X \to E$ be such that $d(x, y) = \|x - y\|_P$. Suppose that $F, T : C \longrightarrow C$ are two mappings such that $TC \subset FC$ and FC is closed and convex. If there exists some constant $1 - \frac{1}{K} < \frac{r}{2} < 1$ such that*

$$d(Tx, Ty) + (1 - \frac{1}{K})d(Fy, Ty) + rd(Fx, Fy) \preceq \frac{1}{2}d(Fx, Tx), \tag{2.10}$$

for all $x, y \in C$, then F and T have at least one point of coincidence. Moreover, if F and T are weakly compatible, then F and T have a unique common fixed point.

Proof. Similar to proof of Theorem 2.10, we construct the sequence $\{Fx_n\}$, therefore

$$Fx_n - Tx_{n-1} = \frac{Fx_{n-1} + Tx_{n-1}}{2} - Tx_{n-1} = \frac{Fx_{n-1} - Tx_{n-1}}{2},$$

which implies that

$$d(Fx_n, Tx_{n-1}) = \frac{1}{2} d(Fx_{n-1}, Tx_{n-1}). \tag{2.11}$$

Using the triangle-type inequality, we get

$$d(Fx_n, Tx_n) - Kd(Fx_n, Tx_{n-1}) \preceq Kd(Tx_{n-1}, Tx_n) \tag{2.12}$$

It follows from (2.3) and (2.11) that

$$\frac{2}{K} d(Fx_n, Fx_{n+1}) - d(Fx_n, Fx_{n-1}) \preceq d(Tx_{n-1}, Tx_n). \tag{2.13}$$

Replacing x and y by x_{n-1} and x_n in (2.10) and using (2.3) and (2.13), we can obtain

$$\begin{aligned} \frac{2}{K} d(Fx_n, Fx_{n+1}) - d(Fx_{n-1}, Fx_n) \quad &+ \quad 2(1 - \frac{1}{K}) d(Fx_n, Fx_{n+1}) \\ &+ \quad rd(Fx_{n-1}, Fx_n) \preceq d(Fx_{n-1}, Fx_n). \end{aligned}$$

Thus,

$$d(Fx_n, Fx_{n+1}) \preceq (1 - \frac{r}{2}) d(Fx_{n-1}, Fx_n),$$

where $1 - \frac{r}{2} < \frac{1}{K}$. Hence by Lemma 2.8, $\{Fx_n\}$ is a Cauchy sequence in FC. Then there exists $z \in C$ such that $Fx_n \longrightarrow Fz$. Substituting $x = x_n$ and $y = z$ in (2.10), we get

$$\begin{aligned} (1 - \frac{1}{K}) d(Fz, Tz) &\preceq d(Tx_n, Tz) + (1 - \frac{1}{K}) d(Fz, Tz) + rd(Fx_n, Fz) \\ &\preceq \frac{1}{2} d(Fx_n, Tx_n). \end{aligned} \tag{2.14}$$

Therefore by taking the limit as $n \to \infty$ in (2.14), we obtain $d(Fz, Tz) = 0$. Then we conclude that z is a point of coincidence of F and T.

Let $w = Fz = Tz$. If F and T are weakly compatible mappings, then $FTz = TFz$, so $Fw = Tw$.

Now, we show that w is a fixed point of F. Putting $x = w$ and $y = z$ in (2.10), we have

$$d(Tw, Tz) + (1 - \frac{1}{K}) d(Fz, Tz) + rd(Fw, Fz) \preceq \frac{1}{2} d(Fw, Tw).$$

Then

$$(r + 1) d(Fw, w) \preceq 0.$$

Therefore $Fw = Tw = w$. So we conclude that $w = Fw = Tw$ is a common fixed point of F and T.

