Hacettepe Journal of Mathematics and Statistics **f** Volume 44(2)(2015), 261-276

Near compactness of ditopological texture spaces

Lawrence M. Brown^{*} and Muhammed M. Gohar[†]

Abstract

The authors consider the notions of near compactness, near cocompactness, near stability, near costability and near dicompactness in the setting of ditopological texture spaces. In particular preservation of these properties under surjective R-dimaps, co-R-dimaps and bi-R-dimaps is investigated and non-trivial characterizations of near dicompactness are given which generalize those for dicompactness. The notions of semiregularization, semicoregularization and semibiregularization are defined and used to give generalizations of Mrówka's Theorem for near compactness and near cocompactness, and of Tychonoff's Theorem for near compactness, near cocompactness and near dicompactness. Also, results related to pseudo-open and pseudo-closed sets are presented. Finally, examples are given of $co-T_1$ nearly dicompact ditopologies on textures which are not nearly plain and an open question is posed.

2000 AMS Classification: 54 A 05, 54 C 10.

Keywords: Texture, Ditopology, Near compactness, Near cocompactness, Near stability, Near costabiliy, Near dicompactness, R-dimaps, co-R-dimaps, Semiregularization, Semicoregularization, Semibiregularization, Tychonoff theorems.

Received 11/01/2013 : Accepted 17/06/2013 Doi : 10.15672/HJMS.2015449101

^{*}Hacettepe University, Faculty of Science, Department of Mathematics, Beytepe, Ankara, Turkey.

Email: brown@hacettepe.edu.tr Corresponding Author.

[†]Department of Mathematics, GC University Lahore, Pakistan.

Email: dr.gohar_01@yahoo.com

1. Introduction

The investigation of various types of generalized open set and generalized continuous function is a rich area of research in general topology, and there are interesting applications in various areas, particularly computer science. An early notion in this area is the now well known concept of regular-open set, and its counterpart regular-closed set. Closely related with this notion is the property of near compactness introduced by Singal and Mathur [16, 17], which has been studied by several other authors (see, for example, [11]). Important in this context is the notion of R-map, see [7]. In this paper these and related notions will be studied in the wider context of ditopological texture spaces.

Textures and ditopological texture spaces were first introduced by the first author as a point-based setting for the study of fuzzy sets, and work continues in this direction, see for example [1, 2, 3, 4, 5], and the recent work of Tiryaki [18], Özçağ and Brown [15], Yıldız and Brown [22]. On the other hand, textures provide a very convenient setting for the investigation of complement-free concepts in general, so much of the recent work does not involve fuzzy sets explicitly. In particular, the notions of diuniformity and dimetric have been introduced in [14], while a textural analogue of the notion of proximity, called a diextremity, is given in [23].

Compactness in ditopological texture spaces was introduced in [1], its study continued in [6, 20], and extended to real compactness in [20] and to strong compactness in [10]. In this paper we place near compactness in a ditopological setting. All the arguments for studying properties related to regular-openness and regular-closedness in the topological setting apply equally well to this case, and since bitopologies and \mathbb{L} -topologies, for \mathbb{L} a Hutton algebra, are special cases of ditopologies, the new concepts of near cocompactness, near stability, near costability, and near dicompactness introduced here, may easily be carried over to these settings also.

To complete this introduction we recall various concepts from [3, 4] that will be needed later on in this paper.

Ditopological Texture Spaces: If S is a set, a *texturing* S of S is a subset of $\mathcal{P}(S)$ which is a point-separating, complete, completely distributive lattice with respect to containment containing S and \emptyset , and for which meet coincides with intersection and finite joins with union. The pair (S, S) is then called a *texture*.

For a texture (S, S), most properties are conveniently defined in terms of the *p*-sets and *q*-sets

$$P_s = \bigcap \{A \in \mathcal{S} \mid s \in A\}, \ Q_s = \bigvee \{A \in \mathcal{S} \mid s \notin A\}.$$

The following are some basic examples of textures that we will need.

1.1. Examples. (1) If X is a set and $\mathcal{P}(X)$ the powerset of X, then $(X, \mathcal{P}(X))$ is the discrete texture on X. For $x \in X$, $P_x = \{x\}$ and $Q_x = X \setminus \{x\}$.

(2) Setting $\mathbb{I} = [0, 1]$, $\mathfrak{I} = \{[0, r), [0, r] \mid r \in \mathbb{I}\}$ gives the unit interval texture $(\mathbb{I}, \mathfrak{I})$. For $r \in \mathbb{I}$, $P_r = [0, r]$ and $Q_r = [0, r)$.

(3) The texture (L, \mathcal{L}) is defined by $L = (0, 1], \mathcal{L} = \{(0, r] \mid r \in \mathbb{I}\}$. For $r \in L$, $P_r = (0, r] = Q_r$.

(4) The real texture (\mathbb{R}, \mathbb{R}) has $\mathbb{R} = \{(-\infty, r), (-\infty, r] \mid r \in \mathbb{R}\} \cup \{\mathbb{R}, \emptyset\}$. For $r \in \mathbb{R}$, $P_r = (-\infty, r]$ and $Q_r = (-\infty, r)$.

Since a texturing S need not be closed under the operation of taking set complement, the notion of topology is replaced by that of *dichotomous topology* or *ditopology*, namely a pair (τ, κ) of subsets of S, where the set of *open sets* τ satisfies

- (1) $S, \emptyset \in \tau$,
- (2) $G_1, G_2 \in \tau \implies G_1 \cap G_2 \in \tau$ and
- (3) $G_i \in \tau, i \in I \implies \bigvee_i G_i \in \tau,$

and the set of $closed~sets~\kappa$ satisfies

- (1) $S, \emptyset \in \kappa$,
- (2) $K_1, K_2 \in \kappa \implies K_1 \cup K_2 \in \kappa$ and
- (3) $K_i \in \kappa, i \in I \implies \bigcap K_i \in \kappa.$

We assume no *a priori* relation between the open and closed sets. Note that in a ditopology, equal emphasis is placed on the open and closed sets.

For $A \in S$ we define the *closure* [A] and the *interior*]A[of A under (τ, κ) by the equalities

$$[A] = \bigcap \{ K \in \kappa \mid A \subseteq K \} \text{ and }]A[= \bigvee \{ G \in \tau \mid G \subseteq A \}.$$

On the other hand, suppose that (S, S) has a complementation σ , that is an inclusion reversing involution $\sigma : S \to S$. Then if τ and κ are related by $\kappa = \sigma(\tau)$ we say that (τ, κ) is a *complemented ditopology* on (S, S, σ) . In this case we have $\sigma([A]) =]\sigma(A)[$ and $\sigma(]A[) = [\sigma(A)].$

We recall the product of textures and of ditopological texture spaces. Let (S_j, S_j) , $j \in J$, be textures and $S = \prod_{i \in J} S_j$. If $A_k \in S_k$ for some $k \in J$ we write

$$E(k, A_k) = \prod_{j \in J} Y_j \text{ where } Y_j = \begin{cases} A_j, & \text{if } j = k \\ S_j, & \text{otherwise.} \end{cases}$$

Then the product texturing $S = \bigotimes_{j \in J} S_j$ of S consists of arbitrary intersections of elements of the set

$$\mathcal{E} = \left\{ \bigcup_{j \in J} E(j, A_j) \mid A_j \in \mathcal{S}_j \text{ for } j \in J \right\}.$$

Let $(S_j, S_j), j \in J$ be textures and (S, S) their product. Then for $s = (s_j) \in S$,

$$P_s = \bigcap_{j \in J} E(j, P_{s_j}) = \prod_{j \in J} P_{s_j}, \text{ and } Q_s = \bigcup_{j \in J} E(j, Q_{s_j}).$$

It is easy to verify that for $A_j \in S_j$, $j \in J$ we have $\prod_{i \in J} A_i \in S$.

