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GEODESICS IN THE TENSOR BUNDLE OF DIAGONAL LIFTS

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Abstract

Let M_n be an n-dimensional differentiable manifold of class C^∞ and $T_q^1(M_n)$ the tensor bundle over M_n of tensor of type $(1, q)$. The purpose of this paper is to define a diagonal lift ${}^D g$ of a Riemannian metric g of a manifold M_n to the tensor bundle $T_q^1(M_n)$ of M_n and and to investigate geodesics in a tensor bundle with respect to the Levi-Civita connection of ${}^D g$.

Key Words: Tensor bundle, Riemannian metric, Diagonal lift, Levi-Civita connection, Geodesics.

Mathematics Subject Classification: Primary: 53 A 45, Secondary: 53 B 21

1. Introduction

Let M_n be an n-dimensional differentiable manifold of class C^∞ and $T_q^1(M_n)$ the tensor bundle over M_n of tensor of type $(1, q)$. If x^i are local coordinates in a neighborhood U of point $x \in M_n$, then a tensor t at x which is an element of $T_q^1(M_n)$ is expressible in the form $(x^i, t^j_{i_1\ldots i_q})$, where $t^j_{i_1\ldots i_q}$ are components of t with respect to the natural frame. We may consider $(x^i, t^j_{i_1\ldots i_q}) = (x^i, x^{\bar{\imath}}) = (x^I)$, $i = 1, \ldots, n, \bar{\imath} = n+1, \ldots, n(1+n^q)$, $I = 1, \ldots, n(1+n^q)$ as local coordinates in a neigbourhood $\pi^{-1}(U)$ (π is the natural projection $T_q^1(M_n)$ onto M_n).

Let now M_n be a Riemannian manifold with non-degenerate metric g whose components in a coordinate neigbourhood U are g_{ji} and denote by Γ^h_{ji} the Christoffel symbols formed with g_{ji} .

We denote by $\mathcal{T}^r_s(M_n)$ the module over $F(M_n)$ ($F(M_n)$ is the ring of C^∞ functions in M_n) all tensor fields of class C^∞ and of type (r, s) in M_n. Let $X \in \mathcal{T}^1_0(M_n)$ and $w \in \mathcal{T}^1_q(M_n)$. Then ${}^C X \in \mathcal{T}^1_0(T^1_q(M_n))$ (complete lift) ${}^H X \in \mathcal{T}^1_0(T^1_q(M_n))$

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(horizontal lift) and ${}^{V}w \in \mathcal{T}_0^1(T_q^1(M_n))$ (vertical lift) have, respectively, components (see [5-7])

$$
{}^{C}X = \begin{pmatrix} X^h \\ t^m_{h_1\ldots h_q}\partial_m X^k - \sum\limits_{\mu=1}^{q} t^k_{h_1\ldots m\ldots h_q}\partial_{h_\mu}X^m \end{pmatrix} \quad (1)
$$
$$
{}^{H}X = \begin{pmatrix} X^h \\ -x^m(\Gamma^k_{ms}t^s_{h_1\ldots h_q} - \sum\limits_{\mu=1}^{q}\Gamma^s_{mh_\mu}t^k_{h_1\ldots m\ldots h_q} \end{pmatrix}
$$
$$
{}^{V}w = \begin{pmatrix} 0 \\ w^k_{h_1\ldots h_q} \end{pmatrix}
$$

with respect to the natural frame $\{\partial_H\} = \{\partial_h, \partial_{\overline{h}}\}$, $x^{\overline{h}} = t^k_{h_1\ldots h_q}$ in $T_q^1(M_n)$, where X^h and $w^k_{h_1\ldots h_q}$ are respectively local components of X and w.

In each coordinate neighborhood $U(x^h)$ of M_n, we put

$$
\begin{aligned}
X_j &= \frac{\partial}{\partial x^j} = \delta^h_j\frac{\partial}{\partial x^h} \in \mathcal{T}_0^1(M_n),\ j = 1,\ldots,n \\
w_{\bar{j}} &= \partial_l \otimes dx^{j_1} \otimes\cdots\otimes dx^{j_q} \\
&= \delta^k_l\delta^{j_1}_{h_1}\cdots\delta^{j_q}_{h_q}\partial_k \otimes dx^{h_1}\otimes\cdots\otimes dx^{h_q} \in \mathcal{T}_q^1(M_n), \\
&\qquad \bar{j} = n+1,\ldots,n(1+n^q)
\end{aligned}
$$

Taking account of (1), we easily see that the components of ${}^{H}X_j$ and ${}^{V}w_{\bar{j}}$ are respectively given by

$$
{}^{H}X_j = (A_j^{\ H}) = \begin{pmatrix} \delta^h_j \\ \sum\limits_{\mu=1}^{q}\Gamma^s_{jh_\mu}t^k_{h_1\ldots s\ldots h_q} - \Gamma^k_{js}t^s_{h_1\ldots h_q} \end{pmatrix},
$$
$$
{}^{V}w_{\bar{j}} = (A_{\bar{j}}^{\ H}) = \begin{pmatrix} 0 \\ \delta^k_l\delta^{j_1}_{h_1}\cdots\delta^{j_q}_{h_q} \end{pmatrix},
$$

with respect to the natural frame $\{\partial_H\}$ where $\delta^i_j$is the Kronecker delta. We call the set $\{{}^{H}X_j, {}^{V}w_j\}$ the frame adapted to the Riemannian connection ∇ in $\pi^{-1}(U) \subset T_q^1(M_n)$. On putting

$$
A_j = {}^{H}X_j,\ A_{\bar{j}} = {}^{V}w_{\bar{j}}
$$

we write the adapted frame as $\{A_\beta\} = \{A_j, A_{\bar{j}}\}$.