To prove the uniqueness of w, suppose that w_1 is another common fixed point of F and T. Replacing x and y by w and w_1 in (2.10), respectively, we get

$$d(Tw, Tw_1) + (1 - \frac{1}{K}) d(Fw_1, Tw_1) + rd(Fw, Fw_1) \preceq \frac{1}{2} d(Fw, Tw). \tag{2.15}$$

Thus,

$$(1 + r) d(w_1, w) \preceq 0.$$

So $w = w_1$. Then w is the unique common fixed point of F and T. □

2.13. Corollary. *Let C be a closed and convex subset of a cone Banach type space $(X, ||.||_P, K)$ such that $1 < K \leq 2$ and $d : X \times X \to E$ be such that $d(x, y) = ||x - y||_P$. Suppose that $T : C \longrightarrow C$ is a mapping which satisfies the condition*

$$d(Tx, Ty) + (1 - \frac{1}{K})d(y, Ty) + rd(x, y) \preceq \frac{1}{2}d(x, Tx),$$

for all $x, y \in C$, where $1 - \frac{1}{K} < \frac{r}{2} < 1$, then T has a unique fixed point.

2.14. Theorem. *Let C be a subset of a cone Banach type space $(X, \|.\|_P, K)$ and $d : X \times X \to E$ be such that $d(x, y) = \|x - y\|_P$. Suppose that $F, T : C \to C$ are two mappings such that $TC \subset FC$ and FC is closed and convex. If there exist a, b, s satisfying*

$$0 < s + |a|K^{\frac{1}{2} - \frac{1}{2}sgn(a)} - 2b < 2(aK^{-sgn(a)} + b), \tag{2.16}$$

and

$$ad(Tx, Ty) + b\Big(d(Fx, Tx) + d(Fy, Ty)\Big) \preceq sd(Fx, Fy), \tag{2.17}$$

for all $x, y \in C$, then F and T have at least one point of coincidence. Moreover if $a > s$ and F and T are weakly compatible, then F and T have a unique common fixed point.

Proof. Similar to proof of Theorem 2.10, we construct the sequence $\{Fx_n\}$. We claim that the inequality (2.17) for $x = x_{n-1}$ and $y = x_n$ implies that

$$\begin{aligned} &2aK^{-\mathrm{sgn}(a)}d(Fx_n,Fx_{n+1}) - |a|K^{\frac{1}{2}-\frac{1}{2}\mathrm{sgn}(a)}d(Fx_{n-1}, Fx_n) \\ &\qquad + 2b\Big(d(Fx_{n-1}, Fx_n) + d(Fx_n, Fx_{n+1})\Big) \preceq sd(Fx_{n-1}, Fx_n), \end{aligned} \tag{2.18}$$

for all a, b, s that satisfy (2.16). To see this, replacing x and y by x_{n-1} and x_n in (2.17), respectively, we obtain

$$ad(Tx_{n-1}, Tx_n) + b\Big(d(Fx_{n-1}, Tx_{n-1}) + d(Fx_n, Tx_n)\Big) \preceq sd(Fx_{n-1}, Fx_n). \tag{2.19}$$

Let $a \geq 0$, using (2.3), (2.13) and (2.19), we have

$$\begin{aligned} &\frac{2a}{K}d(Fx_n, Fx_{n+1}) - ad(Fx_n, Fx_{n-1}) \\ &\qquad + 2b\Big(d(Fx_{n-1}, Fx_n) + d(Fx_n, Fx_{n+1})\Big) \preceq sd(Fx_{n-1}, Fx_n), \end{aligned}$$

which is equivalent to (2.18), since $\mathrm{sgn}(a) = 0$ or 1.

Now suppose that $a < 0$, consider the inequality

$$d(Tx_{n-1}, Tx_n) \preceq K\Big(d(Tx_{n-1}, Fx_n) + d(Fx_n, Tx_n)\Big),$$

which is equivalent to

$$ad(Tx_{n-1}, Tx_n) \succeq Ka\Big(d(Tx_{n-1}, Fx_n) + d(Fx_n, Tx_n)\Big). \tag{2.20}$$