In case (τ_j, κ_j) is a ditopology on (S_j, S_j) , $j \in J$, the product ditopology on the product texture (S, S) has subbase $\{E(j, G) \mid G \in \tau_j, j \in J\}$, cosubbase $\gamma = \{E(j, K) \mid K \in \kappa_j, j \in J\}$.

Let $(S, \mathbb{S}), (T, \mathfrak{T})$ be textures. In the following definition we consider the product texture $\mathcal{P}(S) \otimes \mathfrak{T}$, and denote by $\overline{P}_{(s,t)}, \overline{Q}_{(s,t)}$, respectively the *p*-sets and *q*-sets for the product texture $(S \times T, \mathcal{P}(S) \otimes \mathfrak{T})$.

Direlations: Let (S, S), (T, T) be textures. Then

- r ∈ P(S) ⊗ T is called a relation from (S, S) to (T, T) if it satisfies R1 r ⊈ Q
 (s,t), P{s'} ⊈ Q_s ⇒ r ⊈ Q
 _(s',t). R2 r ⊈ Q
 _(s,t) ⇒ ∃s' ∈ S such that P_s ⊈ Q_{s'} and r ⊈ Q
 _(s',t).
 R ∈ P(S) ⊗ T is called a corelation from (S, S) to (T, T) if it satisfies
- (2) $R \in \mathcal{P}(S) \otimes \mathcal{T}$ is called a *corelation from* (S, \mathbb{S}) to (T, \mathcal{T}) if it satisfies $CR1 \quad \overline{P}_{(s,t)} \not\subseteq R, P_s \not\subseteq Q_{s'} \implies \overline{P}_{(s',t)} \not\subseteq R.$ $CR2 \quad \overline{P}_{(s,t)} \not\subseteq R \implies \exists s' \in S$ such that $P_{s'} \not\subseteq Q_s$ and $\overline{P}_{(s',t)} \not\subseteq R.$
- (3) A pair (r, R), where r is a relation and R a corelation from (S, S) to (T, \mathcal{T}) , is called a *direlation from* (S, S) to (T, \mathcal{T}) .

One of the most useful notions of (ditopological) texture spaces is that of difunction. A difunction is a special type of direlation.

Difunctions: Let (f, F) be a direlation from (S, S) to (T, \mathcal{T}) . Then (f, F) is called a *difunction from* (S, S) to (T, \mathcal{T}) if it satisfies the following two conditions.

DF1 For $s, s' \in S$, $P_s \not\subseteq Q_{s'} \implies \exists t \in T$ with $f \not\subseteq \overline{Q}_{(s,t)}$ and $\overline{P}_{(s',t)} \not\subseteq F$. DF2 For $t, t' \in T$ and $s \in S$, $f \not\subseteq \overline{Q}_{(s,t)}$ and $\overline{P}_{(s,t')} \not\subseteq F \implies P_{t'} \not\subseteq Q_t$.

Image and Inverse Image: Let $(f, F) : (S, S) \to (T, T)$ be a difunction.

- (1) For $A \in S$, the *image* $f^{\rightarrow}A$ and the *co-image* $F^{\rightarrow}A$ are defined by
 - $$\begin{split} f^{\rightarrow} A &= \bigcap \{ Q_t \mid \forall \, s, \ f \not\subseteq \overline{Q}_{(s,t)} \implies A \subseteq Q_s \}, \\ F^{\rightarrow} A &= \bigvee \{ P_t \mid \forall \, s, \ \overline{P}_{(s,t)} \not\subseteq F \implies P_s \subseteq A \}. \end{split}$$
- (2) For $B \in \mathfrak{I}$, the *inverse image* $f^{\leftarrow}B$ and the *inverse co-image* $F^{\leftarrow}B$ are defined by
 - $f^{\leftarrow}B = \bigvee \{ P_s \mid \forall t, \ f \not\subseteq \overline{Q}_{(s,t)} \implies P_t \subseteq B \},$ $F^{\leftarrow}B = \bigcap \{ Q_s \mid \forall t, \ \overline{P}_{(s,t)} \not\subseteq F \implies B \subseteq Q_t \}.$

For a difunction, the inverse image and the inverse co-image are equal, but the image and co-image are generally different. A difunction is called *surjective* if it satisfies

SUR. For $t, t' \in T$, $P_t \not\subseteq Q_{t'} \implies \exists s \in S$ with $f \not\subseteq \overline{Q}_{(s,t')}$ and $\overline{P}_{(s,t)} \not\subseteq F$.

Bicontinuity: The diffunction $(f, F) : (S, S, \tau_S, \kappa_S)) \to (T, \mathfrak{T}, \tau_T, \kappa_T)$ is called *continuous* if $B \in \tau_T \implies F^{\leftarrow}B \in \tau_S$, *cocontinuous* if $B \in \kappa_T \implies f^{\leftarrow}B \in \kappa_S$, and *bicontinuous* if it is both continuous and cocontinuous.

On the other hand (f, F) is open (co-open) if $A \in \tau_S \implies f^{\rightarrow}A \in \tau_T$ $(F^{\rightarrow}A \in \tau_T)$. Also, (f, F) is closed (coclosed) if $A \in \kappa_S \implies f^{\rightarrow}A \in \kappa_T$ $(F^{\rightarrow}A \in \kappa_T)$.

If $(S_j, S_j, \tau_j, \kappa_j)$, $j \in J$, are ditopological texture spaces and (S, S, τ, κ) their product, the projection diffunctions $(\pi_j, \Pi_j) : (S, S, \tau, \kappa) \to (S_j, S_j, \tau_j, \kappa_j)$ are important examples of surjective bicontinouous diffunctions that satisfy $\pi_j^{\leftarrow} B = E(j, B) = \Pi_j^{\leftarrow} B$ for all $B \in S_j$ (see [4] and [6]).

Separation Axioms A comprehensive treatment is given in [5], and the definitions will not be repeated here.

The layout of the paper is as follows. In Section 2 the concepts of near compactness and near cocompactness are given in the setting of ditopological texture spaces, while Section 3 deals with near stability and near costability, which are analogues for near compactness of the notion of stability in bitopological spaces introduced by Ralph Kopperman [12]. Section 4 deals with the preservation of these properties under surjective continuous difunctions via the notions of R-dimap and co-R-dimap, while Section 5 combines them in the notion of near dicompactness and provides important characterizations of this important concept. Finally Section 6 introduces the semibiregularization of a ditopological texture space, gives generalizations of the Mrówka characterizations of near compactness for near compactness and near cocompactness, discuses characterizations of near compact and near cocompact sets in near compact and near cocompact spaces using the notions of pseudo open and pseudo closed set, and gives versions of the Tychonoff product theorem for near compactness and near cocompactness. Here also we look at near dicompactness in the presence of point separation axioms.

