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Fuzzy ideals in right regular LA-semigroups

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Abstract

In this paper, we have discussed fuzzy left (right, two-sided) ideals, fuzzy (generalized) bi-ideals, fuzzy interior ideals, fuzzy $(1,2)$-ideals and fuzzy quasi-ideals of a right regular LA-semigroup. Moreover we have characterized a right regular LA-semigroup in terms of their fuzzy left and fuzzy right ideals.

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1. Introduction

A fuzzy subset (fuzzy set) of a non-empty set $\mathcal{S}$ is an arbitrary mapping $f : \mathcal{S} \to [0,1]$, where $[0,1]$ is the unit segment of the real line. A fuzzy subset is a class of objects with a grades of membership. This important concept of fuzzy sets was first proposed by Zadeh [13] in 1965. Since then, many papers on fuzzy sets appeared which shows its importance and applications to set theory, group theory, groupoids, real analysis, measure theory and topology etc. In one of the recent paper, Zadeh introduced a new idea to explore the relationship between probabilities and fuzzy sets [14].

Rosenfeld [12] was the first who consider the case when $\mathcal{S}$ is a groupoid. He gave the definition of fuzzy subgroupoid and the fuzzy left (right, two-sided) ideal of $\mathcal{S}$ and justified these definitions by showing that a subset $\mathcal{A}$ of a groupoid $\mathcal{S}$ is a subgroupoid or a left (right, two-sided) ideal of $\mathcal{S}$ if the characteristic function of $\mathcal{A}$, that is

$$C_{\mathcal{A}}(x) = \begin{cases} 1, \text{ if } x \in \mathcal{A} \\ 0, \text{ if } x \notin \mathcal{A} \end{cases}$$

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is a fuzzy subgroupoid or a fuzzy left (right, two-sided) ideal of $\mathcal{S}$.

Kuroki and Mordeson have widely explored fuzzy semigroups in [5] and [6].

Fuzzy algebra is going popular day by day due to wide applications of fuzzification in almost every field. Our aim in this paper is to develop some characterizations for a new non-associative algebraic structure known as a left almost semigroup (LA-semigroup in short) which is the generalization of a commutative semigroup (see [2]). An LA-semigroup is an algebraic structure mid way between a groupoid and a commutative semigroup. An LA-semigroup has wide range of applications in theory of flocks (see [9]).

The concept of a left almost semigroup [2] was first introduced by M. A. Kazim and M. Naseeruddin in 1972. A groupoid $\mathcal{S}$ is called an LA-semigroup if it satisfy the following left invertive law

(1) $\quad (ab)c = (cb)a$, for all $a,\ b,\ c \in \mathcal{S}$.

An LA-semigroup is also known as an Abel-Grassmann's groupoid (AG-groupoid) [10]. P. Holgate called it left invertive groupoid [1].

In an LA-semigroup $\mathcal{S}$, the medial law [2] holds

(2) $\quad (ab)(cd) = (ac)(bd)$, for all $a,\ b,\ c,\ d \in \mathcal{S}$.

The left identity in an LA-semigroup if exists is unique [7]. Every LA-semigroup with left identity satisfy the following laws

(3) $\quad (ab)(cd) = (dc)(ba),$ for all $a, b, c, d \in \mathcal{S}$.

(4) $\quad a(bc) = b(ac),$ for all $a, b, c \in \mathcal{S}$.

If an LA-semigroup S contains left identity e then $S = eS \subseteq S^2$. Therefore $S = S^2$.

An LA-semigroup is closely related with a commutative semigroup, because if it contains a right identity, then it becomes a commutative semigroup [7].

Define the binary operation "$\bullet$" on a commutative inverse semigroup S as

$$a \bullet b = ba^{-1}, \text{ for all } a, b \in S,$$

then $(S, \bullet)$ becomes an LA-semigroup [8].

An LA-semigroup $(S, \cdot)$ becomes a semigroup S under new binary operation "$\circ$" defined in [11] as

$$x \circ y = (xa)y, \text{ for all } x, y \in S.$$

It is easy to show that "$\circ$" is associative

$$\begin{aligned} (x \circ y) \circ z &= (((xa)y)a)z = (za)((xa)y) = (xa)((za)y) \\ &= (xa)((ya)z) = x \circ (y \circ z). \end{aligned}$$

Connections discussed above make this non-associative structure interesting and useful.

