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AN ALMOST 2–PARACONTACT STRUCTURE ON THE COTANGENT BUNDLE OF A CARTAN SPACE

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Abstract

A Cartan space is a pair (M, K), where M is a smooth manifold and K an Hamiltonean on the slit cotangent bundle $T_0^*M := TM \setminus \{(x, 0),\ x \in M\}$, that is positively homogeneous of degree 1 in momenta. We show that K induces an almost 2–paracontact Riemannian structure on T_0^*M whose restriction to the figuratrix bundle $\mathbb{K} = \{(x, p) \mid K(x, p) = 1\}$ is an almost paracontact structure. A condition for this almost paracontact structure to be normal is found, and its geometrical meaning is pointed out. Similar results for Finsler spaces can be found in [1] and [3].

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1. Introduction

Let (N, h) be an m–dimensional Riemannian manifold. If on N there exists a tensor field ϕ of type $(1, 1)$, r vector fields $\xi_1, \xi_2, \dots, \xi_r$, $(r < m)$, and r, 1–forms $\eta^1, \eta^2, \dots, \eta^r$ such that

(i) $\eta^a(\xi_b) = \delta^a_b,\ a, b \in (r) = \{1, 2, \dots, r\}$,
(ii) $\phi^2 = I - \sum_a \eta^a \otimes \xi_a$,
(iii) $\eta^a(X) = h(X,\ \xi_a),\ \ a \in (r)$,
(iv) $h(\phi X, \phi Y) = h(X, Y) - \sum_a \eta^a(X)\eta^a(Y)$,

where X, Y are vector fields and I is the identity tensor field on N, then

$$\Sigma = (\phi,\ \xi_a,\ \eta^a)_{a \in (r)}$$

is said to be an *almost r–paracontact Riemannian structure on M*, and the pair (M, Σ) is called an *almost r–paracontact Riemannian manifold.*

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From (i) through (iv) it follows that (see [2]):

$$\phi(\xi_a) = 0,\ \eta^a \circ \phi = 0,\ a \in (r)$$
$$\phi(x, y) := h(\phi X,\ Y) = h(X,\ \phi Y).$$

A Cartan space is a pair $K^n = (M, K)$, where M is a smooth $n-$dimensional manifold and K is a positive function on the cotangent bundle $T_0^*M := T^*M \setminus \{(x, 0) \mid x \in M\}$ such that the function K^2 is a regular Hamiltonian which is homogeneous of degree 2 in momenta.

It is well known that T_0^*M is a Riemannian manifold with a Riemannian metric similar to the Sasaki metric, completely determined by K.

We show in Section 2 that, moreover, T_0^*M can be naturally endowed with an almost 2-paracontact Riemannian structure. Section 1 is devoted to some preliminaries, especially regarding the geometry of T^*M. In Section 3 we consider the figuratrix bundle $\mathbb{K} = \{u \in T^*M \mid K(u) = 1\}$ as a submanifold of T_0^*M of codimension one, and we show that it carries an almost paracontact Riemanian structure induced by the above mentioned almost 2-paracontact Riemannian structure on T_0^*M. A condition for this paracontact Riemannian structure to be normal is established and its geometric meaning is discussed.

2. Preliminaries

We recall from Chapter 6 of [4] some facts from the geometry of the cotangent bundle T^*M. We take (x^i), $i = 1, 2, \ldots, n$ as local coordinates on M. The induced local coordinates on T^*M will be denoted by $(x^i,\ p_i)$, where x^i is in fact $x^i \circ \tau^*$, for $\tau^* : T^*M \to M$ the natural projection, and p_i are the components of a covector from T_x^*M, $x(x^i)$, in the cobasis $(dx^i)_x$. The coordinates p_i will be called momenta.

The kernel of the differential $D\tau^*$ of τ^* is a subbundle of TT_0^*M, known as vertical and denoted by VT_0^*M. The vertical distribution $V : u \in T^*M \to V_uT_0^*M = \ker(D\tau^*)_u$ is locally spanned by $\dot{\partial}^i := \dfrac{\partial}{\partial p_i}$, hence it is integrable.

The vector field $C^* = p_i\, \dot{\partial}^i$ on T^*M is called the Liouville vector field.