This work includes results, many reformulated, from an unpublished section of the PhD thesis of the second author [9] together with interesting new material.

For terms from lattice theory not defined here the reader is referred to [8].

2. Near compactness and near cocompactness

The notion of *nearly compact* topological space was introduced by Singal and Mathur [16, 17], and has been studied by several other authors (see, for example, [11]). The definition of near compactness requires the concepts of regular-open and regular-closed sets in a topological space. We recall the definitions below:

2.1. Definition. Let (X, \mathcal{T}) be a topological space.

- (1) If $A \subseteq X$ satisfies A = int(cl(A)) then A is called a regular-open set.
- (2) If $B \subseteq X$ satisfies B = cl(int(B)) then B is called a regular-closed set.

2.2. Examples. (1) Every regular-open set is open and every regular-closed set is closed. Moreover the complement of a regular-closed set is regular-open and the complement of a regular-open set is regular-closed.

(2) Let $X = \{a, b\}$ and let the topology on X be $\mathcal{T} = \{\emptyset, X, \{a\}\}$. Here $\{b\}$ is a closed set but $cl(int(\{b\})) = \emptyset$, so $\{b\}$ is not regular-closed.

2.3. Lemma. Let (X, \mathcal{T}) be a topological space, $A, B \subseteq X$. Then int(cl(A)) is a regularopen set and cl(int(B)) a regular-closed set

In case of ditopological texture spaces we may give corresponding definitions, as follows.

2.4. Definition. Let (τ, κ) be a ditopology on the texture (S, S).

- (1) An element $A \in S$ will be called *regular-open* if [[A]] = A.
- (2) An element $B \in S$ will be called *regular-closed* if []A[] = A

It is clear that sets of the form $][A][, A \in S$, are regular-open and those of the form $[]A[], A \in S$ are regular-closed. In general there is no relation between regular-open and regular-closed sets in a ditopological texture space, but for complemented ditopological texture spaces we do have the following result.

2.5. Proposition. Let $(S, \mathcal{S}, \sigma, \tau, \kappa)$ be a complemented ditopological texture space and $A \in \mathcal{S}$. Then

- (1) $A \in S$ is regular-open if and only if $\sigma(A)$ is regular-closed.
- (2) $A \in S$ is regular-closed if and only if $\sigma(A)$ is regular-open.

Proof. Straightforward.

Now let us recall the definition of nearly compact topological space.

2.6. Definition. A topological space (X, \mathcal{T}) is said to be *nearly compact* if every open cover $\{U_i \mid i \in I\}$ of X admits a finite subfamily such that $X = \bigcup_{i=1}^{n} \operatorname{int}(\operatorname{cl}(U_i))$. That is, X is nearly compact if and only if every regular-open cover of X has a finite subcover.

The set \mathbb{R} with its usual topology and \mathbb{R} with the discrete topology are not nearly compact. However the indiscrete topology and the left ray topology $\{(-\infty, r) \mid r \in \mathbb{R}\} \cup \{\mathbb{R}, \emptyset\}$ on \mathbb{R} are examples of nearly compact topological spaces.

In the case of ditopological texture spaces we may give a corresponding definition of near compactness, and a dual notion of near cocompactness, as follows.

2.7. Definition. Let $(S, \mathfrak{S}, \tau, \kappa)$ be a ditopological texture space and $A \in \mathfrak{S}$.

(1) A will be called *nearly compact in* S if whenever $A \subseteq \bigvee_{i \in I} G_i, G_i \in \tau$, there exists $I' \subseteq I$ finite, so that $A \subseteq \bigcup_{i \in I'} [G_i][$. The ditopological texture space (S, S, τ, κ) will be called *nearly compact* if S is nearly compact in S.

(2) A will be called *nearly cocompact in* S if whenever $\bigcap_{i \in I} F_i \subseteq A, F_i \in \kappa$, there exists $I' \subseteq I$ finite, so that $\bigcap_{i \in I'} []F_i[] \subseteq A$. The ditopological texture space $(S, \mathfrak{S}, \tau, \kappa)$ will be called *nearly cocompact* if \emptyset is nearly cocompact in S.

As in the topological case we have the following characterizations.

2.8. Proposition. Let $(S, \mathfrak{S}, \tau, \kappa)$ be a ditopological texture space and $A \in \mathfrak{S}$.

- (1) A is nearly compact in S if and only if every cover of A by regular-open sets in S has a finite subcover.
- (2) A is nearly cocompact in S if every cocover of A by regular-open sets in S has a finite sub-cocover.

Proof. (1). Let A be nearly compact in S and suppose G_i , $i \in I$ is cover of A by regularopen sets. Since in particular $G_i \in \tau$, we have I' finite with $A \subseteq \bigcup_{i \in I'} [G_i] [= \bigcup_{i \in I'} G_i$ since the sets G_i are regular-open. Hence $\{G_i \mid i \in I\}$ has a finite subcover.

Conversely, suppose every regular-open cover of A has a finite subcover, and let $\{G_i \mid i \in I\}$ be an open cover of A. For each $i \in I$, $G_i \subseteq][G_i][$, so $\{][G_i][\mid i \in I\}$ is a cover of A by regular-open sets. By hypothesis there exists $I' \subseteq I$ finite with $A \subseteq \bigcup_{i \in I'}][G_i][$, which shows that A is nearly compact.

(2). The proof is dual to the above, and is therefore omitted.

2.9. Corollary. Let $(S, \mathfrak{S}, \tau, \kappa)$ be a ditopological texture space and $A \in \mathfrak{S}$.

- (1) If A is compact in S it is nearly compact in S.
- (2) If A is cocompact in S it is nearly cocompact in S.

Proof. Clear from Proposition 2.8 since every regular-open set is open and every regular-closed set is closed. $\hfill \Box$

The following examples show that the converse of Corollary 2.9 is false. Also, in general, the properties of near compactness and near cocompactness are independent of one another.

2.10. Examples. Consider the texture (L, \mathcal{L}) as in Examples 1.1.

- (1) Let $\tau = \mathcal{L}$ and $\kappa = \{L, \emptyset\}$. If we take r with 0 < r < 1 and set A = (0, r], then [A] = L and so $][A][=]L[\neq A$, from which we see that A is not regular-open. It follows that the only regular-open sets are \emptyset and L, so (τ, κ) is nearly compact because any regular-open cover of L is finite. On the other hand (τ, κ) is not compact by [6, Example 2.2 (2)].
- (2) In just the same way, $\tau = \{\emptyset, L\}$ and $\kappa = \mathcal{L}$ define a ditopology which is nearly cocompact but not cocompact.
- (3) Take τ = {(0, r] | 0 ≤ r ≤ 1/2} ∪ {L} and κ = L. This ditopology (τ, κ) is not nearly co-compact, since for example C = {(0, 1/n] | n = 1, 2...} is a regular-closed co-cover of Ø which has no finite sub co-cover. However (τ, κ) is nearly compact because it is compact.
- (4) In just the same way $\tau = \mathcal{L}$ and $\kappa = \{(0, r] \mid 1/2 \le r \le 1\} \cup \{\emptyset\}$ defines a ditopology which is nearly co-compact but not nearly compact.