It follows from (2.19) and (2.20) that

$$\begin{aligned} &aK(d(Tx_{n-1}, Fx_n) + d(Fx_n, Tx_n)) \\ &\qquad + b\Big(d(Fx_{n-1}, Tx_{n-1}) + d(Fx_n, Tx_n)\Big) \preceq sd(Fx_{n-1}, Fx_n). \end{aligned} \tag{2.21}$$

Using (2.4), (2.11) and (2.21), we get

$$\begin{aligned} &aKd(Fx_{n-1}, Fx_n) + 2aKd(Fx_n, Fx_{n+1}) \\ &\qquad + 2b\Big(d(Fx_{n-1}, Fx_n) + d(Fx_n, Fx_{n+1})\Big) \preceq sd(Fx_{n-1}, Fx_n) \end{aligned}$$

which is equivalent to (2.18), since $\mathrm{sgn}(a) = -1$. Hence, we established our claim.

It follows from (2.18) that

$$d(Fx_n, Fx_{n+1}) \preceq \frac{s + |a|K^{\frac{1}{2}-\frac{1}{2}\mathrm{sgn}(a)} - 2b}{2(aK^{-\mathrm{sgn}(a)} + b)} d(Fx_{n-1}, Fx_n),$$

where $\frac{s + |a|K^{\frac{1}{2}-\frac{1}{2}\mathrm{sgn}(a)} - 2b}{2(aK^{-\mathrm{sgn}(a)} + b)} < 1$. Hence by Lemma 2.8, $\{Fx_n\}$ is a Cauchy sequence in FC. Then there exists $z \in C$ such that $Fx_n \to Fz$, so $Tx_n \to Fz$. Now, using (2.17), we have

$$ad(Tx_n, Tz) + b\Big(d(Fx_n, Tx_n) + d(Fz, Tz)\Big) \preceq sd(Fx_n, Fz). \tag{2.22}$$

Thus by taking the limit as $n \to \infty$ in (2.22), we obtain

$$(a + b)d(Fz, Tz) \preceq 0.$$

Since $aK^{-\mathrm{sgn}(a)} \leq a$, we get $a + b > 0$. Hence, $d(Fz, Tz) = 0$. So z is a point of coincidence of F and T.

If F and T are weakly compatible, then $FTz = TFz$. Therefore $Fw = Tw$, where $w = Fz = Tz$.

Now, we show that w is a unique common fixed point of T and F. Substituting $x = w$ and $y = z$ in (2.17), we obtain

$$ad(Tw, Tz) + b\Big(d(Fw, Tw) + d(Fz, Tz)\Big) \preceq sd(Fw, Fz),$$

which yields that

$$(a - s)d(Tw, w) \preceq 0.$$

Since $a > s$, we have $Tw = w$. Therefore $Fw = Tw = w$. This means w is a common fixed point of F and T.

To prove the uniqueness of w, suppose that w_1 is another common fixed point of F and T. Replacing x and y by w_1 and w in (2.17), we get

$$ad(Tw_1, Tw) + b\Big(d(Fw_1, Tw_1) + d(Fw, Tw)\Big) \preceq sd(Fw_1, Fw).$$

Thus,

$$(a - s)d(w_1, w) \preceq 0.$$

So $w = w_1$. Therefore w is the unique common fixed point of F and T. □

2.15. Corollary. *Let C be a closed and convex subset of a cone Banach type space $(X, \|.\|_P, K)$ and $d : X \times X \to E$ be such that $d(x, y) = \|x-y\|_P$. Suppose that $T : C \to C$ is a mapping for which there exist a, b, s such that*

$$0 < s + |a|K^{\frac{1}{2}-\frac{1}{2}sgn(a)} - 2b < 2(aK^{-sgn(a)} + b),$$

and

$$ad(Tx, Ty) + b\Big(d(x, Tx) + d(y, Ty)\Big) \preceq sd(x, y),$$

for all $x, y \in C$, then T has at least one fixed point. Moreover, if $a > s$, then T has a unique fixed point.