Let $K^n = (M, K)$ be a Cartan space. Then the function $K : T^*M \to \mathbb{R}$ has the properties

(i) K is smooth on T_0^*M and only continuous on the set $\{(x, 0) \mid x \in M\}$,
(ii) $K > 0$ on T^*M,
(iii) $K(x,\ \lambda p) = \lambda K(x, p)$ for any $\lambda > 0$,
(iv) The matrix with entries $g^{ij}(x, p) = \frac{1}{2}\, \dot{\partial}^i \dot{\partial}^j\ K^2$ is positive definite.

If one sets $p^i = \frac{1}{2}\, \dot{\partial}^i\ K^2$then $g^{ij} = \dot{\partial}^j\ p^i$, and from the homogeneity condition (iii) there results

$$p^i = g^{ij}p_j,\ K^2 = g^{ij}p_ip_j = p_ip^j, \tag{2.1}$$
$$C^{ijk}p_k = 0,\ \text{where } C^{ijk} := \tfrac{1}{2}\, \dot{\partial}^i\ g^{jk}.$$

As $\det(g^{ij}) \neq 0$, from the first equation in (1.1) it follows that $p_i = g_{ij}p^j$, where (g_{ij}) is the inverse of the matrix (g^{ij}).

In the following we restrict our considerations to the open submanifold T_0^*M of T^*M.

A nonlinear connection on T_0^*M is a distribution $u \to H_uT_0^*M$, called horizontal, which is supplementary to the vertical distribution. This is usually given by a local basis $\delta_i = \partial_i + N_{ij}(x, p)\, \dot{\partial}^j$ for some functions (N_{ij}) having a special behavior, by a change of coordinates on T^*M. It was proved by R. Miron [4, Chapter 4] that any Cartan space has a canonical nonlinear connection, completely determined by K, whose coefficients $(N_{ij}(x,p))$ are positively homogeneous of degree 1 in momenta, and have the property $N_{ij} = N_{ji}$.

Thus we have a decomposition

(2.2) $$T_u T_0^* M = V_u T_0^* M \oplus H_u T_0^* M,$$

and $(\delta_i, \dot{\partial}^i)$ is a basis adapted to it.

This suggests the following definition of an almost product structure Q on T^*M.

(2.3) $$Q(\delta_i) = g_{ij}\,\dot{\partial}^j,\ Q(\dot{\partial}^i) = g^{ij}\delta_j.$$

It is easy to check that $Q^2 = I$.

Using the matrices (g_{ij}) and (g^{ij}) the following Riemannian metric on T_0^*M is defined

(2.4) $$G = g_{ij}dx^i \otimes dx^j + g^{ij}\delta p_i \otimes \delta p_j,$$

where $\delta p_i = dp_i - N_{ij}(x,p)dx^j$ and $(dx^i, \delta p_i)$ is the dual basis of $(\delta_i, \dot{\partial}^i)$.

The Riemannian metric G is similar to the Sasaki metric on the tangent bundle. An easy calculation gives:

(2.5) $$G(QX, QY) = G(X, Y), X, Y \in \mathcal{X}(T_0^*M).$$

Here $\mathcal{X}(T^*M)$ is the $\mathcal{F}(M)$–module of vector fields on T_0^*M.

3. An almost 2-paracontact Riemannian structure on T_0^*M when (M, K) is a Cartan space

We already know that (T_0^*M, G) is a Riemannian manifold. On T_0^*M there exist two globally defined vector fields:

$$\xi_1 = \frac{1}{K}p^i\delta_i \text{ and } \xi_2 = \frac{1}{K}p_i\,\dot{\partial}^i.$$

They are linearly independent. The second one is collinear with the Liouville vector field, while the first one is nothing but the Hamiltonian vector field of the function K.

Indeed, the Hamiltonian vector field of K is

$$\overrightarrow{K} = (\dot{\partial}^i K)\frac{\partial}{\partial x^i} - (\partial_i K)\frac{\partial}{\partial p_i} = (\dot{\partial}^i K)\delta_i - (\delta_i K)\,\dot{\partial}^i = \xi_1,$$

because for a Cartan space $\delta_i K = 0$ and $\dot{\partial}^i K = \dfrac{p^i}{K}$.