For complemented ditopologies, however, near compactness and near cocompactness are equivalent.

2.11. Proposition. Let $(S, S, \sigma, \tau, \kappa)$ be a complemented ditopological texture space. Then (τ, κ) is nearly compact if and only if (τ, κ) is nearly cocompact.

Proof. Straightforward.

3. Near stability and near costability

We now wish to generalize the notions of stability and costability. The following definition would appear to be appropriate.

3.1. Definition. Let (τ, κ) be a ditopology on the texture space (S, \mathfrak{S}) .

- (1) (τ, κ) will be called *nearly stable* if every set $F \in \mathcal{S} \setminus \{S\}$ of the form $F = \bigcap_{\alpha \in A} K_{\alpha}$, K_{α} regular-closed for each $\alpha \in A$, is nearly compact in S.
- (2) (τ, κ) will be called *nearly costable* if every set $G \in S \setminus \emptyset$ of the form $G = \bigvee_{\alpha \in A} G_{\alpha}$, G_{α} regular-open for each $\alpha \in A$, is nearly cocompact in S.

3.2. Example. Consider the texture (L, \mathcal{L}) , as in previous examples,

- (1) Take $\tau = \mathcal{L}$ and $\kappa = \{\emptyset, (0, 1/2], L\}$. As we have noted earlier, (τ, κ) is not stable because (0, 1/2] is a closed set which is not compact. On the other hand the only regular-open sets in this space are \emptyset , (0, 1/2] and L, so it is nearly stable.
- (2) Dually, (τ, κ) defined by $\tau = \{\emptyset, (0, 1/2], L\}$ and $\kappa = \mathcal{L}$ is nearly costable but not costable.
- (3) Let $\tau = \kappa = \{(0, 1/2 1/n] \mid n \geq 2\} \cup \{(0, 1/2], L\}$. The regular-closed set (0, 1/2] is not nearly compact since $\{(0, 1/2 1/n] \mid n \geq 2\}$ is a cover by regular-open sets which has no finite subcover, so (τ, κ) is not nearly stable. On the other hand it is clearly nearly costable.
- (4) Dually, (τ, κ) defined by $\tau = \kappa = \{(0, 1/2 + 1/n] \mid n \ge 2\} \cup \{\emptyset, (0, 1/2]\}$ is nearly stable but not nearly costable.

The last two examples show that in general near stability and near costability are independent of one another. However for complemented ditopological texture spaces these concepts are equivalent, as we now show.

3.3. Proposition. Let (S, S, σ) be a texture with complementation σ and (τ, κ) a complemented ditopology on (S, S, σ) . Then (τ, κ) is nearly stable if and only if (τ, κ) is nearly costable.

Proof. Suppose that (τ, κ) is nearly stable. Let $\emptyset \neq G = \bigvee_{\alpha \in A} G_{\alpha}$ with G_{α} regular-open for each α . Also, let \mathcal{D} be a regular-closed cocover of G, i.e.,

$$\bigcap \mathcal{D} \subseteq G = \bigvee G_{\alpha}.$$

If we let $K_{\alpha} = \sigma(G_{\alpha})$ then K_{α} is regular-closed and $K = \sigma(G) = \sigma(\bigvee G_{\alpha}) = \bigcap_{\alpha \in A} K_{\alpha}$ satisfies $K \neq S$. Also,

$$\mathcal{C} = \{ \sigma(F) \mid F \in \mathcal{D} \}$$

is a cover of K by regular-open sets. By hypothesis K is nearly compact so there exists $\alpha_1, \ldots, \alpha_n \in A$ such that

$$K \subseteq \sigma(F_{\alpha_1}) \cup \sigma(F_{\alpha_2}) \cup \ldots \cup \sigma(F_{\alpha_n}) = \sigma(F_{\alpha_1} \cap F_{\alpha_2} \cap \ldots \cap F_{\alpha_n}),$$

so $F_{\alpha_1} \cap F_{\alpha_2} \cap \ldots \cap F_{\alpha_n} \subseteq \sigma(K) = G$, which proves that G is nearly cocompact. Hence, (τ, κ) is nearly costable.

Conversely, if (τ, κ) is nearly costable it is nearly stable. The proof is dual to the above, and is omitted.

4. *R*-dimaps and co-*R*-dimaps

Now let us consider the preservation of the above properties under surjective diffunctions. It is known [7] that near compactness of topological spaces is preserved under R-maps, so we begin by recalling the definition of R-map. **4.1. Definition.** Let (X_1, \mathcal{T}_1) , (X_2, \mathcal{T}_2) be topological spaces and $f : X_1 \to X_2$ a function. If for every regular-open set $B \subseteq X_2$ the set $f^{-1}(B)$ is regular-open in X_1 the function f is called an R-map.

We make corresponding definitions for ditopological texture spaces.

4.2. Definition. Let $(S_1, \mathfrak{S}_1, \tau_1, \kappa_1)$, $(S_2, \mathfrak{S}_2, \tau_2, \kappa_2)$ be ditopological texture spaces and $(f, F) : (S_1, \mathfrak{S}_1) \to (S_2, \mathfrak{S}_2)$ a difunction.

- (1) (f, F) will be said to be an *R*-dimap if for every regular-open set $B \in S_2$ the set $F^{\leftarrow}(B) \in S_1$ is regular-open.
- (2) (f, F) will be said to be a *co-R-dimap* if for every regular-closed set $B \in S_2$ the set $f^{\leftarrow}(B) \in S_1$ is regular-closed.
- (3) (f, F) will be said to be a *bi-R-dimap* if it is an *R*-dimap and a co-*R*-dimap.

Now we may state and prove the following theorems, which generalize the topological case.

4.3. Theorem. Let $(f, F) : (S_1, S_1, \tau_1, \kappa_1) \to (S_2, S_2, \tau_2, \kappa_2)$ be a surjective *R*-dimap. If $(S_1, S_1, \tau_1, \kappa_1)$ is nearly compact then $(S_2, S_2, \tau_2, \kappa_2)$ is nearly compact.

Proof. Assume $S_2 = \bigvee_{\alpha \in A} G_{\alpha}$, where the sets $G_{\alpha} \in S_2$ are regular-open. Since (f, F) is a difunction we have $F^{\leftarrow}(S_2) = S_1$ by [3, Proposition 2.28 (1 c)], so

$$S_1 = F^{\leftarrow}(S_2) = F^{\leftarrow}\left(\bigvee_{\alpha \in A} G_{\alpha}\right) = \bigvee_{\alpha \in A} F^{\leftarrow}(G_{\alpha}),$$

by [3, Corollary 2.12(2)]. Also, $F^{\leftarrow}(G_{\alpha})$ is regular-open for each $\alpha \in A$ since (f, F) is a *R*-bimap. Hence, by the near compactness of $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ there exists $A' \subseteq A$ finite such that $S_1 = \bigcup_{\alpha \in A'} F^{\leftarrow}(G_{\alpha})$. Hence

$$f^{\leftarrow}(S_2) = S_1 = \bigcup_{\alpha \in A'} F^{\leftarrow}(G_{\alpha}) = F^{\leftarrow}\Big(\bigcup_{\alpha \in A'} G_{\alpha}\Big).$$

Since (f, F) is surjective $f^{\leftarrow}(S_2) \subseteq F^{\leftarrow}(\bigcup_{\alpha \in A'} G_{\alpha})$ implies that $S_2 \subseteq \bigcup_{\alpha \in A'} G_{\alpha}$ by [3, Corollary 2.33 (1 ii)]. Therefore, $S_2 = \bigcup_{\alpha \in A'} G_{\alpha}$, and so $(S_2, S_2, \tau_2, \kappa_2)$ is nearly compact.