Here we have given some examples of LA-semigroups in terms of abelian groups.

1.1. Example. Let us consider the abelian group $(\mathbb{R}, +)$ of all real numbers under the binary operation of addition. If we define

$$a * b = b - a - r, \text{ where } a, b, r \in \mathbb{R},$$

then $(\mathbb{R}, *)$ becomes an LA-semigroup. Indeed

$$(a * b) * c = c - (a * b) - r = c - (b - a - r) - r = c - b + a + r - r = c - b + a,$$

and

$$(c * b) * a = a - (c * b) - r = a - (b - c - r) - r = a - b + c + r - r = a - b + c.$$

Since $(\mathbb{R}, +)$ is commutative, so $(a * b) * c = (c * b) * a$ and therefore $(\mathbb{R}, *)$ satisfies a left invertive law. It is easy to observe that $(\mathbb{R}, *)$ is non-commutative and non-associative.

The same is hold for set of integers and rationals. Thus $(\mathbb{R}, *)$ is an LA-semigroup which is the generalization of an LA-semigroup given in [8].

1.2. Example. Consider the abelian group $(\mathbb{R}\backslash\{0\}, .)$ of all real numbers except zero under the binary operation of multiplication. If we define

$$a * b = ba^{-1}r^{-1}, \text{ where } a, b, r \in \mathbb{R},$$

then $(\mathbb{R}\backslash\{0\}, *)$ becomes an LA-semigroup. Indeed

$$(a * b) * c = ba^{-1}r^{-1} * c = c(ba^{-1}r^{-1})^{-1}r^{-1} = crab^{-1}r^{-1} = cab^{-1},$$

and

$$(c * b) * a = bc^{-1}r^{-1} * a = a(bc^{-1}r^{-1})^{-1}r^{-1} = arcb^{-1}r^{-1} = acb^{-1}.$$

As $(\mathbb{R}\backslash\{0\}, .)$ is commutative, therefore $(a * b) * c = (c * b) * a$ and thus $(\mathbb{R}, *)$ satisfies a left invertive law. Clearly $(\mathbb{R}, *)$ is non-commutative and non-associative. The same is hold for set of integers and rationals. This LA-semigroup is also the generalization of an LA-semigroup given in [8].

2. Preliminaries

Let $\mathcal{S}$ be an LA-semigroup, by an LA-subsemigroup of $\mathcal{S}$, we means a non-empty subset $\mathcal{A}$ of $\mathcal{S}$ such that $\mathcal{A}^2 \subseteq \mathcal{A}$.

A non-empty subset $\mathcal{A}$ of an LA-semigroup $\mathcal{S}$ is called a left (right) ideal of $\mathcal{S}$ if $\mathcal{S}\mathcal{A} \subseteq \mathcal{A}$ $(\mathcal{A}\mathcal{S} \subseteq \mathcal{A})$.

A non-empty subset $\mathcal{A}$ of an LA-semigroup $\mathcal{S}$ is called a two-sided ideal or simply an ideal if it is both a left and a right ideal of $\mathcal{S}$.

A non-empty subset $\mathcal{A}$ of an LA-semigroup $\mathcal{S}$ is called a generalized bi-ideal of $\mathcal{S}$ if $(\mathcal{A}\mathcal{S})\mathcal{A} \subseteq \mathcal{A}$.

An LA-subsemigroup $\mathcal{A}$ of $\mathcal{S}$ is called a bi-ideal of $\mathcal{S}$ if $(\mathcal{A}\mathcal{S})\mathcal{A} \subseteq \mathcal{A}$.

A non-empty subset $\mathcal{A}$ of an LA-semigroup $\mathcal{S}$ is called an interior ideal of $\mathcal{S}$ if $(\mathcal{S}\mathcal{A})\mathcal{S} \subseteq \mathcal{A}$.

A non-empty subset $\mathcal{A}$ of an LA-semigroup $\mathcal{S}$ is called a quasi ideal of $\mathcal{S}$ if $\mathcal{S}\mathcal{A} \cap \mathcal{A}\mathcal{S} \subseteq \mathcal{A}$.