Now we consider the 2 1-forms

$$\eta^1 = \frac{1}{K}p_i dx^i \text{ and } \eta^2 = \frac{1}{K}p^i\delta p_i.$$

These are globally defined. It quickly follows that

(3.1) $$\eta^a(\xi_b) = \delta^a_b,\ a, b \in \{1, 2\}.$$

One easily checks that

(3.2) $$Q(\xi_1) = \xi_2,\ Q(\xi_2) = \xi_1,$$

(3.3) $$\eta^1 \circ Q = \eta^2,\ \eta^2 \circ Q = \eta^1.$$

Using $Q,\ \xi_a,\ \eta^a,\ a \in \{1, 2\}$, we construct the tensor field

$$q = Q - \eta^2 \otimes \xi_1 - \eta^1 \otimes \xi_2.$$

Based on (3.1) - (3.3) it comes out that

(3.4) $$q^2(X) = X - \eta^1(X)\xi_1 - \eta^2(X)\xi_2.$$

In the adapted frame $(\delta_i, \dot{\partial}^i)$ we have

$$G(\delta_i, \delta_j) = g_{ij}, G(\delta_i, \dot{\partial}^j) = 0,\ G(\dot{\partial}^i, \dot{\partial}^j) = g^{ij}.$$

These equations are used to verify that

$$\eta^a(X) = G(X, \xi_a),\ a \in \{1,2\}, \tag{3.5}$$

holds for $X = \delta_i, \dot{\partial}^i$.

A direct calculation using (2.5), (3.5), as well as $G(\xi_a, \xi_b) = \delta_{ab}$, gives

$$G(qX, qY) = G(X,Y) - \eta^1(X)\eta^1(Y) - \eta^2(X)\eta^2(Y),\ X, Y \in \mathcal{X}(T_0^*M). \tag{3.6}$$

The equations (3.1), (3.4) - (3.6) show that the following theorem holds good.

3.1. Theorem. *Let $K^n = (M, K)$ be a Cartan space. Then T_0^*M is an almost 2-paracontact Riemannian manifold with the almost 2-paracontact Riemannian structure $(q,\ \xi_a,\ \eta^a, G),\ a \in \{1,2\}$.*

The next equations follow easily from (3.1) and (3.2)

$$q(\xi_a) = 0,\ \eta^a \circ q = 0. \tag{3.7}$$

By (3.4) and (3.7) we have

$$q^3 - q = 0. \tag{3.8}$$

Now we prove

3.2. Lemma. *The rank of q is $2n-2$.*

Proof. By the first equation (3.7), the subspace span $\{\xi_1, \xi_2\}$ is contained in $\ker q$. Let now $X \in \ker q$. If $X = X^i\delta_i + Y_i\ \dot{\partial}^i$, the condition $q(X) = 0$ gives $X^i = \dfrac{X^j p_j}{K^2} p^i$, $Y_i = \dfrac{Y_i p^j}{K^2} p_i$, hence $X \in \text{span}\{\xi_1, \xi_2\}$ □

Let $h(X,Y) = G(qX, Y),\ X, Y \in \mathcal{X}(T_0^*M)$.

3.3. Theorem. *The mapping h is bilinear and symmetric and its null space is* $\ker q$.

Proof. The bilinearity is obvious. The symmetry holds even in a more general setting cf. Section 1. The null space of h is $\{X \mid h(X,Y) = 0,\ \forall Y\} = \{X \mid G(qX, Y) = 0,\ \forall Y\} = \{X \mid qX = 0\} = \ker q$. □

In the adapted basis $(\delta_i, \dot{\partial}^i)$ we have

$$\begin{aligned} &q(\delta_i) = A_{ij}\ \dot{\partial}^j,\ q(\dot{\partial}^i) = B^{ij}\delta_j, \\ &A_{ij} = g_{ij} - \frac{1}{K^2} p_i p_j,\ B^{ij} = g^{ij} - \frac{1}{K^2} p^i p^j. \end{aligned} \tag{3.9}$$

We notice that these matrices have rank $n-1$ because of

$$A_{ij}p^j = 0,\ B^{ij}p_j = 0. \tag{3.10}$$

The mapping h has the form

$$h = A_{ij}dx^i \otimes dx^j - B^{ij}\delta p_i \otimes \delta p_j.$$

Thus it is a singular pseudo-Riemannian metric on T_0^*M.