It is easily verified that $n(1+n^q)$ local 1-forms

$$
\begin{aligned}
\tilde{A}^i &= (\tilde{A}^i_H) = (\delta^i_h, 0) = dx^i,\ i = 1,\ldots,n \\
\tilde{A}^{\bar{\imath}} &= (\tilde{A}^{\bar{\imath}}_H) = \left(\Gamma^r_{hs}t^s_{i_1\ldots i_q} - \sum_{\mu=1}^{q}\Gamma^s_{hi_\mu}t^r_{i_1\ldots s\ldots i_q},\ \delta^r_k\delta^{h_1}_{i_1}\ldots\delta^{h_q}_{i_q}\right) \qquad (2) \\
&= \left(\Gamma^r_{hs}t^s_{i_1\ldots i_q} - \sum_{\mu=1}^{q}\Gamma^s_{hi_\mu}t^r_{i_1\ldots s\ldots i_q}\right)dx^h + \delta^r_k\delta^{h_1}_{i_1}\ldots\delta^{h_q}_{i_q}dx^{\bar{h}} \\
&= \delta t^s_{i_1\ldots i_q},\ \bar{\imath} = n+1,\ldots,n(1+n^q)
\end{aligned}
$$

form a coframe $\{\widetilde{A}^{\ \alpha}\} = \{\widetilde{A}^{\ i}, \widetilde{A}^{\ \bar{\imath}}\}$ dual to the adapted frame $\{A_\beta\}$, i.e. $\widetilde{A}^{\ \alpha}{}_{H} A_{\beta}{}^{H} = \delta_\beta^\alpha$.

2. Lift ^Dg of a Riemannian g to $T_q^1(M_n)$

On putting locally

$$\begin{aligned} {}^Dg &= {}^Dg_{ji}\,\widetilde{A}^{\ j} \otimes \widetilde{A}^{\ i} + {}^Dg_{\bar{\jmath}\bar{\imath}}\,\widetilde{A}^{\ \bar{\jmath}} \otimes \widetilde{A}^{\ \bar{\imath}} \\ &= g_{ji} dx^j \otimes dx^i + g_{lr}\delta^{j_1 i_1}\cdots\delta^{j_q i_q}\delta t^l_{j_1\ldots j_q} \otimes \delta t^r_{i_1\ldots i_q} \end{aligned} \tag{3}$$

where δ^{ji} is the Kronecker delta, we see that ^Dg defines a tensor field of type $(0,2)$ in $T_q^1(M_n)$, which is called the diagonal lift of the tensor field g to $T_q^1(M_n)$ with respect to ∇. From (3) we prove that ^Dg has components of the form

$$ {}^Dg = ({}^Dg_{\beta\alpha}) = \begin{pmatrix} g_{ji} & 0 \\ 0 & g_{lr}\delta^{j_1 i_1}\ldots\delta^{j_q i_q} \end{pmatrix} \tag{4}$$

with respect to the adapted frame and components

$$\begin{aligned} {}^Dg &= ({}^Dg_{JI}) = \begin{pmatrix} {}^Dg_{ji} & {}^Dg_{j\bar{\imath}} \\ {}^Dg_{\bar{\jmath}i} & {}^Dg_{\bar{\jmath}\bar{\imath}} \end{pmatrix} \\ {}^Dg_{ji} &= g_{ji} + g_{lr}\delta^{j_1 i_1}\cdots\delta^{j_q i_q}\left(\Gamma^l_{jm} t^m_{j_1\ldots j_q} - \sum_{\mu=1}^{q}\Gamma^m_{jj_\mu} t^l_{j_1\ldots m\ldots j_q}\right)\cdot \\ &\qquad\qquad \left(\Gamma^r_{is} t^s_{i_1\ldots i_q} - \sum_{\mu=1}^{q}\Gamma^s_{ii_\mu} t^r_{i_1\ldots s\ldots i_q}\right) \\ {}^Dg_{j\bar{\imath}} &= g_{lr}\delta^{j_1 i_1}\cdots\delta^{j_q i_q}\left(\Gamma^l_{jm} t^m_{j_1\cdots j_q} - \sum_{\mu=1}^{q}\Gamma^m_{jj_\mu} t^l_{j_1\ldots m\ldots j_q}\right) \\ {}^Dg_{\bar{\jmath}i} &= g_{lr}\delta^{j_1 i_1}\cdots\delta^{j_q i_q}\left(\Gamma^r_{is} t^s_{i_1\cdots i_q} - \sum_{\mu=1}^{q}\Gamma^s_{ii_\mu} t^r_{i_1\ldots s\ldots i_q}\right) \\ {}^Dg_{\bar{\jmath}\bar{\imath}} &= g_{lr}\delta^{j_1 i_1}\cdots\delta^{j_q i_q} \end{aligned}$$

with respect to the natural frame. The indices $\alpha = (i,\bar{\imath})$, $\beta = (j,\bar{\jmath}) = 1,\ldots,n(1+n^q)$ and $I = (i,\bar{\imath})$, $J = (j,\bar{\jmath}) = 1,\ldots,n(1+n^q)$ indicate the indices with respect to the adapted frame and natural frame respectively.