An LA-subsemigroup $\mathcal{A}$ of an LA-semigroup $\mathcal{S}$ is called a $(1, 2)$-ideal of $\mathcal{S}$ if $(\mathcal{A}\mathcal{S})\mathcal{A}^2 \subseteq \mathcal{A}$.

The following definitions are available in [6].

A fuzzy subset f of an LA-semigroup $\mathcal{S}$ is called a fuzzy LA-subsemigroup of $\mathcal{S}$ if $f(xy) \geq f(x) \wedge f(y)$ for all x, $y \in \mathcal{S}$.

A fuzzy subset f of an LA-semigroup $\mathcal{S}$ is called a fuzzy left (right) ideal of $\mathcal{S}$ if $f(xy) \geq f(y)$ $(f(xy) \geq f(x))$ for all x, $y \in \mathcal{S}$.

A fuzzy subset f of an LA-semigroup $\mathcal{S}$ is called a fuzzy two-sided ideal of $\mathcal{S}$ if it is both a fuzzy left and a fuzzy right ideal of $\mathcal{S}$.

A fuzzy subset f of an LA-semigroup $\mathcal{S}$ is called a fuzzy generalized bi-ideal of $\mathcal{S}$ if $f((xa)y) \geq f(x) \wedge f(y)$, for all x, a and $y \in \mathcal{S}$.

A fuzzy LA-subsemigroup f of an LA-semigroup $\mathcal{S}$ is called a fuzzy bi-ideal of $\mathcal{S}$ if $f((xa)y) \geq f(x) \wedge f(y)$, for all x, a and $y \in \mathcal{S}$.

A fuzzy subset f of an LA-semigroup $\mathcal{S}$ is called a fuzzy interior ideal of $\mathcal{S}$ if $f((xa)y) \geq f(a)$, for all x, a and $y \in \mathcal{S}$.

Characteristic function of an LA-semigroup $\mathcal{S}$ is denoted by $C_{\mathcal{S}}(x)$ and defined as $C_{\mathcal{S}}(x) = 1$ for all x in $\mathcal{S}$.

Note that for any two fuzzy subsets f and $\mathcal{S}$ of $\mathcal{S}$, $f \subseteq \mathcal{S}$ means that $f(x) \leq C_{\mathcal{S}}(x)$ for all x in $\mathcal{S}$.

A fuzzy subset f of an LA-semigroup $\mathcal{S}$ is called a fuzzy quasi-ideal of $\mathcal{S}$ if $(f \circ C_{\mathcal{S}}(x)) \cap (C_{\mathcal{S}}(x) \circ f) \subseteq f$.

A fuzzy LA-subsemigroup f of an LA-semigroup $\mathcal{S}$ is called a fuzzy $(1,2)$-ideal of $\mathcal{S}$ if $f((xa)(yz)) \geq f(x) \wedge f(y) \wedge f(z)$ for all x, a, y and $z \in \mathcal{S}$.

Let f and g be any two fuzzy subsets of an LA-semigroup $\mathcal{S}$, then the product $f \circ g$ is defined by,

$$(f \circ g)(a) = \begin{cases} \bigvee\limits_{a=bc} \{f(b) \wedge g(c)\}, & \text{if there exist } b, c \in \mathcal{S}, \text{ such that } a = bc. \\ 0, & \text{otherwise.} \end{cases}$$

The symbols $f \cap g$ and $f \cup g$ will means the following fuzzy subsets of $\mathcal{S}$

$$(f \cap g)(x) = \min\{f(x),\ g(x)\} = f(x) \wedge g(x), \text{ for all } x \text{ in } \mathcal{S}$$

and

$$(f \cup g)(x) = \max\{f(x),\ g(x)\} = f(x) \vee g(x), \text{ for all } x \text{ in } \mathcal{S}.$$

For a fuzzy subset f of an LA-semigroup $\mathcal{S}$ and $\alpha \in (0,1]$, the set $f_\alpha = \{x \in \mathcal{S} : f(x) \geq \alpha\}$ is called a level cut of f.

A fuzzyleft ideal f is called idempotent if $f \circ f = f$.