4. An almost paracontact structure on the figuratrix bundle of a Cartan space K^n

The set $\mathbb{K} = \{(x, p) \in T_0^*M \mid K(x, p) = 1\}$ will be called the *figuratrix bundle* of the Cartan space K^n. It will be thought of as a hypersurface (submanifold of codimension 1) of T_0^*M, endowed with the almost 2-paracontact Riemannian structure from Theorem 3.1.

Let

$$\text{(4.1)} \qquad \begin{cases} x^i = x^i(u^\alpha), \\ p_i = p_i(u^\alpha), \;\; \alpha = 1, 2, \ldots, 2n-1, \end{cases}$$

with rank $\left(\dfrac{\partial x^i}{\partial u^\alpha}, \dfrac{\partial p_i}{\partial u^\alpha}\right) = 2n - 1$, a parametrization of $\mathbb{K}$.

We consider the local vector fields $\dfrac{\partial}{\partial u^\alpha} = \dfrac{\partial x^i}{\partial u^\alpha}\partial_i + \dfrac{\partial p_i}{\partial u^\alpha}\,\dot{\partial}^i$, which provide a local basis in the tangent bundle to $\mathbb{K}$, and put them into the form

$$\frac{\partial}{\partial u^\alpha} = \frac{\partial x^i}{\partial u^\alpha}\delta_i + \left(\frac{\partial p_j}{\partial u^\alpha} - N_{ij}\frac{\partial x^i}{\partial u^\alpha}\right)\dot{\partial}^j .$$

It follows that $G\left(\dfrac{\partial}{\partial u^\alpha}, \xi_2\right) = \dfrac{1}{K}p^j\left(\dfrac{\partial p_j}{\partial u^\alpha} - N_{ij}\dfrac{\partial x^i}{\partial u^\alpha}\right)$. On the other hand, by deriving the identity $K^2(x^i(u^\alpha), p_i(u^\alpha)) = 1$ with respect to u^α we find

$$\begin{aligned} 0 &= (\partial_i K^2)\frac{\partial x^i}{\partial u^\alpha} + (\dot{\partial}^i K^2)\frac{\partial p_i}{\partial u^\alpha} \\ &= (\delta_i K^2)\frac{\partial x^i}{\partial u^\alpha} + (\dot{\partial}^i K^2)\left(\frac{\partial p_i}{\partial u^\alpha} - N_{ij}\frac{\partial x^j}{\partial u^\alpha}\right) \\ &= 2p^i\left(\frac{\partial p_i}{\partial u^\alpha} - N_{ij}\frac{\partial x^j}{\partial u^\alpha}\right) \end{aligned}$$

because, for a Cartan space, $\delta_i K = 0$. Thus on $\mathbb{K}$ we have $G = \left(\dfrac{\partial}{\partial u^\alpha}, \xi_2\right) = 0$ for every $\alpha = 1, 2, \ldots, 2n-1$. In other words, the vector field ξ_2 restricted to $\mathbb{K}$ is normal to $\mathbb{K}$.

Recall that ξ_2 restricted to $\mathbb{K}$ is $\overline{\xi_2} = p_i(u^\alpha)\,\dot{\partial}^i$.

4.1. Lemma. *The hypersurface $\mathbb{K}$ is invariant with respect to q i.e. $q(T_u\mathbb{K}) \subset T_u\mathbb{K}$, $\forall u \in \mathbb{K}$.*

Proof. $G\left(q\left(\dfrac{\partial}{\partial u^\alpha}\right), \xi_2\right) = (\eta^2 \circ q)\left(\dfrac{\partial}{\partial u^\alpha}\right) = 0 \;\forall\, \alpha = 1, 2, \ldots, 2n-1.$ □

By item (i) in Theorem 3.1 from [2], and Lemma 4.1, there follows:

4.2. Theorem. *The almost 2-paracontact Riemannian structure $(q,\ \xi_a,\ \eta^a, G)$, $a \in \{1, 2\}$, on T_0^*M induces by restriction an almost paracontact Riemannian structure on the figuratrix bundle $\mathbb{K}$.*

If we use overlines to denote the restrictions we have

- $\overline{\xi_1} = p^i\delta_i$, is tangent to $\mathbb{K}$
- $\overline{\eta_2} = 0$ because of $\eta^2(X) = G(X, \xi_2) = 0$ for every X tangent to $\mathbb{K}$,
- $\overline{G} = G|_{\mathbb{K}}$,
- $\overline{q} = \overline{Q} - \overline{\eta}^1 \otimes \overline{\eta}^2$ is an automorphism of $T_u\mathbb{K}$, $\forall u \in \mathbb{K}$.