From (4) it easily follows that if g is a Riemannian metric in M_n, then ^Dg is a Riemannnian metric in $T_q^1(M_n)$.

2.1. Remark. The metric ^Dg is similar to that of the Riemannian extension studied by S. Sasaki in the tangent bundle $T_0^1(M_n)$ $(q=0)$ [8] (see also [9], p.155, for the frame bundle, see [1, 4]. O. Kowalski [2] studied the Levi-Civita connection of the Sasaki metric on the tangent bundle. Section 3 in this paper will be devoted to a study of the Levi-Civita connection of ^Dg in $T_q^1(M_n)$.

From (1) and (2) we see that components of ${}^{C}X$, ${}^{H}X$ and ${}^{V}w$:

$${}^{C}X^{\alpha} = \widetilde{A}\ {}^{\alpha}{}_{H}\ {}^{C}X^{H}, \quad {}^{H}X^{\alpha} = \widetilde{A}\ {}^{\alpha}{}_{H}\ {}^{H}X^{H}, \quad {}^{V}w^{\alpha} = \widetilde{A}\ {}^{\alpha}{}_{H}\ {}^{V}w^{H}$$

with respect to the adapted frame are given respectively by

$$\begin{cases} ({}^{C}X^{\alpha}) = \begin{pmatrix} X^{i} \\ t^{m}_{i_1\ldots i_q}\nabla_m X^{r} - \sum_{\mu=1}^{q} t^{r}_{i_1\ldots m\ldots i_q}\nabla_{i_\mu}X^{m} \end{pmatrix}, \\ ({}^{H}X^{\alpha}) = \begin{pmatrix} X^{i} \\ 0 \end{pmatrix}, \\ ({}^{V}w^{\alpha}) = \begin{pmatrix} 0 \\ w^{r}_{i_1\ldots i_q} \end{pmatrix}. \end{cases} \tag{5}$$

From (4) and (5) we have

$${}^{D}g({}^{H}X, {}^{H}Y) = g(X,Y) \tag{6}$$

$${}^{D}g({}^{V}w, {}^{H}Y) = 0 \tag{7}$$

from (6). Hence we have:

2.2. Theorem: *Let $X, Y \in \mathcal{T}^1_0(M_n)$. Then the inner product of the horizontal lifts ${}^{H}X$ and ${}^{H}Y$ to $T^1_q(M_n)$ with metric ${}^{D}g$ is equal to the vertical lift of the inner product of X and Y in M_n.*

From (4) and (5) we have also

$${}^{D}g({}^{V}w, {}^{V}\theta) = \sum_{(i)}^{V}(g(w_{(i)}, \theta_{(i)})), \tag{8}$$

$${}^{D}g({}^{V}w, {}^{C}Y) = \sum_{(i)}^{V} g(w_{(i)}, \imath(\nabla Y)_{(i)})), \tag{9}$$

$${}^{D}g({}^{C}X, {}^{C}Y) = {}^{V}(g(X,Y)) + \sum_{(i)}^{V}(g(\imath(\nabla X)_{(i)}, \imath(\nabla Y)_{(i)})), \tag{10}$$

where $(i) = (i_1 \ldots i_q)$ and $\imath(\nabla X)_{(i)} = (t^{m}_{(i)}\nabla_m X^{r} - \sum_{\mu=1}^{q} t^{r}_{i_1\ldots m\ldots i_q}\nabla_{i_\mu}X^{m})\partial_r$.

Since the horizontal (or complete) and the vertical lifts to $T^1_q(M_n)$ of vector fields in M_n span the module of vector fields in $T^1_q(M_n)$, formulas (6)–(8) (or (8)–(10)) completely determine the diagonal lift ${}^{D}g$ of the Riemannian metric g to the tensor bundle $T^1_q(M_n)$.

2.3. Remark. We recall that any element $\widetilde{g} \in \mathcal{T}^0_2(T^1_0(M_n))$ of type $(0,2)$ in the tangent bundle $T^1_0(M_n)$, $(q = 0)$ is completely determined by its action on lifts of the type ${}^{C}X_1$, ${}^{C}X_2$, where X_i, $i = 1,2$ are arbitrary vector fields in M_n ([9], p.33). Then ${}^{D}g \in \mathcal{T}^0_2(T^1_0(M_n))$ is completely determined by (10) alone.