2.1. Example. Let $\mathcal{S} = \{a, b, c, d, e\}$ be an LA-semigroup with left identity d with the following multiplication table.

.	a	b	c	d	e
a	a	a	a	a	a
b	a	b	b	b	b
c	a	b	d	e	c
d	a	b	c	d	e
e	a	b	e	c	d

Note that $\mathcal{S}$ is non-commutative as $ed \neq de$ and also $\mathcal{S}$ is non-associative because $(cc)d \neq c(cd)$.

Define a fuzzy subset f of $\mathcal{S}$ as follows: $f(a) = 1$ and $f(b) = f(c) = f(d) = f(e) = 0$, then clearly f is a fuzzy two-sided ideal of $\mathcal{S}$.

It is easy to see that every fuzzy left (right, two-sided) ideal of an LA-semigroup $\mathcal{S}$ is a fuzzy LA-subsemigroup of $\mathcal{S}$ but the converse is not true in general. Let us define a fuzzy subset f of $\mathcal{S}$ as follows: $f(a) = 1$, $f(b) = 0$ and $f(c) = f(d) = f(e) = 0.5$, then by routine calculation one can easily check that f is a fuzzy LA-subsemigroup of $\mathcal{S}$ but it is not a fuzzy left (right, two-sided) ideal of $\mathcal{S}$ because $f(bd) \ngeq f(d)$ or $f(db) \ngeq f(d)$.

2.2. Theorem. *For an LA-semigroup $\mathcal{S}$, the following statements are true.*

(i) f_α is a right (left, two-sided) ideal of $\mathcal{S}$ if f is a fuzzy right (left) ideal of $\mathcal{S}$.

(ii) f_α is a bi-(generalized bi-) ideal of $\mathcal{S}$ if f is a fuzzy bi-(generalized bi-) ideal of $\mathcal{S}$.

Proof. (i): Let $\mathcal{S}$ be an LA-semigroup and let f be a fuzzy right ideal of $\mathcal{S}$. If $x, y \in \mathcal{S}$ such that $x \in f_\alpha$, then $f(x) \geq \alpha$ and therefore $f(xy) \geq f(x) \geq \alpha$ implies that $xy \in f_\alpha$. This shows that f_α is a right ideal of $\mathcal{S}$. If f is a fuzzy left ideal of $\mathcal{S}$, then $f(yx) \geq f(x) \geq \alpha$ implies that $yx \in f_\alpha$. This shows that f_α is a left ideal of $\mathcal{S}$.

(ii): Let $\mathcal{S}$ be an LA-semigroup and let f be a fuzzy bi-(generalized bi-) ideal of $\mathcal{S}$. If x, y and $z \in \mathcal{S}$ such that x and $z \in f_\alpha$, then $f(x) \geq \alpha$ and $f(z) \geq \alpha$, therefore $f((xy)z) \geq f(x) \wedge f(z) \geq \alpha$ implies that $(xy)z \in f_\alpha$. Which shows that f_α is a generalized bi ideal of $\mathcal{S}$. Now let $x, y \in f_\alpha$, then $f(x) \geq \alpha$ and $f(y) \geq \alpha$ and therefore $f(xy) \geq f(x) \wedge f(y) \geq \alpha$ implies that $xy \in f_\alpha$. Thus f_α is a bi ideal of $\mathcal{S}$. ∎

Note that the converses of (i) and (ii) are not true in general. Define a fuzzy subset f of an LA-semigroup $\mathcal{S}$ in Example 2.1 as follows: $f(a) = 0.2$, $f(b) = 0.9$, $f(c) = f(d) =$

$f(e) = 0$. Let $\alpha = 0.2$, then it is easy to see that $f_\alpha = \{a, b\}$ and one can easily verify from Example 2.1 that $\{a, b\}$ is a right (left, generalized bi-, bi-) ideal of $\mathcal{S}$ but $f(ba) \ngeq f(b)$ $(f(ab) \ngeq f(b)$, $f((ba)b) \ngeq f(b))$ implies that f is not a fuzzy right (left, generalized bi-, bi) ideal of $\mathcal{S}$.