We put $\xi = \overline{\xi}_1$, $\eta = \overline{\eta}_1$, so the almost paracontact Riemannian structure given by Theorem 4.2 is $(\overline{q}, \xi, \eta, \overline{G})$. We mention that

$$\overline{q}^2 = I - \eta \otimes \xi,\ \eta(X) = G(X, \xi), \tag{4.2}$$

$$\overline{G}(\overline{q}X, \overline{q}Y) = \overline{G}(X,Y) - \eta(X)\eta(Y),\ X, Y \in \mathfrak{X}(\mathbb{K}). \tag{4.3}$$

By Theorem 1.1 in [2], the almost paracontact Riemannian structure $(\overline{q}, \xi, \eta, \overline{G})$ is **normal** if and only if

$$N(X,Y) := N_{\overline{q}}(X,Y) - 2d\eta(X,Y)\xi = 0, \tag{4.4}$$

where $N_{\overline{q}}$ is the Nijenhuis tensor field of $\overline{q}$, i.e.

$$N_{\overline{q}}(X,Y) = [\overline{q}X, \overline{q}Y] + \overline{q}^2[X,Y] - \overline{q}[\overline{q}X, Y] - \overline{q}[X, \overline{q}Y],\ \forall X, Y \in \mathfrak{X}(\mathbb{K}). \tag{4.5}$$

Now we look for conditions under which $(\overline{q}, \xi, \eta, \overline{G})$ is normal.

If we put $\dot{\delta}_j = \overline{q}(\delta_j)$ we get n local vector fields that are tangent to $\mathbb{K}$, because $\mathbb{K}$ is an invariant hypersurface. These, together with δ_i, $i = 1, 2, \ldots, n$, are all tangent to $\mathbb{K}$ and they are not linearly independent. But if we consider δ_i, $i = 1, 2, \ldots, n$ and $\dot{\delta}_j$ with $j = 1, 2, \ldots, n-1$, we obtain a set $(\delta_i, \dot{\delta}_j)$ of local vector fields that form a local basis in the tangent bundle to $\mathbb{K}$. We shall compute N from (4.4) in this basis.

First, we note that

$$\begin{aligned} &\dot{\delta}_j = \overline{q}(\delta_j) = a_{jk}\, \dot{\partial}^k,\ a_{jk} = g_{jk} - p_j p_k \\ &\overline{q}(\dot{\partial}^k) = g^{ki}\delta_i. \end{aligned} \tag{4.6}$$

$$\begin{aligned} &\overline{q}^2(\delta_i) = b_i^k \delta_k,\ b_i^k = \delta_i^k - p_i p^k \\ &\overline{q}^2(\dot{\partial}^k) = b_i^k\, \dot{\partial}^i\ . \end{aligned} \tag{4.7}$$

Secondly, we recall some formulae related to K^n from [4],

$$\begin{aligned} &[\delta_i, \delta_j] = R_{kij}\, \dot{\partial}^k,\ R_{kij} = \delta_j N_{ki} - \delta_i N_{kj}, \\ &\left[\delta_i, \dot{\partial}^j\right] = -(\dot{\partial}^j N_{ik})\, \dot{\partial}^k\ . \end{aligned} \tag{4.8}$$

$$p_k\, \dot{\partial}^k\, N_{ij} = N_{ij}\ \text{(the homogeneity of } N_{ij} \text{ in momenta).} \tag{4.9}$$