As expected, we have a dual theorem for near cocompactness.

4.4. Theorem. Let $(f, F) : (S_1, S_1, \tau_1, \kappa_1) \to (S_2, S_2, \tau_2, \kappa_2)$ be a surjective co *R*-dimap. If $(S_1, S_1, \tau_1, \kappa_1)$ is nearly cocompact then $(S_2, S_2, \tau_2, \kappa_2)$ is nearly cocompact.

Proof. This is dual to the proof of Theorem 4.3, and is omitted.

Next let us investigate the preservation of near stability and near costability under surjective difunctions.

4.5. Theorem. Let $(S_1, S_1\tau_1, \kappa_1)$, $(S_2, S_2, \tau_2, \kappa_2)$ be ditopological texture spaces with (τ_1, κ_1) nearly stable, and $(f, F) : (S_1, S_1, \tau_1, \kappa_1) \rightarrow (S_2, S_2, \tau_2, \kappa_2)$ a surjective bi-*R*-dimap. Then (τ_2, κ_2) is nearly stable.

Proof. Take $K \in S_2 \setminus \{S_2\}$ of the form $K = \bigcap_{\alpha \in A} K_\alpha$ with K_α regular-closed for each $\alpha \in A$. We must prove that K is nearly compact for the ditopology (τ_2, κ_2) so take regular-open sets $G_i, i \in I$, satisfying

$$(4.1) K \subseteq \bigvee \{G_i \mid i \in I\}.$$

Since (f, F) is a co-*R*-dimap, $f^{\leftarrow}(K) = f^{\leftarrow}(\bigcap_{\alpha \in A} K_{\alpha}) = \bigcap_{\alpha \in A} f^{\leftarrow}(K_{\alpha})$ is an intersection of regular-closed sets in (S_1, \mathcal{S}_1) . As in the proof of [6, Theorem 3.17], $f^{\leftarrow}(K) \neq S_1$. Hence $f^{\leftarrow}(K)$ is nearly compact in $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$, since this space is nearly stable. From (4.1) we have

$$f^{\leftarrow}(K) \subseteq f^{\leftarrow} \left(\bigvee \{ G_i \mid i \in I \} \right) = F^{\leftarrow} \left(\bigvee \{ G_i \mid i \in I \} \right) = \bigvee \{ F^{\leftarrow}(G_i) \mid i \in I \}$$

by [3, Corollary 2.12(2)], and $F^{\leftarrow}(G_i)$ is regular-open since (f, F) is an *R*-dimap. By near compactness we have $i_1, i_2, \ldots, i_n \in I$ with

$$\begin{split} f^{\leftarrow}(K) &\subseteq \bigcup_{k=1}^{n} F^{\leftarrow}(G_{i_k}) \\ &= F^{\leftarrow} \Big(\bigcup_{k=1}^{n} G_{i_k}\Big) = f^{\leftarrow} \Big(\bigcup_{k=1}^{n} G_{i_k}\Big) \end{split}$$

Since (f, F) is surjective we have

$$K = F(f^{\leftarrow}(K)) \subseteq F\left(f^{\leftarrow}\left(\bigcup_{k=1}^{n} G_{i_k}\right)\right) = \bigcup_{k=1}^{n} G_{i_k}.$$

This shows that K is nearly compact and completes the proof that (τ_2, κ_2) is nearly stable.

4.6. Theorem. Let $(S_1, S_1\tau_1, \kappa_1)$, $(S_2, S_2, \tau_2, \kappa_2)$ be ditopological texture spaces with (τ_1, κ_1) nearly costable, and $(f, F) : (S_1, S_1, \tau_1, \kappa_1) \rightarrow (S_2, S_2, \tau_2, \kappa_2)$ a surjective bi-*R*-dimap. Then (τ_2, κ_2) is nearly costable.

Proof. The proof is dual to that of Theorem 4.5, and is omitted.

5. Near dicompactness

The notion of dicompactness may be generalized for near compactness as follows.

5.1. Definition. A ditopological texture space will be called *nearly dicompact* if it is nearly compact, nearly cocompact, nearly stable and nearly costable.

As a consequence of Theorems 4.3, 4.4, 4.5 and 4.6 we may state the following:

5.2. Theorem. Near dicompactness is preserved under a surjective bi-R-dimap.

To give non-trivial characterizations of near dicompactness analogous to those for dicompact ditopological texture spaces we require the following definitions that are adapted from [1, 6].

5.3. Definition. Let (τ, κ) be a ditopology on (S, δ) .

- (1) A set $\mathcal{D} \subseteq S \times S$ is called a *difamily* on (S, S). A difamily \mathcal{D} is called *regular-closed*, *co-regular-open* if $A \mathcal{D} B$ implies A is regular-closed and B is regular-open, while it is called *regular-open*, *co-regular-closed* if A is regular-open and B is regular-closed.
- (2) A difamily \mathcal{D} has the *finite exclusion property* (fep) if whenever $(F_i, G_i) \in \mathcal{D}$, $i = 1, 2, \ldots, n$ we have $\bigcap_{i=1}^n F_i \not\subseteq \bigcup_{i=1}^n G_i$.
- (3) A regular-closed, co-regular-open difamily \mathcal{D} with $\bigcap \{F \mid F \in \text{dom } \mathcal{D}\} \not\subseteq \bigvee \{G \mid G \in \text{ran } \mathcal{D}\}$ is said to be *bound* in (S, S, τ, κ) .
- (4) A difamily $\mathcal{D} = \{(G_i, F_i) \mid i \in I\}$ is called a *dicover* of (S, S) if for all partitions I_1, I_2 of I (including the trivial partitions) we have

$$\bigcap_{i \in I_1} F_i \subseteq \bigvee_{i \in I_2} G_i$$

(5) A difamily \mathcal{D} is called *finite* (*co-finite*) if dom \mathcal{D} (resp. ran \mathcal{D}) is finite.

5.4. Theorem. The following are equivalent for a ditopological texture space (S, S, τ, κ) .

- (1) $(S, \mathfrak{S}, \tau, \kappa)$ is nearly dicompact.
- (2) Every regular-closed, co-regular-open difamily with the finite exclusion property is bound.
- (3) Every regular-open, co-regular-closed dicover has a sub-dicover which is finite and co-finite.