2.3. Lemma. [3] *Every fuzzy right ideal of an LA-semigroup $\mathcal{S}$ with left identity becomes a fuzzy left ideal of $\mathcal{S}$.*

Note that the converse of the above is not true in general. If we define a fuzzy subset f of an LA-semigroup $\mathcal{S}$ in Example 3.1 as follows: $f(a) = 0.8, f(b) = 0.5$, $f(c) = 0.4, f(d) = 0.3$ and $f(e) = 0.6$, then it is easy to observe that f is a fuzzy left ideal of $\mathcal{S}$ but it is not a fuzzy right ideal of $\mathcal{S}$, because $f(bd) \ngeq f(b)$.

Assume that $\mathcal{S}$ is an LA-semigroup and let $F(\mathcal{S})$ denote the set of all fuzzy subsets of $\mathcal{S}$, then $(F(\mathcal{S}), \circ)$ is an LA-semigroup and satisfies all the basic laws of an LA-semigroup [3].

2.4. Lemma. [3] *For a fuzzy subset f of an LA-semigroup $\mathcal{S}$, the following conditions are true.*

(i) f is a fuzzy left (right) ideal of $\mathcal{S}$ if and only if $C_\mathcal{S}(x) \circ f \subseteq f$ $(f \circ C_\mathcal{S}(x) \subseteq f)$.
(ii) f is a fuzzy LA-subsemigroup of $\mathcal{S}$ if and only if $f \circ f \subseteq f$.

2.5. Lemma. [3] *For any non-empty subsets $\mathcal{A}$ and $\mathcal{B}$ of an LA-semigroup $\mathcal{S}$, the following conditions are true.*

(i) $C_\mathcal{A} \circ C_\mathcal{B} = C_{\mathcal{A}\mathcal{B}}$
(ii) $C_\mathcal{A} \cap C_\mathcal{B} = C_{\mathcal{A}\cap \mathrm{B}}$

2.6. Lemma. [3] *Let $\mathcal{A}$ be a non-empty subset of an LA-semigroup $\mathcal{S}$. Then the following properties holds.*

(i) $\mathcal{A}$ is an LA-subsemigroup of $\mathcal{S}$ if and only if $C_\mathcal{A}$ is a fuzzy LA-subsemigroup of $\mathcal{S}$.
(ii) $\mathcal{A}$ is a left (right, two-sided) ideal of $\mathcal{S}$ if and only if $C_\mathcal{A}$ is a fuzzy left (right, two-sided) ideal of $\mathcal{S}$.

3. Fuzzy ideals in Right Regular LA-semigroups

An element a of an LA-semigroup $\mathcal{S}$ is called a right regular if there exists $x \in \mathcal{S}$ such that $a = a^2x$ and $\mathcal{S}$ is called right regular if every element of $\mathcal{S}$ is right regular.

An LA-semigroup considered in Example 2.1 is right regular because, $a = a^2d$, $b = b^2c$, $c = c^2c$, $d = d^2d$, $e = e^2e$.

Note that in an LA-semigroup $\mathcal{S}$ with left identity, $\mathcal{S} = \mathcal{S}^2$.

3.1. Example. Let us consider an LA-semigroup $\mathcal{S} = \{a, b, c, d, e\}$ with left identity d in the following Cayley's table.

.	a	b	c	d	e
a	a	a	a	a	a
b	a	e	e	c	e
c	a	e	e	b	e
d	a	b	c	d	e
e	a	e	e	e	e

Note that $\mathcal{S}$ is not right regular because for $c \in \mathcal{S}$, there does not exists $x \in \mathcal{S}$ such that $c = c^2x$.

Note that if f is any fuzzy subset of an LA-semigroup $\mathcal{S}$ with left identity then $\mathcal{S}$ is right regular if $f(x) = f(x^2)$ holds for all x in $\mathcal{S}$. But the converse is not true in general.

3.2. Example. Let $\mathcal{S} = \{a, b, c, d, e\}$ be a right regular LA-semigroup with left identity d in the following multiplication table.

.	a	b	c	d	e
a	b	a	a	a	a
b	a	b	b	b	b
c	a	b	c	d	e
d	a	b	e	c	d
e	a	b	d	e	c

Let us consider a right regular LA-semigroup $\mathcal{S}$ in Example 3.2. Define a fuzzy subset f of $\mathcal{S}$ as follows: $f(a) = 0.6$, $f(b) = 0.2$ and $f(c) = f(d) = f(e) = 0.9$, then it is easy to see that $f(a) \neq f(a^2)$ for $a \in \mathcal{S}$.