Assume that K^n is endowed with the linear Cartan connection $(H^i_{jk}, C^j k_i = g_{is}C^{sjk})$. Denote by $_{|k}$ and $|^k$ the horizontal and vertical covariant derivatives, respectively. Then we have

$$\begin{aligned} &K^2_{|j} := \delta_j K^2 = 0, K^2|^j = 2p^j, p_{i|j} = 0,\ p_i|^j = \delta_i^j \\ &p^i_{|j} = 0, p^i|^j = g^{ij}. \end{aligned} \tag{4.10}$$

$$R_{kij}p^k = 0, P^i_{jk}p^j = 0,\ P^i_{jk} := H^i_{jk} - \dot{\partial}^i\, N_{jk}. \tag{4.11}$$

$$\delta_i g_{jk} = H^s_{ji} g_{sk} + H^s_{ki} g_{js}. \tag{4.12}$$

Now we compute:

$$\begin{aligned} &2d\eta(\delta_i, \delta_j) = \delta_i p_j - \delta_j p_i = 0 \text{ since by (3.9), } \delta_i p_j = H^s_{ij} p_s, \\ &2d\eta(\delta_i, \dot{\delta}_j) = -\dot{\delta}_j p_i = -a_{ij}, \\ &2d\eta(\dot{\partial}^i, \dot{\partial}^j) = 0. \end{aligned}$$

And further,

$$(4.13)\quad \begin{aligned} &N(\delta_i,\delta_j) = A_{hij}g^{hk}\delta_k + (B_{kij} - R_{kij})\ \dot{\partial}^j \\ &N(\delta_i,\dot{\delta}_j) = (a_{ih}\ \dot{\partial}^h\ b^s_j - b^k_j R_{hik}g^{hs} - B_{hij}g^{hs} - B_{hij}g^{hs} + a_{ij}p^s)\delta_s - \\ &\qquad - \left[A_{kij} - p_j p^r(\delta_r a_{ik} - a_{ih}\ \dot{\partial}^h\ N_{rk}) + a_{kr}\delta_i b^r_j\right]\ \dot{\partial}^k, \\ &N(\dot{\delta}_i,\dot{\delta}_j) = (D^s_{ij} - D^s_{ji})\delta_s + (b^k_i b^h_j R_{rkh} - B_{rij} + a_{rk}E^k_{ij})\ \dot{\partial}^r, \end{aligned}$$

where

$$(4.14)\quad \begin{aligned} &A_{kij} = \delta_i a_{jk} - \delta_j a_{ik} + a_{ih}\ \dot{\partial}^h\ N_{jk} - a_{jh}\ \dot{\partial}^h\ N_{ik}, \\ &B_{kij} = a_{ih}\ \dot{\partial}^h\ a_{jk} - a_{jh}\ \dot{\partial}^h\ a_{ik}, \\ &D^s_{ij} = b^k_i\delta_k b^s_j + (b^k_j\delta_k a_{ih} + b^k_i a_{jh}\dot{\partial}^h N_{kr})g^{hs}, \\ &E^k_{ij} = a_{ih}\ \dot{\partial}^h\ b^k_j - a_{jh}\ \dot{\partial}^h\ b^k_i. \end{aligned}$$

The expressions from (4.14) can be simplified as follows.

First, using (1.1) one easily obtains that $B_{kij} = p_i g_{jk} - p_j g_{ik}$.

From $p^k a_{kr} = 0$ it follows that $(\delta_i p^k)a_{kr} = -(\delta_i a_{kr})p^r$, and $p_{i|k} = 0$ is equivalent to $\delta_i p_k = N_{ik}$. Based on these formulae we find that the vertical part of $N(\delta_i,\dot{\delta}_j)$ is $-b^k_j A_{rik}\ \dot{\partial}^r$ and its horizontal part takes the form $b^k_j(p_i g_{kh} - p_k g_{ih} - R_{hik})g^{hs}\delta_s$.

A tedious computation gives

$$D^s_{ij} - D^s_{ji} = A_{hij}g^{hs} + p^k P^r_{kh}(p_i g_{rj} - p_j g_{ir}) = A_{hij}g^{hs}$$

by (4.11).

We have $E^k_{ij} = p_i\delta^k_j - p_j\delta^k_i$, and using this we get that the vertical part of $N(\dot{\delta}_i,\dot{\delta}_j)$ is $(R_{kij} + p_i R_{kjo} - p_j R_{rio})\partial^r$, where $R_{kjo} = R_{kjs}p^s$.

The tensor field A_{kij} takes the form $A_{kij} = \delta_i g_{jk} - \delta_j g_{ik} + g_{ih}\ \dot{\partial}^h\ N_{jk} - g_{jh}\ \dot{\partial}^h\ N_{ik}$ and by (iii) of Prop. 2.3 in Chapter 7 of [4], it vanishes for Cartan spaces.