3. Levi-Civita Connection of ${}^{D}g$

We now need the components of the non-holonomic object which is important when we use a frame of reference such as $\{A_\beta\}$. They are defined by

$$[A_\gamma, A_\beta] = \Omega_{\gamma\beta}{}^{\alpha} A_\alpha, \text{ or,}$$
$$\Omega_{\gamma\beta}{}^{\alpha} = (A_\gamma A_\beta^{\ H} - A_\beta A_\gamma^{\ H})\, \widetilde{A}\ {}^{\alpha}{}_{H}.$$

According to $A_j = \partial_j + (\sum_{\mu=1}^{q} \Gamma^s_{jh_\mu} t^k_{h_1\ldots s\ldots h_q} - \Gamma^k_{js} t^s_{h_1\cdots h_q})\partial_{\overline{h}}$, $A_{\overline{j}} = \partial_{\overline{j}}$, the components $\Omega_{\gamma\beta}{}^{\alpha}$ are given by

$$\begin{cases} \Omega_{\overline{i}j}{}^{\overline{s}} = -\Omega_{j\overline{i}}{}^{\overline{s}} = \sum_{\mu=1}^{q} \Gamma^t_{js_\mu} \delta^n_r \delta^{i_1}_{s_1} \cdots \delta^{i_\mu}_t \cdots \delta^{i_q}_{s_q} - \Gamma^n_{jr} \delta^{i_1}_{s_1} \ldots \delta^{i_q}_{s_q} \\ \Omega_{ij}{}^{\overline{s}} = -\Omega_{ji}{}^{\overline{s}} = \sum_{\mu=1}^{q} R_{ijs_\mu}{}^{t} t^r_{s_1\ldots t\ldots s_q} - R_{ijt}{}^{r} t^t_{s_1\ldots s_q} \\ \Omega_{ij}{}^{s} = \Omega_{\overline{i}j}{}^{s} = \Omega_{i\overline{j}}{}^{s} = \Omega_{\overline{i}\overline{j}}{}^{s} = \Omega_{\overline{i}\overline{j}}{}^{\overline{s}} = 0 \ , \end{cases} \tag{11}$$

where $R_{kji}{}^{h}$ are components of the curvature tensor of the Riemannian connection ∇.

Components of the Riemannian connection determined by the metric ${}^{D}g$ are given by:

$${}^{D}\Gamma^{\alpha}_{\gamma\beta} = \frac{1}{2}\, {}^{D}g^{\alpha\epsilon}(A_\gamma\, {}^{D}g_{\epsilon\beta} + A_\beta\, {}^{D}g_{\gamma\epsilon} - A_\epsilon\, {}^{D}g_{\gamma\beta}) + \frac{1}{2}(\Omega_{\gamma\beta}{}^{\alpha} + \Omega^{\alpha}{}_{\gamma\beta} + \Omega^{\alpha}{}_{\beta\gamma}), \tag{12}$$

where $\Omega^{\alpha}{}_{\gamma\beta} = {}^{D}g^{\alpha\epsilon}\, {}^{D}g_{\delta\beta}\, \Omega_{\epsilon\gamma}{}^{\delta}$, ${}^{D}g^{\alpha\epsilon}$ are the contravariant components of the metric ${}^{D}g$ with respect to the adapted frame:

$$({}^{D}g^{\beta\alpha}) = \begin{pmatrix} g^{ji} & 0 \\ 0 & g^{lr}\delta_{j_1 i_1} \cdots \delta_{j_q i_q} \end{pmatrix} \tag{13}$$

Then, taking account of (11), (12), and (13), we have

$$\begin{cases} {}^{D}\Gamma^{h}_{ij} = \Gamma^{h}_{ij}\ , \quad {}^{D}\Gamma^{\overline{h}}_{\overline{i}j} = \sum_{\mu=1}^{q} \Gamma^t_{jh_\mu} \delta^{i_1}_{h_1} \cdots \delta^{i_\mu}_t \cdots \delta^{i_q}_{h_q} \delta^k_r \\ {}^{D}\Gamma^{h}_{i\overline{j}} = \frac{1}{2} g^{hn} g_{\theta r} \delta^{m_1 j_1} \cdots \delta^{m_q j_q} \left(\sum_{\mu=1}^{q} R_{nim_\mu}{}^{t} t^{\theta}_{m_1\ldots t\ldots m_q} - R_{nit}{}^{\theta} t^t_{m_1\ldots m_q} \right) \\ {}^{D}\Gamma^{h}_{\overline{i}j} = \frac{1}{2} g^{hn} g_{\theta r} \delta^{m_1 i_1} \cdots \delta^{m_q i_q} \left(\sum_{\mu=1}^{q} R_{njm_\mu}{}^{t} t^{\theta}_{m_1\ldots t\ldots m_q} - R_{njt}{}^{\theta} t^t_{m_1\ldots m_q} \right) \\ {}^{D}\Gamma^{\overline{h}}_{ij} = \frac{1}{2} \left(\sum_{\mu=1}^{q} R_{ijh_\mu}{}^{t} t^k_{h_1\ldots t\ldots h_q} - R_{ijt}{}^{k} t^t_{h_1\ldots h_q} \right) \\ {}^{D}\Gamma^{\overline{h}}_{i\overline{j}} = \Gamma^k_{il} \delta^{j_1}_{h_1} \cdots \delta^{j_q}_{h_q}\ , \quad {}^{D}\Gamma^{h}_{\overline{i}\overline{j}} = 0\ , \quad {}^{D}\Gamma^{\overline{h}}_{\overline{i}\overline{j}} = 0 \end{cases} \tag{14}$$