3.3. Lemma. *If f is a fuzzy interior ideal of a right regular LA-semigroup $\mathcal{S}$ with left identity, then $f(ab) = f(ba)$ holds for all a, b in $\mathcal{S}$.*

Proof. Assume that f is a fuzzy interior ideal of a right regular LA-semigroup $\mathcal{S}$ with left identity and let $a \in \mathcal{S}$, then $a = a^2x$ for some x in $\mathcal{S}$. Now by using (1), (4) and (3), we have

$$f(a) = f((aa)x) = f((xa)a) = f((xa)((aa)x)) = f((aa)((xa)x))$$

$$\begin{aligned} &= f((ea^2)((xa)x)) \geq f(a^2) = f(aa) = f(a((aa)x)) \\ &= f((aa)(ax)) = f((xa)(aa)) = f((xa)a^2) \geq f(a). \end{aligned}$$

Which implies that $f(a) = f(a^2)$ for all a in $\mathcal{S}$.
Now by using (3), (4) and (2), we have

$$\begin{aligned} f(ab) &= f((ab)^2) = f((ab)(ab)) = f((ba)(ba)) \\ &= f((e(ba))(ba)) \geq f(ba) = f(b((aa)x)) \\ &= f((aa)(bx)) = f((ab)(ax)) \\ &= f((e(ab))(ax)) \geq f(ab). \end{aligned}$$

∎

The converse is not true in general. For this, let us define a fuzzy subset f of a right regular LA-semigroup $\mathcal{S}$ in Example 2.1 as follows: $f(a) = 0.1$, $f(b) = 0.2, f(c) = 0.6, f(d) = 0.4$ and $f(e) = 0.6$, then it is easy to see that $f(ab) = f(ba)$ holds for all a and b in $\mathcal{S}$ but f is not a fuzzy interior ideal of $\mathcal{S}$ because $f((ab)c) \ngeq f(b)$.

3.4. Lemma. *For any fuzzy subset f of a right regular LA-semigroup $\mathcal{S}$, $C_{\mathcal{S}}(x) \circ f = f$.*

Proof. Since S is right regular, therefore for each a in S there exists x such that $a = a^2x$, now using left invertive law, we get $a = (xa)a$. Then

$$C_{\mathcal{S}}(x) \circ f(a) = \bigvee_{a=(xa)a} \{C_{\mathcal{S}}(x)(xa) \wedge f(a)\} = \bigvee_{a=(xa)a} \{1 \wedge f(a) = f(a).$$

Hence $C_{\mathcal{S}}(x) \circ f = f$. ∎

3.5. Lemma. *In a right regular LA-semigroup $\mathcal{S}$, $f \circ C_{\mathcal{S}}(x) = f$ and $C_{\mathcal{S}}(x) \circ f = f$ holds for every fuzzy two-sided ideal f of $\mathcal{S}$.*

Proof. Let $\mathcal{S}$ be a right regular LA-semigroup. Now for every $a \in \mathcal{S}$ there exists $x \in \mathcal{S}$ such that $a = a^2x$. Then by using (1), we have $a = (aa)x = (xa)a$, therefore

$$\begin{aligned}(f \circ C_{\mathcal{S}}(x))(a) &= \bigvee_{a=(xa)a} \{f(xa) \wedge C_{\mathcal{S}}(x)(a)\} \geq f(xa) \wedge C_{\mathcal{S}}(x)(a) \\ &\geq f(a) \wedge 1 = f(a).\end{aligned}$$

It is easy to observe from Lemma 3.4 that $C_{\mathcal{S}}(x) \circ f = f$ holds for every fuzzy two-sided ideal f of $\mathcal{S}$. ∎

3.6. Corollary. *In a right regular LA-semigroup* $\mathcal{S}$, $C_{\mathcal{S}}(x) \circ C_{\mathcal{S}}(x) = C_{\mathcal{S}}(x)$.