Gathering together the above facts we obtain

4.3. Theorem. *In the frame $(\delta_i,\dot{\delta}_j)$, $j = 1,2,\ldots,n-1$, the tensor field N given by* (4.4) *has the form*

$$\begin{aligned} &N(\delta_i,\delta_j) = (p_i g_{jk} - p_j g_{ik} - R_{kij})\ \dot{\partial}^j, \\ &N(\delta_i,\dot{\delta}_j) = -b^k_j(p_i g_{jk} - p_k g_{ih} - R_{hik})g^{hs}\delta_s, \\ &N(\dot{\delta}_i,\dot{\delta}_j) = (R_{kij} + p_i R_{kjo} - p_j R_{kio})\ \dot{\partial}^k\ . \end{aligned}$$

4.4. Corollary. *The almost paracontact Riemannian structure $(\overline{q},\ \xi,\ \eta,\ \overline{G})$ is normal if and only if*

$$(4.15)\quad R_{kij} = p_i g_{jk} - p_j g_{ik}.$$

Proof. One easily checks that if R_{kij} has the form given by (4.15), then $R_{kij} + p_i R_{kjo} - p_j R_{kio} = 0$. Then the conclusion is obvious. □

We give a geometrical meaning to (4.15), showing that it implies that the Cartan space K^n is of constant scalar curvature -1.

A Cartan space K^n is of constant scalar curvature c if

$$(4.16)\quad H_{hijk}p^i p^j X^h X^k = c(g_{hj}g_{ik} - g_{hk}g_{ij})p^i p^j X^h X^k$$

for every $(x,p) \in T_0^*M$ and $X = (X^i) \in T_xM$. Here, H_{hijk} is the $(hh)h$–curvature of the linear Cartan connection of K^n.

We replace H_{hijk} in (4.16) with $g_{is}\ H^s_{hjk}$ and so it reduces to

(4.16') $\quad p_s H^s_{hjk} p^j X^h X^k = c(p_h p_k - g_{hk}) X^h X^k.$

on $\mathbb{K}$.

By Proposition 5.1 (ii) in Chapter 7 of [4], $p_s H^s_{hjk} = -R_{khj}$, hence we get $R_{kho} X^h X^k = c(g_{hk} - p_h p_k) X^h X^k$, or equivalently

(4.17) $\quad R_{kho} = c(g_{hk} - p_h p_k),$

because $(X^h),\ (X^k)$ are arbitrary vector fields on M.

Now it is easy to check that (4.17) follows from (4.15) when $c = -1$.

In general, (4.17) does not imply (4.15). But this happens when the (g^{ij}) do not depend on p. Indeed, in this case $N_{ij}(x,p) = \gamma^k_{ij} p_k$, where (γ^k_{ij}) are the Christoffel symbols constructed with $g_{ij}(x)$, and then $R_{kij} = R^h_{kij} p_h$, where R^h_{kij} is the curvature tensor derived from $g_{ij}(x)$.

The equation (4.17) now reads as follows:

(4.17') $\quad R^s_{khj}(x) p_s p^j = c(g_{kh} - p_k p_h).$

On the other hand we can write $g_{kh} - p_k p_h = (\delta^s_j g_{kh}(x) - \delta^s_h g_{kj}(x)) p_s p^j$, and making use of (4.17') we get

(4.18) $\quad R^s_{khj}(x) = c(\delta^s_j g_{kh} - \delta^s_h g_{kj}).$

Equation (4.15) becomes $R^h_{kij} = p_h(\delta^h_i g_{jk} - \delta^h_j g_{ik})$, or equivalently

(4.19) $\quad R^h_{kij} = \delta^h_i g_{jk} - \delta^h_j g_{ik},$

which is equivalent to (4.18) for $c = -1$.

Thus we have obtained:

4.5. Theorem. *Let (M,K) be the Cartan space with $K^2 = g^{ij}(x) p_i p_j$. Then the almost paracontact Riemannian structure $(\overline{q},\ \xi,\ \eta,\ \overline{G})$ on the figuratrix bundle $\mathbb{K}$ is normal if and only if the Riemannian manifold $(M, g_{ij}(x))$ is of constant curvature -1.*

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