Proof. (1) ⇒ (2). Suppose that (1) holds, but that we have a regular-closed, co-regularopen difamily $\mathcal{B} = \{(F_i, G_i) \mid i \in I\}$ with the fep, which is not bound in (S, S, τ, κ). Let $F = \bigcap_{i \in I} F_i$. Then *F* is an intersection of regular-closed sets, and $F \subseteq \bigvee_{i \in I} G_i$ since \mathcal{B} is not bound. According as $F \neq S$ or F = S we may use near stability or near compactness, respectively, to show the existence of a finite subset J_1 of *I* with $F \subseteq \bigcup_{j \in J_1} G_j$. Now let $G = \bigvee_{j \in J_1} G_j$. Then *G* is a join of regular-open sets and $\bigcap_{i \in I} F_i \subseteq G$. Hence, according as $G \neq \emptyset$ or $G = \emptyset$, we may use near costability or near cocompactness, respectively, to show that $\bigcap_{j \in J_2} F_j \subseteq G$ for some finite subset J_2 of *I*. Since now $\bigcap_{j \in J_1 \cup J_2} F_j \subseteq \bigcup_{j \in J_1 \cup J_2} G_j$ we have a contradiction to the fact that \mathcal{B} has the fep.

(2) \implies (3). Suppose that $\mathcal{C} = \{(G_i, F_i) \mid i \in I\}$ is a regular-open, co-regular-closed dicover with no finite, co-finite sub-dicover. As in the proof of [1, Theorem 3.5] we consider the set \mathcal{F} of functions f satisfying

(a) dom f is a set of finite subsets of I.

- (b) $\forall J \in \operatorname{dom} f, f(J) = (f_1(J), f_2(J)) \in \mathcal{P}_J^{\star\star}.$
- (c) $J_1, \ldots, J_n \in \operatorname{dom} f \implies J_1 \cup \ldots \cup J_n \in \operatorname{dom} f.$
- (d) $J, K \in \text{dom } f, J \subseteq K \implies f_l(J) = J \cap f_l(K), l = 1, 2.$

Here

$$\mathcal{P}_J^{\star\star} = \{ (J_1, J_2) \in \mathcal{P}_J^{\star} \mid \forall K \text{ finite, } J \subseteq K \subseteq I, \exists (K_1, K_2) \in \mathcal{P}_K^{\star} \\ \text{with } J \cap K_l = J_l, \ l = 1, 2 \}.$$

where

$$\mathcal{P}_J = \{ (J_1, J_2) \mid J = J_1 \sqcup J_2 \}, \text{ and}$$

$$\mathcal{P}_J^* = \{ (J_1, J_2) \in \mathcal{P}_J \mid \bigcap_{j \in J_1} F_j \not\subseteq \bigvee_{j \in J_2} G_j \},$$

and $J = J_1 \sqcup J_2$ denotes that $\{J_1, J_2\}$ is a partition of J. Just as in the proof of [1, Theorem 3.5] we may establish that \mathcal{F} contains an element g satisfying $\bigcup \text{dom } g = I$.

Now consider the family $\mathcal{B} = \{(F_j, G_k) \mid j \in g_1(J), k \in g_2(J), J \in \text{dom } g\}$. It is easy to show that \mathcal{B} has the fep. Also F_j is regular-closed and G_k is regular-open, so by (2) we have

$$\bigcap_{I \in \text{dom } g} (\bigcap_{j \in g_1(J)} F_j) \not\subseteq \bigvee_{J \in \text{dom } g} (\bigcup_{j \in g_2(J)} G_j).$$

Let $I_1 = \bigcup \{g_1(J) \mid J \in \text{dom } g\}$, $I_2 = I \setminus I_1$. Then (I_1, I_2) is a partition of I, and $I_2 \subseteq \bigcup \{g_2(J) \mid J \in \text{dom } g\}$. This gives us

$$\bigcap_{J \in \mathrm{dom}\; g} \big(\bigcap_{j \in g_1(J)} F_j \big) = \bigcap_{i \in I_1} F_i \subseteq \bigvee_{i \in I_2} G_i \subseteq \bigvee_{J \in \mathrm{dom}\; g} \big(\bigcup_{j \in g_2(J)} G_j \big),$$

which is a contradiction.

(3) \implies (1). First take regular-open sets G_i , $i \in I$, with $S = \bigvee_{i \in I} G_i$. For $i \in I$ let $F_i = \emptyset$. Then $\mathcal{C} = \{(G_i, F_i) \mid i \in I\}$ is a regular-open, co-regular-closed dicover, so has

a finite, co-finite sub-dicover $\{(G_j, F_j) \mid j \in J\}$. For the partition $J_1 = \emptyset$, $J_2 = J$ of J,

$$S = \bigcap_{j \in J_1} F_j \subseteq \bigcup_{j \in J_2} G_j,$$

whence $S = \bigcup_{j \in J} G_j$, and $(\mathfrak{S}, \tau, \kappa)$ is nearly compact. Near cocompactness is proved in an analogous way.

To establish near stability let $F \neq S$ have the form $F = \bigcap_{\alpha \in A} F_{\alpha}$ where each F_{α} is regular-closed and let $G_i, i \in I$, be regular-open sets with $F \subseteq \bigvee_{i \in I} G_i$. Define

 $\mathcal{C} = \{ (S, F_{\alpha}) \mid \alpha \in A \} \cup \{ (G_i, \emptyset) \mid i \in I \}.$

It is clear that C is a regular-open , co-regular-closed dicover, and hence has a finite, co-finite sub-dicover C_1 . Without loss of generality we may assume

$$\mathcal{C}_1 = \{ (S, F_\beta) \mid \beta \in B \} \cup \{ (G_j, \emptyset) \mid j \in J \},\$$

where $B \subseteq A$ and $J \subseteq I$ are finite. Now we obtain

$$F \subseteq \bigcap_{\beta \in B} F_{\beta} \subseteq \bigcup_{j \in J} G_j,$$

which shows that F is nearly compact in S. Hence (τ, κ) is nearly stable, and near costability can be proved in a similar way.

Hence (τ, κ) is nearly dicompact.

6. Semibiregulization and the Tychonoff Theorems

As we have noted before, for any topological space (X, \mathcal{T}) the intersection of a finite number of regular-open sets is regular-open. It follows that the regular-open sets in any topological space (X, \mathcal{T}) form a base for a topology on X. This gives rise to the following definition.

6.1. Definition. The topology generated by the regular-open sets in the topological space (X, \mathcal{T}) is called the *semiregularization topology* of (X, \mathcal{T}) and is denoted by (X, \mathcal{T}^*) .

It is clear that $\mathfrak{T}^* \subseteq \mathfrak{T}$, but the converse is not true in general. Topologies for which $\mathfrak{T} = \mathfrak{T}^*$ are given a special name.

6.2. Definition. The topological space (X, \mathcal{T}) is called *semiregular* if the regular-open sets form a base for the topology. Equivalently, if $\mathcal{T}^* = \mathcal{T}$.

6.3. Example. For any set X, all the open sets in the discrete topology on X and in the indiscrete topology on X are regular-open. Hence these spaces are semiregular. For the usual topology on \mathbb{R} the basic open sets (a, b) are also regular-open, so this topology is semiregular. However the only regular-open sets for the finite complement topology on an infinite set X are \emptyset and X, which generate the indiscrete topology on X. Hence this space is not semiregular.

In the case of ditopological texture spaces we may define analogous concepts.