From (5) and (14) we see that ${}^{D}\nabla_{{}^{V}\theta}{}^{V}w$, ${}^{D}\nabla_{{}^{H}X}{}^{H}Y$, ${}^{D}\nabla_{{}^{V}\theta}{}^{H}Y$ and ${}^{D}\nabla_{{}^{H}X}{}^{V}w$ have, respectively, components of the form

$$\left\{\begin{array}{rcl} {}^{D}\nabla_{{}^{V}\theta}{}^{V}w &=& 0 \\ {}^{D}\nabla_{{}^{H}X}{}^{H}Y &=& \begin{pmatrix} X^i\nabla_i Y^k \\ \frac{1}{2}X^iY^j(\sum\limits_{\mu=1}^{q} R_{ijk_\mu}{}^{t}t^{s}_{k_1\ldots t\ldots k_q} - R_{ijt}{}^{s}t^{t}_{k_1\ldots k_q}) \end{pmatrix} \\ &=& {}^{H}(\nabla_X Y) - \frac{1}{2}\overline{R(X,Y)} \\ {}^{D}\nabla_{{}^{V}\theta}{}^{H}Y &=& \begin{pmatrix} \frac{1}{2}Y^j\theta^{r}_{i_1\ldots i_q}g^{hn}g_{lr}\delta^{m_1 i_1}\cdots\delta^{m_q i_q} \\ \qquad\left(\sum\limits_{\mu=1}^{q} R_{njm_\mu}{}^{t}t^{l}_{m_1\ldots t\ldots m_q} - R_{njt}{}^{l}t^{t}_{m_1\ldots m_q}\right) \\ Y^j\sum\limits_{\mu=1}^{q}\Gamma^{t}_{jh_\mu}\theta^{k}_{h_1\ldots t\ldots h_q} \end{pmatrix} \\ {}^{D}\nabla_{{}^{H}X}{}^{V}w &=& \begin{pmatrix} \frac{1}{2}X^i w^{l}_{j_1\ldots j_q}g^{hn}g_{lr}\delta^{m_1 j_1}\cdots\delta^{m_q j_q} \\ \qquad\left(\sum\limits_{\mu=1}^{q} R_{nim_\mu}{}^{t}t^{r}_{m_1\ldots t\ldots m_q} - R_{nit}{}^{r}t^{t}_{m_1\ldots m_q}\right) \\ X^i(\partial_i w^{k}_{h_1\ldots h_q} + \Gamma^{k}_{il}w^{l}_{h_1\ldots h_q}) \end{pmatrix} \end{array}\right. \tag{15}$$

where $\overline{R(X,Y)}$ has components of the form

$$\overline{R(X,Y)} = \begin{pmatrix} 0 \\ \left(\sum\limits_{\mu=1}^{q} R_{ijk_\mu}{}^{t}t^{s}_{k_1\ldots t\ldots k_q} - R_{ijt}{}^{s}t^{t}_{k_1\ldots k_q}\right)X^iY^j \end{pmatrix}$$

with respect to the adapted frame (also the natural frame). Since the horizontal and vertical lifts to $T^1_q(M_n)$ span the module of vector field in $T^1_q(M_n)$, formulae (15) completely determine the Riemannian connection ${}^{D}\nabla$ of the metric ${}^{D}g$.

We will now define the horizontal lift ${}^{H}\nabla$ of the Riemannian connection ∇ in M_n to $T^1_q(M_n$ by the conditions

$$\left\{\begin{array}{ll} {}^{H}\nabla_{{}^{V}w}{}^{V}\theta = 0\,, & {}^{H}\nabla_{{}^{V}w}{}^{H}Y = 0\,, \\ {}^{H}\nabla_{{}^{H}X}{}^{V}\theta = {}^{V}(\nabla_X\theta)\,, & {}^{H}\nabla_{{}^{H}X}{}^{H}Y = {}^{H}(\nabla_X Y) \end{array}\right. \tag{16}$$

for any $X, Y \in \mathcal{T}^1_q(M_n)$, $w, \theta \in \mathcal{T}^1_q(M_n)$. The horizontal lift ${}^{H}\nabla$ has the components