2.16. Theorem. *Let C be a subset of a cone Banach type space $(X, \|.\|_P, K)$ and $d : X \times X \to E$ be such that $d(x, y) = \|x - y\|_P$. Suppose that $F, T : C \to C$ are two mappings such that $TC \subset FC$ and FC is closed and convex. If there exist a, b satisfying*

$$1 < b < 1 + \frac{(2a-1)K - 1}{2K^2} \ \ \& \ \ a > \frac{K+1}{2K}, \tag{2.23}$$

and

$$(2.24)\qquad ad(Fy,Ty)+d(Fy,Tx)\preceq bd(Fx,Tx)+\frac{1}{K}d(Fx,Fy),$$

for all $x,y\in C$, then F and T have at least one point of coincidence. Moreover, if $K>1$ and F and T are weakly compatible, then F and T have a unique common fixed point.

Proof. Let $x_0\in C$ be arbitrary, we define a sequence $\{Fx_n\}$ in the following relation:

$$(2.25)\qquad Fx_{n+1}:=\frac{(2K-1)Fx_n+Tx_n}{2K},\quad n=0,1,2,\cdots,$$

we see that

$$(2.26)\qquad Fx_n-Tx_n=2K\Bigg(Fx_n-\Big(\frac{(2K-1)Fx_n+Tx_n}{2K}\Big)\Bigg)=2K(Fx_n-Fx_{n+1}),$$

which implies

$$(2.27)\qquad d(Fx_n,Tx_n)=2Kd(Fx_n,Fx_{n+1}).$$

Similarly

$$Fx_n-Tx_{n-1}=\frac{(2K-1)Fx_{n-1}+Tx_{n-1}}{2K}-Tx_{n-1}=(\frac{2K-1}{2K})(Fx_{n-1}-Tx_{n-1}),$$

then

$$(2.28)\qquad d(Fx_n,Tx_{n-1})=(\frac{2K-1}{2K})d(Fx_{n-1},Tx_{n-1}).$$

Replacing x and y by x_{n-1} and x_n in (2.24), respectively, we get

$$(2.29)\qquad ad(Fx_n,Tx_n)+d(Fx_n,Tx_{n-1})\preceq bd(Fx_{n-1},Tx_{n-1})+\frac{1}{K}d(Fx_{n-1},Fx_n).$$

It follows from (2.27), (2.28) and (2.29) that

$$2aKd(Fx_n,Fx_{n+1})+(2K-1)d(Fx_n,Fx_{n-1})\preceq 2bKd(Fx_{n-1},Fx_n)+\frac{1}{K}d(Fx_{n-1},Fx_n).$$

Therefore

$$d(Fx_n,Fx_{n+1})\preceq\frac{(2bK+\frac{1}{K}-2K+1)}{2aK}d(Fx_{n-1},Fx_n),$$

where $\frac{(2bK+\frac{1}{K}-2K+1)}{2aK}<\frac{1}{K}$. Hence by Lemma 2.8, $\{Fx_n\}$ is a Cauchy sequence in FC. Then there exists $z\in C$ such that $Fx_n\to Fz$, so $Tx_n\to Fz$. Replacing x and y by x_n and z in (2.24), respectively, we get

$$(2.30)\qquad ad(Fz,Tz)+d(Fz,Tx_n)\preceq bd(Fx_n,Tx_n)+\frac{1}{K}d(Fx_n,Fz).$$

Then by taking the limit as $n\to\infty$ in (2.30), we obtain $d(Fz,Tz)=0$. So we conclude that z is a point of coincidence of F and T.

If F and T are weakly compatible, then $FTz=TFz$. Therefore $Fw=Tw$, where $w=Fz=Tz$.

Now, we show that w is a unique common fixed point of F and T. Substituting $x=w$ and $y=z$ in (2.24), we obtain

$$ad(Fz,Tz)+d(Fz,Tw)\preceq bd(Fw,Tw)+\frac{1}{K}d(Fw,Fz),$$

which implies that

$$(1-\frac{1}{K})d(w,Tw)\preceq 0.$$

Hence $w=Tw$, therefore w is a common fixed point of F and T.