6.4. Definition. Let $(S, \mathfrak{S}, \tau, \kappa)$ be a ditopological texture space.

- (1) Denote by τ^* the topology generated by the regular-open sets of (τ, κ) . Then (τ^*, κ) is called the *semiregularization* of (τ, κ) . The ditopology (τ, κ) is called *semiregular* if $\tau^* = \tau$.
- (2) Denote by κ^* the cotopology generated by the regular-closed sets of (τ, κ) . Then (τ, κ^*) is called the *semicoregularization* of (τ, κ) . The ditopology (τ, κ) is called *semicoregular* if $\kappa^* = \kappa$.

(3) The ditopology (τ^*, κ^*) is called the *semibiregularization* of (τ, κ) and (τ, κ) is called *semibiregular* if $(\tau^*, \kappa^*) = (\tau, \kappa)$.

An important result for topological spaces is that a topological space is nearly compact if and only if the semiregularization topology is compact. For ditopological texture spaces we have the following results.

6.5. Theorem. Let (τ, κ) be a ditopology on (SS).

- (1) The following are equivalent:
 - (a) (τ, κ) is nearly compact.
 - (b) (τ^*, κ) is compact.
 - (c) $(\tau^{\star}, \kappa^{\star})$ is compact.
- (2) The following are equivalent:
 - (a) (τ, κ) is nearly cocompact.
 - (b) (τ, κ^*) is cocompact.
 - (c) $(\tau^{\star}, \kappa^{\star})$ is cocompact.

Proof. (1) (a) \Longrightarrow (b). Let C be a cover of S by members of τ^* . Then C is also a cover of S by members of τ so there exists a finite subfamily of C, say C_i , $i = 1 \dots n$, such that

$$S = \bigcup_{i=1}^{n} [C_i][.$$

Since $][C_i][= C_i$ we obtain $S = \bigcup_{i=1}^n C_i$, so (τ^*, κ) is compact.

(b) \implies (c). Clear since compactness depends only on the topology.

(c) \implies (a). Let \mathcal{C} be a cover of S by member of τ . Then $C \subseteq][C][\forall C \in \mathcal{C}$, where the interior and closure are taken in (τ, κ) , and therefore $\{][C][\mid C \in \mathcal{C}\}$ is a covering of S by members of τ^* . By (c) there exists a finite subfamily of \mathcal{C} , say C_i , $i = 1 \dots n$, such that

$$S = \bigcup_{i=1}^{n}][C_i][.$$

Hence (τ, κ) is nearly compact.

(2). The proof is dual to (1), and is omitted.

We may present the following generalization of Mrówka's characterization of compact topological spaces [13].

6.6. Theorem. Let $(S_1, S_1, \tau_1, \kappa_1)$ be a ditopological texture space. The following are equivalent:

- (1) $(S_1, S_1, \tau_1, \kappa_1)$ is nearly compact.
- (2) For all ditopological spaces (S₂, S₂, τ₂, κ₂) the projection difunction (π₂, Π₂): (S₁, S₁, τ₁^{*}, κ₁) × (S₂, S₂, τ₂, κ₂) → (S₂, S₂, τ₂, κ₂) is co-open.
- (3) Condition (2) holds for all normal ditopological spaces $(S_2, S_2, \tau_2, \kappa_2)$.

Proof. In view of Theorem 6.5 (1 b) it is sufficient to apply [6, Corollary 2.18] with τ_1^* in place of τ_1 .

Dually,

6.7. Theorem. Let $(S_1, S_1, \tau_1, \kappa_1)$ be a ditopological texture space. The following are equivalent:

(1) $(S_1, \mathfrak{S}_1, \tau_1, \kappa_1)$ is nearly cocompact.

(2) For all ditopological spaces $(S_2, S_2, \tau_2, \kappa_2)$ the projection diffunction

 $(\pi_2, \Pi_2) : (S_1, S_1, \tau_1, \kappa_1 *) \times (S_2, S_2, \tau_2, \kappa_2) \to (S_2, S_2, \tau_2, \kappa_2)$ is closed.

(3) Condition (2) holds for all normal ditopological spaces $(S_2, S_2, \tau_2, \kappa_2)$.

6.8. Theorem. Let (τ, κ) be a ditopology on (S, S). Then:

- (1) (τ, κ) is nearly stable iff (τ^*, κ^*) is stable.
- (2) (τ, κ) is nearly costable iff (τ^*, κ^*) is costable.

Proof. (1). Suppose (τ, κ) is nearly stable. Take $F \in \kappa^*$ with $F \neq S$. Then $F = \bigcap K_{\alpha}$, where the K_{α} are regular-closed sets. By near stability F is nearly compact in (τ, κ) . However, as in the proof of Theorem 6.5 this means that F is compact in (τ^*, κ^*) . Hence (τ^*, κ^*) is stable.

The converse is proved is exactly the same way.

(2). This is dual to (1), and the proof is omitted. \Box

6.9. Corollary. The ditopological texture space (S, S, τ, κ) is nearly dicompact if and only if (S, S, τ^*, κ^*) is dicompact.

We recall the following notions from [6, Definition 3.10]:

6.10. Definition. Let $(S, \mathfrak{S}, \tau, \kappa)$ be a ditopological texture space and $A \in \mathfrak{S}$.

- (1) (a) $Q(A) = \bigcap \{ Q_s [| P_s \not\subseteq A \}.$
 - (b) A is called *pseudo open* if Q(A) = |A|.
- (2) (a) $P(A) = \bigvee \{ [P_s] \mid A \not\subseteq Q_s \}.$
- (b) A is called *pseudo closed* if P(A) = [A].

It is shown in [6, Lemma 3.11] that $]A[\subseteq Q(A) \subseteq A$ and $A \subseteq P(A) \subseteq [A]$ for all $A \in S$. Now we have:

6.11. Theorem. Let (S, S, τ, κ) be a nearly compact nearly stable ditopological space for which (τ^*, κ^*) is R_0 . Then every pseudo closed set in (τ^*, κ^*) is nearly compact in (τ, κ) .

Proof. By Theorem 6.5 (1c) we have that (S, S, τ^*, κ^*) is compact, while by Theorem 6.8 it is stable. Hence by [6, Theorem 3.14 (1)] a pseudo closed set F in (S, S, τ^*, κ^*) is compact. It follows as in the proof of Theorem 6.5 that F is nearly compact in (S, S, τ, κ) .

Dually,

6.12. Theorem. Let $(S, \mathfrak{S}, \tau, \kappa)$ be a nearly cocompact nearly costable ditopological space for which (τ^*, κ^*) is co- R_0 . Then every pseudo open set in (τ^*, κ^*) is nearly cocompact in (τ, κ) .

In the opposite direction the following is an immediate consequence of [6, Theorem 3.15].

6.13. Theorem. Let (S, S, τ, κ) be a ditopological texture space.

- (1) Suppose (τ^*, κ^*) is co- R_1 . Then if A is nearly compact in (τ, κ) it is pseudo closed in (τ^*, κ^*) .
- (2) Suppose (τ^*, κ^*) is R_1 . Then if A is cocompact in (τ, κ) it is pseudo open in (τ^*, κ^*) .

Now we may give Tychonoff Theorems for near compactness, near cocompactness and near dicompactness.