$$\begin{aligned} {}^{H}\Gamma^{i}_{ms} &= \Gamma^{i}_{ms}, \quad {}^{H}\Gamma^{i}_{\overline{m}s} = 0, \quad {}^{H}\Gamma^{i}_{m\overline{s}} = 0, \quad {}^{H}\Gamma^{i}_{\overline{m}\overline{s}} = 0, \\ {}^{H}\Gamma^{\overline{\imath}}_{\overline{m}s} &= \Gamma^{j_1}_{sl_1}\delta^{m_1}_{i_1}\cdots\delta^{m_q}_{i_q} - \sum_{c=1}^{q}\delta^{j_1}_{l_1}\delta^{m_1}_{i_1}\cdots\Gamma^{m_c}_{si_c}\cdots\delta^{m_q}_{i_q}, \\ {}^{H}\Gamma^{\overline{\imath}}_{m\overline{s}} &= \Gamma^{j_1}_{mk_1}\delta^{s_1}_{i_1}\cdots\delta^{s_q}_{i_q} - \sum_{c=1}^{q}\delta^{j_1}_{k_1}\delta^{s_1}_{i_1}\cdots\Gamma^{s_c}_{mi_c}\cdots\delta^{s_q}_{i_q}, \\ {}^{H}\Gamma^{\overline{\imath}}_{ms} &= (\partial_m\Gamma^{j_1}_{sa} + \Gamma^{j_1}_{mr}\Gamma^{r}_{sa} - \Gamma^{r}_{mc}\Gamma^{j_1}_{ra})t^{a}_{i_1\ldots i_q} \\ &\quad + \sum_{c=1}^{q}(-\partial_m\Gamma^{a}_{si_c} + \Gamma^{j_1}_{mi_c}\Gamma^{a}_{sr} + \Gamma^{r}_{ms}\Gamma^{a}_{ri_c})t^{a}_{i_1\ldots a\ldots i_q} \end{aligned}$$

$$-\sum_{c=1}^{q}(\Gamma^{j_1}_{mr}\Gamma^{a}_{si_c}+\Gamma^{a}_{mi_c}\Gamma^{j_1}_{sr})t^{r}_{i_1\ldots a\ldots i_q}$$
$$+\frac{1}{2}\sum_{b=1}^{q}\sum_{c=1}^{q}(\Gamma^{l}_{mi_c}\Gamma^{r}_{si_b}+\Gamma^{r}_{mi_b}\Gamma^{l}_{si_c})t^{j_1}_{i_1\ldots r\ldots l\ldots i_q},$$
$${}^{H}\Gamma^{\bar{\imath}}_{\overline{m}\,\overline{s}} \;=\; 0, \quad (x^{\bar{\imath}}=t^{j_1}_{i_1\ldots i_q},\; x^{\overline{m}}=t^{l_1}_{m_1\ldots m_q},\; x^{\overline{s}}=t^{k_1}_{s_1\ldots s_q})$$

with respect to the natural frame in $T^1_q(M_n)$.

Since the local vector fields ${}^{H}X_i$ and ${}^{V}w_{\bar{\imath}}$ span the module of vector fields in $\pi^{-1}(U)\subset T^1_q(M_n)$, any tensor field of type (1,2) is determined in $\pi^{-1}(U)$ by its action on ${}^{H}X_i$ and ${}^{V}w_{\bar{\imath}}$. We now define a tensor field ${}^{H}S\in(\mathcal{T}^1_2(M_n))$ by

$$\begin{cases} {}^{H}S({}^{H}X,{}^{H}Y) \;=\; {}^{H}(S(X,Y)),\; \forall X,Y\in\mathcal{T}^1_0(M_n)\\ {}^{H}S({}^{V}w,{}^{H}Y) \;=\; {}^{V}(S_Y(w)),\; \forall w\in\mathcal{T}^1_q(M_n)\\ {}^{H}S({}^{H}X,{}^{V}\theta) \;=\; {}^{V}(S_X(\theta)),\; \forall\theta\in\mathcal{T}^1_q(M_n)\\ {}^{H}S({}^{V}w,{}^{V}\theta) \;=\; 0, \end{cases} \tag{17}$$

where $S_X(w)$, $S_X(\theta)\in\mathcal{T}^1_q(M_n)$ and ${}^{H}S$ is called the horizontal lift of $S\in\mathcal{T}^1_2(M_n)$ to $T^1_q(M_n)$ [5] .

Denote by T and $\widetilde{T}$, respectively, the torsion tensors of ∇ and ${}^{H}\nabla$. Directly from the definition of the torsion tensor, we get

$$\begin{aligned} \widetilde{T}\,({}^{V}w,{}^{V}\theta) &= {}^{H}\nabla_{{}^{V}w}{}^{V}\theta - {}^{H}\nabla_{{}^{V}\theta}{}^{V}w-[{}^{V}w,{}^{V}\theta]\\ \widetilde{T}\,({}^{V}w,{}^{H}Y) &= {}^{H}\nabla_{{}^{V}w}{}^{H}Y - {}^{H}\nabla_{{}^{H}Y}{}^{V}w-[{}^{V}w,{}^{H}Y]\\ \widetilde{T}\,({}^{H}X,{}^{H}Y) &= {}^{H}\nabla_{{}^{H}X}{}^{H}Y - {}^{H}\nabla_{{}^{H}Y}{}^{H}X-[{}^{H}X,{}^{H}Y]. \end{aligned}$$

On other hand, let R denote the curvature tensor field of the connection ∇. Then [3],

$$\begin{cases} [{}^{V}w,{}^{V}\theta]=0, \quad [{}^{V}w,{}^{H}Y]=-{}^{V}(\nabla_Y w),\\ [{}^{H}X,{}^{H}Y]={}^{H}[X,Y]+\overline{R(X,Y)} \end{cases} \tag{18}$$

