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Meromorphic subordination results for p-valent functions associated with convolution

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Abstract

In this paper, by making use of the convolution and subordination principals, we obtain some subordination results for certain family of meromorphic p-valent functions defined by using a new linear operator.

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1. Introduction

Let $\sum_p$ be the class of functions of the form:

$$f(z) = z^{-p} + \sum_{k=0}^{\infty} a_k z^k \qquad (p \in \mathbb{N} = \{1, 2, ...\}), \tag{1.1}$$

which are analytic and p-valent in the punctured unit disk $U^* = U \backslash \{0\}$, where $U = \{z : z \in \mathbb{C}\ ,\ |z| < 1\}$. If f and g are analytic functions in U, we say that f is subordinate to g, written $f \prec g$ if there exists a Schwarz function w, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$, such that $f(z) = g(w(z)),\ z \in U$. Furthermore, if the function g is univalent in U, then we have the following equivalence ([5] and [10]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

For functions $f,\ g\ \in \Sigma_p$, Aouf et al. [3] defined the linear operator $D^n_{\lambda,p}(f * g)(z) : \Sigma_p \longrightarrow \Sigma_p$ $(\lambda \geq 0,\ p \in \mathbb{N},\ n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$ by

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$$D^0_{\lambda,p}(f*g)(z) = (f*g)(z),$$

$$\begin{aligned} D^1_{\lambda,p}(f*g)(z) &= D_{\lambda,p}(f*g)(z) = (1-\lambda)(f*g)(z) + \lambda z^{-p}\ (z^{p+1}(f*g)(z))' \\ &= z^{-p} + \sum_{k=0}^{\infty}[1+\lambda(k+p)]a_k b_k z^k\ (\lambda \geq 0;\ p \in \mathbb{N}), \\ D^2_{\lambda,p}(f*g)(z) &= D_{\lambda,p}(D_{\lambda,p}(f*g))(z) \\ &= z^{-p} + \sum_{k=0}^{\infty}[1+\lambda(k+p)]^2 a_k b_k z^k\ (\lambda \geq 0;\ p \in \mathbb{N}) \end{aligned}$$

and (in general)

$$\begin{aligned} D^n_{\lambda,p}(f*g)(z) &= D_{\lambda,p}(D^{n-1}_{\lambda,p}(f*g)(z)) \\ &= z^{-p} + \sum_{k=0}^{\infty}[1+\lambda(k+p)]^n a_k b_k z^k\ (\lambda \geq 0;\ p \in \mathbb{N};\ n \in \mathbb{N}_0). \end{aligned}$$

(1.2)

From (1.2) it is easy to verify that [3]:

(1.3) $$z(D^n_{\lambda,p}(f*g)(z))' = \frac{1}{\lambda}D^{n+1}_{\lambda,p}(f*g)(z) - (p+\frac{1}{\lambda})D^n_{\lambda,p}(f*g)(z)\ (\lambda > 0).$$

Specializing the parameters n, l, p, λ and g in (1.2), we have:

(i) For $n = 0$ and $g(z)$ is in the form:

(1.4) $$g(z) = z^{-p} + \sum_{k=0}^{\infty} \frac{(\alpha_1)_{k+p}...(\alpha_q)_{k+p}}{(\beta_1)_{k+p}...(\beta_s)_{k+p}(1)_{k+p}} z^k,$$

$\alpha_1, \alpha_2, ..., \alpha_q$ and $\beta_1, \beta_2, ..., \beta_s$ are complex or real ($\beta_j \notin Z_0^- = \{0, -1, -2, ...\}$, $j = 1, 2, ..., s$), we have, $D^n_{\lambda,p}(f*g)(z) = H_{p,q,s}(\alpha_1)f(z)$,where the linear operator $H_{p,q,s}(\alpha_1)$ was investigated recently by Liu and Srivastava [9] and Aouf [2] and contains in turn the operator $L_p(a,c)$ (see [8]) for $q = 2,\ s = 1,\ \alpha_1 = a > 0,\ \beta_1 = c\ (c \neq 0, -1, ...)$ and $\alpha_2 = 1$ and also contains the operator $D^{\nu+p-1}$ (see [13]) for $q = 2,\ s = 1,$ $\alpha_1 = \nu + p\ (\nu > -p,\ p \in \mathbb{N})$ and $\alpha_2 = \beta_1 = p$;

(ii) For $n = 0$ and $g(z)$ is in the form:

(1.5) $$g(z) = z^{-p} + \sum_{k=1-p}^{\infty} [\frac{l+\lambda(k+p)}{l}]^m a_k b_k z^k\ \ (\lambda, l \geq 0; m \in \mathbb{N}_0),$$

we have $D^0_{\lambda,p}(f*g)(z) = I^m_p(l,\lambda)f(z)$, where the operator $I^m_p(l,\lambda)$ was introduced and studied by El-Ashwah [6] and El-Ashwah and Aouf [7];

(iii) For $n = 0$ and $g(z)$ is in the form:

(1.6) $$g(z) = z^{-p} + \frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)} \sum_{k=1}^{\infty} \frac{\Gamma(k+p+\beta)}{\Gamma(k+p+\beta+\alpha)} z^k\ (\alpha \geq 0; \beta > -1),$$

we have $D^n_{\lambda,p}(f*g)(z) = Q^{\alpha}_{\beta,p}f(z)$ where the operator $Q^{\alpha}_{\beta,p}$ was introduced and studied by Aqlan et al.[4].

To prove our main results we need the next lemmas.

Lemma 1 [11]. *Let $q(z)$ be univalent in U and let $\varphi(z)$ be analytic in a domain containing $q(U)$. If $zq'(z)\varphi(q(z))$ is starlike and*

$$z\psi'(z)\varphi(\psi(z)) \prec zq'(z)\varphi(q(z)),$$

then $\psi(z) \prec q(z)$ *and* $q(z)$ *is the best dominant.*

Lemma 2 [12]. *Let* β, ν *be any complex numbers,* $\nu \neq 0$ and $q(z) = 1 + q_1 z + q_2 z^2 + \ldots$ *be univalent in* $U, q(z) \neq 0$. *Suppose that* $Q(z) = \gamma z q'(z)/q(z)$ *be starlike, and*

$$\Re\left\{\frac{\beta}{\nu} q(z) + \frac{zQ'(z)}{Q(z)}\right\} > 0.$$

If $\psi(z) = 1 + c_1 z + c_2 z^2 + \ldots$ *is analytic in* U *and satisfies*

$$\beta\psi(z) + \nu \frac{z\psi'(z)}{\psi(z)} \prec \beta q(z) + \nu \frac{zq'(z)}{q(z)},$$

then $\psi(z) \prec q(z)$ *and* $q(z)$ *is the best dominant.*

2. Main Results

Unless otherwise mentioned, we assume throughout this paper that, $\gamma \in \mathbb{C}, \lambda > 0, p \in \mathbb{N}, n \in \mathbb{N}_0, f,\, g \in \sum_p$ and the powers are the principal ones.

2.1. Theorem. *Let* $q(z) \neq 0$ *be univalent in* U *and* $zq'(z)/q(z)$*, be starlike. If* f *satisfies:*

$$\frac{1}{\lambda} \frac{D_{\lambda,p}^{n+1}(f*g)(z)}{D_{\lambda,p}^{n}(f*g)(z)} - \frac{\gamma}{\lambda} \frac{D_{\lambda,p}^{n+2}(f*g)(z)}{D_{\lambda,p}^{n+1}(f*g)(z)} \prec \frac{1-\gamma}{\lambda} + \frac{zq'(z)}{q(z)}, \tag{2.1}$$

then

$$\frac{z^{p(1-\gamma)} D_{\lambda,p}^{n+1}(f*g)(z)}{\left[D_{\lambda,p}^{n}(f*g)(z)\right]^{\gamma}} \prec q(z)$$

and $q(z)$ *is the best dominant.*

Proof. Let the function $p(z)$ defined by

$$p(z) = \frac{z^{p(1-\gamma)} D_{\lambda,p}^{n+1}(f*g)(z)}{\left[D_{\lambda,p}^{n}(f*g)(z)\right]^{\gamma}} \ (z \in U). \tag{2.2}$$

Differentiating (2.2) logarithmically with respect to z and using the identity (1.3), we have

$$\frac{zp'(z)}{p(z)} = \frac{1}{\lambda} \frac{D_{\lambda,p}^{n+2}(f*g)(z)}{D_{\lambda,p}^{n+1}(f*g)(z)} - \frac{\gamma}{\lambda} \frac{D_{\lambda,p}^{n+1}(f*g)(z)}{D_{\lambda,p}^{n}(f*g)(z)} - \frac{1}{\lambda}(1-\gamma),$$

that is, that

$$\frac{zp'(z)}{p(z)} + \frac{1}{\lambda}(1-\gamma) = \frac{1}{\lambda} \frac{D_{\lambda,p}^{n+2}(f*g)(z)}{D_{\lambda,p}^{n+1}(f*g)(z)} - \frac{\gamma}{\lambda} \frac{D_{\lambda,p}^{n+1}(f*g)(z)}{D_{\lambda,p}^{n}(f*g)(z)}. \tag{2.3}$$