6.14. Theorem. Let $(S_i, S_i, \tau_i, \kappa_i)$, $i \in I$, be non-empty ditopological texture spaces and (S, S, τ, κ) their product.

- (i) (S, S, τ, κ) is nearly compact if and only if $(S_i, S_i, \tau_i, \kappa_i)$ is nearly compact for all $i \in I$.
- (ii) (S, S, τ, κ) is nearly cocompact if and only if $(S_i, S_i, \tau_i, \kappa_i)$ is nearly cocompact for all $i \in I$.
- (iii) (S, S, τ, κ) is nearly dicompact if and only if $(S_i, S_i, \tau_i, \kappa_i)$ is nearly dicompact for all $i \in I$.

Proof. We will need the following lemma.

6.15. Lemma. Let $(S_i, S_i, \tau_i, \kappa_i)$, $i \in I$, be ditopological texture spaces and (S, S, τ, κ) their product. Let (S, S, τ', κ') denote the product of the spaces $(S_i, S_i, \tau_i^*, \kappa_i^*)$. Then $(\tau^*, \kappa^*) = (\tau', \kappa')$.

Proof. The sets E(i,G), $i \in I$, $G \in \tau_i^*$, form a subbase for the topology τ' . Since the regular-open sets in τ_i are a base for the topology τ_i^* it follows that the family

 $\mathcal{G} = \{ E(i, G) \mid i \in I, G \text{ regular-open in } \tau_i \}$

is also a subbase for τ' . However E(i, G) is regular-open in τ if and only if G is regularopen in τ_i , so it is not difficult to verify that \mathcal{G} is also a subbase for the topology τ^* . Hence $\tau' = \tau^*$, and the proof of $\kappa' = \kappa^*$ is dual to this.

We now turn to the proof of the theorem.

(i). Suppose that (τ, κ) is nearly compact. Then, by Theorem 6.5 (1 c), the ditopology (τ^*, κ^*) is compact. However, by Lemma 6.15, (S, S, τ^*, κ^*) is the product of the spaces $(S_i, S_i, \tau_i^*, \kappa_i^*)$ so by [6, Theorem 2.15 (i)] $(S_i, S_i, \tau_i^*, \kappa_i^*)$ is compact for all $i \in I$. Hence, by Theorem 6.5 (1 c), $(S_i, S_i, \tau_i, \kappa_i)$ is nearly compact for all $i \in I$.

Conversely, suppose that the spaces $(S_i, S_i, \tau_i, \kappa_i)$ are nearly compact for all $i \in I$. Then by Theorem 6.5 (1 c) the spaces $(S_i, S_i, \tau_i^*, \kappa_i^*)$ are compact, so by [6, Theorem 2.15 (i)], $(\tau', \kappa') = (\tau^*, \kappa^*)$ is a compact ditopology on (S, S). The product ditopological texture (S, S, τ, κ) is therefore nearly compact, again by Theorem 6.5 (1 c).

(ii). The proof is dual to (i), and is omitted.

(iii). Straightforward from (i) and (ii).

We end by recalling an important result from [20]. First we will require the following definition.

6.16. Definition. Let (S, S) be a texture.

- (1) $s \in S$ will be called a *plain point* if $P_s \not\subseteq Q_s$. The set of plain points of S will be denoted by S_p .
- (2) The texture (S, S) will be called *nearly-plain* if given $s \in S$ there exists a point in S_p with the same q-set as s.

It is shown in [20, Proposition 4.1] that every dicompact bi- T_2 ditopological texture space has a nearly plain texture. This restriction rules out the possibility of a dicompact bi- T_2 ditopology on, for example, the texture (L, \mathcal{L}) of Examples 1.1 (1) for this texture has no plain points. Indeed, this texture does not even have the weaker almost plain property discussed in [21]. It is natural to ask if similar restrictions hold for near compactness. This is currently an open question, but the following examples show that there can exist co- T_1 nearly dicompact ditopologies on such textures. **6.17. Example.** Consider (L, \mathcal{L}) with the ditopology (τ, κ) defined by $\tau = \mathcal{L}$, $\kappa = \{L, \emptyset\}$. As in Examples 2.10 (1) we see that the only regular-open sets are L, \emptyset , whence $\tau^* = \kappa^* = \{L, \emptyset\}$ and so $(L, \mathcal{L}, \tau, \kappa)$ is nearly dicompact. With regard to separation we see that $Q_s \not\subseteq Q_t \implies t < s$ so we may choose t < r < s and then $G = (0, r] \in \tau$, $P_s \not\subseteq G \not\subseteq Q_t$, so $(L, \mathcal{L}, \tau, \kappa)$ is co- T_1 by [5, Theorem 4.11 (2 ii)]. On the other hand by [5, Theorem 4.11 (1 i)] it is clear that this space is not T_1 and hence, in particular not bi- T_2 .

6.18. Example. Consider the Hutton texture of the real texture $(\mathbb{R}, \mathcal{R})$ given in Examples 1.1 (4). As shown in [22] this may be represented by $(M_{\mathbb{R}}, \mathcal{M}_{\mathbb{R}})$ where $M_{\mathbb{R}} = (\mathbb{R} \times \{0\}) \cup (\mathbb{R} \times \{1\}) \cup \{\infty\}$ and $\mathcal{M}_{\mathbb{R}} = \{U_r \mid r \in \mathbb{R}\} \cup \{V_r \mid r \in \mathbb{R}\} \cup \{M_{\mathbb{R}}\}$, where

$$U_r = (-\infty, r] \times \{0, 1\}, \ V_r = (-\infty, r) \times \{0\} \cup (-\infty, r] \times \{1\}.$$

Moreover, $P_{(r,0)} = U_r$, $Q_{(r,0)} = V_r$; $P_{(r,1)} = Q_{(r,1)} = V_r$; $P_{\infty} = Q_{\infty} = M_{\mathbb{R}}$. As noted in [22] this texture is not nearly plain since ∞ is a non-plain point whose q-set is not equal to the q-set of any of the plain points. If we take $\tau = \{V_r \mid r \in \mathbb{R}\} \cup \{M_{\mathbb{R}}, \emptyset\}$, $\kappa = \{M_{\mathbb{R}}, \emptyset\}$ then much as in the above example we see that $\tau^* = \kappa^* = \{M_{\mathbb{R}}, \emptyset\}$, whence $(M_{\mathbb{R}}, M_{\mathbb{R}}, \tau, \kappa)$ is nearly dicompact. Again, this space is co- T_1 . To see this using [5, Theorem 4.11 (2 ii)] note that we have $Q_{\infty} \not\subseteq Q_{(s,k)}$ for any $s \in \mathbb{R}$, k = 0, 1, and $Q_{(s,m)} = V_s \not\subseteq V_t = Q_{(t,n)}$ for t < s and m, n = 0, 1. In the first case we may take r > s, $G = V_r \in \tau$ to give $P_{\infty} \not\subseteq G \not\subseteq Q_{(s,k)}$ and in the second case t < r < s, $G = V_r$ to give $P_{(s,m)} \not\subseteq G \not\subseteq Q_{(t,n)}$.

As in the above example this space also is not T_1 and hence in particular not bi- T_2 .

Acknowledgement

The authors thank the referees for their helpful comments

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