Taking into account (16), (17) and (18), we obtain

$$\begin{aligned} \widetilde{T}\,({}^{V}w,{}^{V}\theta) &= 0,\; \widetilde{T}\,({}^{V}w,{}^{H}Y)=0\\ \widetilde{T}\,({}^{H}X,{}^{H}Y) &= {}^{H}T({}^{H}X,{}^{H}Y)-\overline{R(X,Y)}\\ &= {}^{H}(T(X,Y))-\overline{R(X,Y)}=-\overline{R(X,Y)} \end{aligned}$$

Therefore we have

3.1. Theorem: *When ∇ is a Riemannian connection, ${}^{H}\nabla$ is torsionless if ∇ is locally flat, i.e. $T=0$ and $R=0$.*

We put

$${}^{D}\tilde{g}=\tilde{g}^{\,ji}A_j\otimes A_i+\delta_{j_1i_1}\cdots\delta_{j_qi_q}\tilde{g}^{lr}A_{\bar{\jmath}}\otimes A_{\bar{\imath}} \tag{19}$$

in $T_q^1(M_n)$.

From (16), we have

$$\begin{cases} {}^H\nabla_{{}^HX} A_i = X^s \Gamma_{si}^h A_h \\ {}^H\nabla_{{}^HX} A_{\bar{i}} = X^s (\Gamma_{sr}^k \delta_{h_1}^{i_1} \cdots \delta_{h_q}^{i_q} - \sum_{\mu=1}^q \Gamma_{sh_\mu}^t \delta_r^k \delta_{h_1}^{i_1} \cdots \delta_t^{i_\mu} \cdots \delta_{h_q}^{i_q}) A_{\bar{h}} \\ {}^H\nabla_{{}^Vw} A_i = 0, \quad {}^H\nabla_{{}^Vw} A_{\bar{i}} = 0 \end{cases} \tag{20}$$

for any $X \in \mathcal{T}_0^1(M_n)$, $w \in \mathcal{T}_q^1(M_n)$. Thus, according to (19) and (20), we obtain

$$\begin{cases} {}^H\nabla_{{}^HX} {}^D\widetilde{g} = {}^D(\nabla_X \widetilde{g}), \\ {}^H\nabla_{{}^Vw} {}^D\widetilde{g} = 0 \end{cases} \tag{21}$$

Let $\nabla_X g = 0$, then $\nabla_X \widetilde{g} = 0$. Thus, taking account of (21), $\nabla_X g = 0$ and ${}^Dg_{\alpha\gamma} {}^D\widetilde{g}^{\gamma\beta} = \delta_\alpha^\beta$, we obtain

$$\begin{aligned} {}^H\nabla_{{}^HX} {}^Dg &= 0, \\ {}^H\nabla_{{}^Vw} {}^Dg &= 0. \end{aligned}$$

Thus we have

3.2. Theorem: *Let M_n be a Riemannian manifold with metric g. Then the horizontal lift ${}^H\nabla$ of the Riemannian connection ∇ is a metric connection with respect to ${}^Dg$.*

4. Geodesics in $T_q^1(M_n)$ with metric ${}^Dg$

Let C be a curve in M_n expressed locally by $x^h = x^h(t)$ and $w_{h_1 \ldots h_q}^k(t)$ be a tensor field of type $(1, q)$ along C. Then, in the tensor bundle $T_q^1(M_n)$, we define a curve $\tilde{C}$ by

$$x^h = x^h(t), \; x^{\widetilde{h}} \overset{def}{=} t_{h_1 \ldots h_q}^k = w_{h_1 \ldots h_q}^k(t) \tag{22}$$

If the curve C satisfies at all points the relation

$$\frac{\delta w_{h_1 \ldots h_q}^k}{dt} = 0, \tag{23}$$

where δ denotes absolute differentiation, then the curve $\tilde{C}$ is said to be a *horizontal lift* of the curve C in M_n. Thus, if the initial condition $w_{h_1 \ldots h_q}^k = (w_{h_1 \ldots h_q}^k)_0$ for $t = t_0$ is given, there exists a unique horizontal lift expressed by (22).

We now consider differential equations of the geodesics of the tensor bundle $T_q^1(M_n)$ with the metric ${}^{\mathcal{D}}g$. If t is the arc length of a curve $x^A = x^A(t)$ in $T_q^1(M_n)$, equations of geodesics in $T_q^1(M_n)$ have the usual form

$$\frac{\delta^2 x^A}{dt^2} = \frac{d^2 x^A}{dt^2} + {}^D\Gamma_{CB}^A \frac{dx^C}{dt} \frac{dx^B}{dt} = 0 \tag{24}$$

with respect to the natural coordinates $(x^i, x^{\bar{i}}) = (x^i, t_{j_1 \ldots j_q}^l)$ in $T_q^1(M_n)$.