Therefore, in view of (2.3), the subordination (2.1) becomes

$$\frac{zp'(z)}{p(z)} \prec \frac{zq'(z)}{q(z)}.$$

By an application of Lemma 1, with $\varphi(w) = \frac{1}{w}, w \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$, we have $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

Taking $n = 0$ and $g(z)$ of the form (1.4) and using the identity (see [9]):

$$z\left(H_{p,q,s}(\alpha_1) f(z)\right)' = \alpha_1 H_{p,q,s}(\alpha_1 + 1) f(z) - (\alpha_1 + p) H_{p,q,s}(\alpha_1) f(z), \tag{2.4}$$

we have the following corollary. □

2.2. Corollary. *Let $q(z) \neq 0$ be univalent in U and $zq'(z)/q(z)$, be starlike . If f satisfies*

$$(\alpha_1+1)\frac{H_{p,q,s}(\alpha_1+2)f(z)}{H_{p,q,s}(\alpha_1+1)f(z)} - \gamma\alpha_1\frac{H_{p,q,s}(\alpha_1+1)f(z)}{H_{p,q,s}(\alpha_1)f(z)} \prec \alpha_1(1-\gamma)+1+\frac{zq'(z)}{q(z)},$$

then

$$\frac{z^{p(1-\gamma)}H_{p,q,s}(\alpha_1+1)f(z)}{[H_{p,q,s}(\alpha_1)f(z)]^\gamma} \prec q(z)$$

and $q(z)$ is the best dominant.

Taking $q=2, s=1, \alpha_1=a>0, \beta_1=c>0$ and $\alpha_2=1$, in Corollary 1, we have the following result which correctes the result obtained by Ali and Ravichandran [1, Theorems 2.3].

2.3. Corollary. *Let $q(z) \neq 0$ be univalent in U and $zq'(z)/q(z)$, be starlike. If f satisfies*

$$(a+1)\frac{L_p(a+2;c)f(z)}{L_p(a+1;c)f(z)} - \gamma a\frac{L_p(a+1;c)f(z)}{L_p(a;c)f(z)} \prec a(1-\gamma)+1+\frac{zq'(z)}{q(z)},$$

then

$$\frac{z^{p(1-\gamma)}L_p(a+1;c)f(z)}{[L_p(a;c)f(z)]^\gamma} \prec q(z)$$

and $q(z)$ is the best dominant.

Taking $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \le B < A \le 1)$ in Theorem 1, we have

2.4. Corollary. *Let $-1 \le B < A \le 1$. If $f \in \sum_p$ satisfies*

$$\frac{1}{\lambda}\frac{D^{n+2}_{\lambda,p}(f*g)(z)}{D^{n+1}_{\lambda,p}(f*g)(z)} - \frac{\gamma}{\lambda}\frac{D^{n+1}_{\lambda,p}(f*g)(z)}{D^{n}_{\lambda,p}(f*g)(z)}$$

$$\prec \frac{1}{\lambda}(1-\gamma) + \frac{(A-B)z}{(1+Az)(1+Bz)},$$

then

$$\frac{z^{p(1-\gamma)}D^{n+1}_{\lambda,p}(f*g)(z)}{\left[D^{n}_{\lambda,p}(f*g)(z)\right]^\gamma} \prec \frac{1+Az}{1+Bz}.$$

Taking $n=0$ and $g(z)$ in the form (1.4) with $q=2, s=1, \alpha_1=a, \beta_1=c, a,c>0$ and $\alpha_2=1$, in Corollary 3, we have the following result which corrects the result obtained by Ali and Ravichandran [1, Corollary 2.4].

2.5. Corollary. *Let $-1 \le B < A \le 1$. If $f \in \sum_p$ satisfies*

$$(a+1)\frac{L_p(a+2;c)f(z)}{L_p(a+1;c)f(z)} - \gamma a\frac{L_p(a+1;c)f(z)}{L_p(a;c)f(z)}$$

$$\prec a(1-\gamma)+1+\frac{(A-B)z}{(1+Az)(1+Bz)},$$

then

$$\frac{z^{p(1-\gamma)}L_p(a+1;c)f(z)}{[L_p(a;c)f(z)]^\gamma} \prec \frac{1+Az}{1+Bz}.$$

By appealing to Lemma 2, we prove the following theorem.