To prove the uniqueness of w, suppose that w_1 is another common fixed point of F and T. Replacing x and y by w and w_1 in (2.24), respectively, we have

$$ad(Fw_1, Tw_1) + d(Fw_1, Tw) \preceq bd(Fw, Tw) + \frac{1}{K}d(Fw, Fw_1).$$

Thus,

$$(1 - \frac{1}{K})d(w, w_1) \preceq 0.$$

So $w = w_1$. Therefore w is the unique common fixed point of F and T. □

2.17. Corollary. *Let C be a closed and convex subset of a cone Banach type space $(X, \|.\|_P, K)$ and $d : X \times X \to E$ be such that $d(x, y) = \|x - y\|_P$. Suppose that $T : C \to C$ is a mapping for which there exist a, b satisfying*

$$1 < b < 1 + \frac{(2a-1)K - 1}{2K^2} \ \ \& \ \ a > \frac{K+1}{2K},$$

and

$$ad(y, Ty) + d(y, Tx) \preceq bd(x, Tx) + \frac{1}{K}d(x, y),$$

for all $x, y \in C$, then T has a fixed point. Moreover, if $K > 1$, then T has a unique fixed point.

References

[1] T. Abdeljawad, E. Karapinar and K. Tas, *Common fixed point theorems in cone banach spaces*, Hacet. J. Math. Stat. **40** (2) (2011), 211–217.

[2] S. Alizadeh, F. Moradlou and Peyman Salimi, *Some fixed point results for (α, β)-(ψ, ϕ)-contractive mappings*, Filomat **28** (2014), 635–647.

[3] A. S. Cvetkovic, M. P. Stanic, S. Dimitrijevic and S. Simic, *Common Fixed Point Theorems for Four Mappings on Cone Metric Type Space*, Fixed point Theory Appl. 15 pages (2011) Article ID 589725.

[4] K. Deimling, *Nonlinear Functional Analysis,* Springer-Verlag, (1985).

[5] E. Karapinar, *Fixed point theorems in cone Banach spaces*, Fixed Point Theory and Appl. 9 pages (2009) Article ID 60928.

[6] G. Jungck. and B. E. Rhoades, *Fixed points for set valued functions without continuity*, Indian J. Pure and Appl. Math. **29** (3) (1998), 227–238.

[7] M. S. Khan and M. A. Imdad, *A common fixed point theorem for a class of mappings*, Indian J. Pure Appl. Math. **14** (1983), 1220–1227.

[8] L. G. Huang and X. Zhang, *Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl. **332** (2007), 1468–1476.

[9] S. D. Lin, *A common fixed point theorem in abstract spaces*, Indian J. Pure Appl. Math. **18** (8) (1987) 685–690.

[10] M. G. Maia, *Un'Osservazione sulle contrazioni metriche*, Ren. Sem. Mat. Univ. Padova **40** (1968), 139–143.

[11] N. Hussain and M. H. Shah, *KKM maps in cone b-metric spaces*, Comput. Math. Appl. **73** (2010) 3123–3129.

[12] F. Moradlou, P. Salimi and P. Vetro, *Some new extensions of Edelstein-Suzuki-type fixed point theorem to G-metric and G-cone metric spaces,* Acta Math. Sci. Ser. B Engl. Ed. 2013, **33** B (4):1049–1058.

[13] F. Moradlou, P. Salimi and P. Vetro, *Fixed point results for r-(η, ξ, ψ)-contractive mappings of type (I), (II) and (III),* Filomat **27** (2013), 403–410.

[14] Sh. Rezapour and R. Hamlbarani, *Some notes on the paper "one metric spaces and fixed point theorems of contractive mappings"*, J. Math. Anal. Appl. **347** (2008), 719–724.

[15] B. Rzepecki, *On fixed point theorems of Maia type*, Publications De Linstitut Mathématique **28** (1980), 179-186.

[16] D. Turkoglu and M. Abuloha, *Cone metric spaces and fixed point theorems in diametrically contractive mappings*, Acta. Math. Sin. **26** (3) (2010), 489–496.