We find it more convenient to refer the equations (24) to the adapted frame $\{A_i, A_{\bar{i}}\}$. Using (2), we now write

$$\theta^h = A^{(h)}{}_A\, dx^A = dx^h,$$

$$\theta^{\bar{h}} = A^{(\bar{h})}{}_A\, dx^A = \delta t^k_{h_1\ldots h_q},$$

and put

$$\frac{\theta^h}{dt} = A^{(h)}{}_A\, \frac{dx^A}{dt} = \frac{dx^h}{dt},$$

$$\frac{\theta^{\bar{h}}}{dt} = A^{(\bar{h})}{}_A\, \frac{dx^A}{dt} = \frac{\delta t^k_{h_1\ldots h_q}}{dt}$$

along a curve $x^A = x^A(t)$, i.e., $x^h = x^h(t)$, $t^k_{h_1\ldots h_q} = t^k_{h_1\ldots h_q}(t)$ in $T^1_q(M_n)$.

If we therefore write down the form equivalent to (24), namely,

$$\frac{d}{dt}(\frac{\theta^\alpha}{dt}) +{}^D\,\Gamma_\delta{}^\alpha_\beta(\frac{\theta^\gamma}{dt})(\frac{\theta^\beta}{dt}) = 0$$

with respect to the adapted frame and take account of (14), then we have

$$\begin{cases} \frac{\delta^2 x^h}{dt^2} + g_{\theta l}\delta^{m_1 i_1}\cdots\delta^{m_q i_q}\left(\sum_{\mu=1}^q R^n_{\ im_\mu}{}^t\, t^\theta_{m_1\ldots t\ldots m_q} - \right. \\ \qquad\qquad \left. -R_{nit}{}^\theta t^t_{m_1\ldots m_q}\right)\frac{dx^i}{dt}\frac{dt^l_{j_1\ldots j_q}}{dt} = 0, \\ \frac{d}{dt}(\frac{\delta t^k_{h_1\ldots h_q}}{dt}) + \frac{1}{2}\left(\sum_{\mu=1}^q R_{ijh_\mu}{}^n t^k_{h_1\ldots n\ldots h_q} - R_{ijn}{}^k t^n_{h_1\ldots h_q}\right)\frac{dx^i}{dt}\frac{dx^j}{dt} \\ + \sum_{\mu=1}^q \Gamma^n_{jh_\mu}\left(\frac{\delta t^k_{h_1\ldots n\ldots h_q}}{dt}\right)\frac{dx^j}{dt} + \Gamma^k_{il}\frac{dx^i}{dt}\frac{\delta t^l_{h_1\ldots h_q}}{dt} = 0 \end{cases} \tag{25}$$

Since we have

$$R_{j\,i\,h}{}^m\,\frac{dx^j}{dt}\frac{dx^i}{dt} = 0$$

as a consequence of $R_{(j\,i)\,h}{}^m = 0$, we conclude by means of (25) that a curve $x^i = x^i(t)$, $t^k_{h_1\ldots h_q} = t^k_{h_1\ldots h_q}(t)$ in $T^1_q(M_n)$ with the metric ${}^D g$ is a geodesic in $T^1_q(M_n)$, if and only if

$$\begin{cases} \frac{\delta^2 x^h}{dt^2} + g_{\theta l}\delta^{m_1 i_1}\cdots\delta^{m_q i_q}\left((\sum_{\mu=1}^q R^n_{\ im_\mu}{}^t\, t^\theta_{m_1\ldots t\ldots m_q}\right. \\ \qquad\qquad \left. -R_{nit}{}^\theta t^t_{m_1\ldots m_q}\right)\frac{dx^i}{dt}\frac{dt^l_{j_1\ldots j_q}}{dt} = 0, \quad (a) \\ \frac{d}{dt}\left(\frac{\delta t^k_{h_1\ldots h_q}}{dt}\right) + \sum_{\mu=1}^q \Gamma^n_{jh_\mu}\frac{\delta t^k_{h_1\ldots h_q}}{dt}\frac{dx^j}{dt} + \Gamma^k_{jl}\frac{\delta t^l_{j_1\ldots j_q}}{dt}\frac{dx^j}{dt} = 0. \quad (b) \end{cases} \tag{26}$$

If a curve satisfying (26) lies on the fibre given by $x^h =$ const, then (24, (b)) reduces to

$$\frac{d^2 t^k_{h_1\ldots h_q}}{dt^2} = 0,$$

so that $t^k_{h_1\dots h_q} = a^k_{h_1\dots h_q}\, t + b^k_{h_1\dots h_q}$, $a^k_{h_1\dots h_q}$ and $b^k_{h_1\dots h_q}$ being constant. Thus we have

4.1. Theorem: *If the geodesic $x^h = x^h(t)$, $t^k_{h_1\dots h_q} = t^k_{h_1\dots h_q}(t)$ lies in a fibre of $T^1_q(M_n)$ with the metric ${}^D g$, the geodesic is expressed by the linear equations $x^h = c^h$, $t^k_{h_1\dots h_q} = a^k_{h_1\dots h_q}\, t + b^k_{h_1\dots h_q}$, where $a^k_{h_1\dots h_q}$, $b^k_{h_1\dots h_q}$ and c^h are constant.*

From (23) and (26), we have

4.2. Theorem: *The horizontal lift of a geodesic in M_n is always a geodesic in $T^1_q(M_n)$ with the metric ${}^D g$.*

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