2.6. Theorem. *Let $\gamma \neq 0$ and $q(z)$ be univalent in U, $q(z) \neq 0$, $Q(z) = \gamma z q'(z)/q(z)$ be starlike and*

$$\Re\left\{\frac{1}{\lambda\gamma} q(z) + \frac{zQ'(z)}{Q(z)}\right\} > 0 \quad (z \in U). \tag{2.5}$$

If $f(z) \in \sum_p$ satisfies

$$(1-\gamma)\frac{D^{n+1}_{\lambda,p}(f*g)(z)}{D^{n}_{\lambda,p}(f*g)(z)} + \gamma\frac{D^{n+2}_{\lambda,p}(f*g)(z)}{D^{n}_{\lambda,p}(f*g)(z)} \prec q(z) + \lambda\gamma\frac{zq'(z)}{q(z)},$$

then

$$\frac{D^{n+1}_{\lambda,p}(f*g)(z)}{D^{n}_{\lambda,p}(f*g)(z)} \prec q(z)$$

and $q(z)$ is the best dominant.

Proof. Let the function $p(z)$ defined by

$$p(z) = \frac{D^{n+1}_{\lambda,p}(f*g)(z)}{D^{n}_{\lambda,p}(f*g)(z)} \quad (z \in U). \tag{2.6}$$

Differentiating (2.6) logarithmically with respect to z and using the identity (1.3), we have

$$\frac{D^{n+2}_{\lambda,p}(f*g)(z)}{D^{n+1}_{\lambda,p}(f*g)(z)} = p(z) + \lambda\frac{zp'(z)}{p(z)},$$

therefore, we have

$$(1-\gamma)\frac{D^{n+1}_{\lambda,p}(f*g)(z)}{D^{n}_{\lambda,p}(f*g)(z)} + \gamma\frac{D^{n+2}_{\lambda,p}(f*g)(z)}{D^{n+1}_{\lambda,p}(f*g)(z)} = p(z) + \lambda\gamma\frac{zp'(z)}{p(z)}.$$

From (2.5), we have

$$p(z) + \lambda\gamma\frac{zp'(z)}{p(z)} \prec q(z) + \lambda\gamma\frac{zq'(z)}{q(z)}.$$

By an application of Lemma 2, it follows that $p(z) \prec q(z)$ and $q(z)$ is the best dominant. □

Taking $n = 0$ and $g(z)$ of the form (1.4) and using the identity (2.4) we have the following corollary.

2.7. Corollary. *Let $\gamma \neq 0, \alpha_1 \neq -1$ and $q(z)$ be univalent in U, $q(z) \neq 0$, $Q(z) = \gamma z q'(z)/q(z)$ be starlike and*

$$\Re\left\{\frac{\alpha_1 + 1 - \gamma}{\gamma} q(z) + \frac{zQ'(z)}{Q(z)}\right\} > 0 \quad (z \in U).$$

If $f(z) \in \sum_p$ satisfies

$$(1-\gamma)\frac{H_{p,q,s}(\alpha_1+1)f(z)}{H_{p,q,s}(\alpha_1)f(z)} + \gamma\frac{H_{p,q,s}(\alpha_1+2)f(z)}{H_{p,q,s}(\alpha_1+1)f(z)}$$

$$\prec \frac{1}{\alpha_1+1}\left[\gamma + (1+\alpha_1-\gamma)q(z) + \gamma\frac{zq'(z)}{q(z)}\right], \tag{2.7}$$

then

$$\frac{H_{p,q,s}(\alpha_1+1)f(z)}{H_{p,q,s}(\alpha_1)f(z)} \prec q(z)$$

and $q(z)$ is the best dominant.

Remarks. (*i*) *Taking* $n = 0$ *and* $g(z)$ *in the form* (1.4) *with* $q = 2, s = 1, \alpha_1 = a > 0, \beta_1 = c > 0$ *and* $\alpha_2 = 1$, *in Corollary 5, we have the result obtained by Ali and Ravichandran [1, Theorem 2.5];*

(*ii*) *Taking* $n = 0$ *and* $g(z)$ *of the form* (1.5) and using the identity [6]:

$$\lambda z \left(I_p^m(\lambda, l) f(z) \right)' = l I_p^{m+1}(\lambda, l) f(z) - (\lambda p + l) I_p^m(\lambda, l) f(z), \lambda > 0,$$

in our results, we have the results corresponding to the operator $I_p^m(\lambda, l)$*;*

(*iii*) *Taking* $n = 0$ *and* $g(z)$ *of the form* (1.6) *and using the identity [4]:*

$$z(Q_{\beta,p}^{\alpha} f(z))' = (\alpha + \beta - 1) Q_{\beta,p}^{\alpha-1} f(z) - (\alpha + \beta + p - 1) Q_{\beta,p}^{\alpha} f(z), \alpha \geq 0; \beta > -1,$$

in our results, we have the results corresponding to the operator $Q_{\beta,p}^{\alpha}$.

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