Proof. It is simple. ∎

3.7. Lemma. *A fuzzy subset f of a right regular LA-semigroup $\mathcal{S}$ is a fuzzy left ideal of $\mathcal{S}$ if and only if it is a fuzzy right ideal of $\mathcal{S}$.*

Proof. Assume that f is a fuzzy left ideal of a right regular LA-semigroup $\mathcal{S}$ with left identity and let $a, b \in \mathcal{S}$, then $a = a^2x$ for some x in $\mathcal{S}$. Now by using (1), we have

$$f(ab) = f((a^2x)b) = f((bx)a^2) \geq f(a^2) = f(aa) \geq f(a).$$

This shows that$f$ is a fuzzy right ideal of $\mathcal{S}$.

Similarly we can show that every fuzzy right ideal of $\mathcal{S}$ is a fuzzy left ideal of $\mathcal{S}$. ∎

3.8. Theorem. *In a right regular LA-semigroup $\mathcal{S}$ with left identity, the following statements are equivalent.*

(i) f is a fuzzy $(1, 2)$-ideal of $\mathcal{S}$.
(ii) f is a fuzzy two-sided ideal of $\mathcal{S}$.

Proof. $(i) \Longrightarrow (ii)$: Assume that f is a fuzzy $(1, 2)$-ideal of a right regular LA-semigroup $\mathcal{S}$ with left identity and let $a \in \mathcal{S}$, then there exists $y \in \mathcal{S}$ such that $a = a^2y$. Now by using (4), (1) and (3), we have

$$\begin{aligned}f(xa) &= f(x((aa)y)) = f((aa)(xy)) = f((((aa)y)a)(xy)) \\ &= f(((ay)(aa))(xy)) = f(((aa)(ya))(xy)) \\ &= f(((xy)(ya))(aa)) = f(((ay)(yx))a^2) \\ &= f(((((aa)y)y)(yx))a^2) = f((((yy)(aa))(yx))a^2) \\ &= f((((aa)y^2)(yx))a^2) = f((((yx)y^2)(aa))a^2) \\ &= f((a(((yx)y^2)a))(aa)) \geq f(a) \wedge f(a) \wedge f(a) = f(a).\end{aligned}$$

This shows that f is a fuzzy left ideal of $\mathcal{S}$ and by using Lemma 3.7, f is a fuzzy two-sided ideal of $\mathcal{S}$.

$(ii) \Longrightarrow (i)$ is obvious. ∎

3.9. Theorem. *In a right regular LA-semigroup $\mathcal{S}$ with left identity, the following statements are equivalent.*

(i) f is a fuzzy $(1, 2)$-ideal of $\mathcal{S}$.
(ii) f is a fuzzy quasi ideal of $\mathcal{S}$.

Proof. $(i) \Longrightarrow (ii)$ is an easy consequence of Theorem 3.8 and Lemma 3.5.

$(ii) \Longrightarrow (i)$: Assume that f is a fuzzy quasi ideal of a right regular LA-semigroup $\mathcal{S}$ with left identity and let $a \in \mathcal{S}$, then there exists $x \in \mathcal{S}$ such that $a = a^2x$. Now by using $(1), (3)$ and (4), we have

$$a = (aa)x = (xa)a = (xa)(ea) = (ae)(ax) = a((ae)x),$$

therefore

$$(f \circ C_{\mathcal{S}}(x))(a) = \bigvee_{a=a((ae)x)} \{f(a) \wedge C_{\mathcal{S}}(x)((ae)x)\} \geq f(a) \wedge 1 = f(a) \; .$$

Now by using Lemmas 3.4, 3.6 and (2), we have

$$\begin{aligned} f \circ C_{\mathcal{S}}(x) &= (C_{\mathcal{S}}(x) \circ f) \circ (C_{\mathcal{S}}(x) \circ C_{\mathcal{S}}(x)) = (C_{\mathcal{S}}(x) \circ C_{\mathcal{S}}(x)) \circ (f \circ C_{\mathcal{S}}(x)) \\ &= C_{\mathcal{S}}(x) \circ (f \circ C_{\mathcal{S}}(x)) \supseteq C_{\mathcal{S}}(x) \circ f. \end{aligned}$$

Which shows that $C_{\mathcal{S}}(x) \circ f \subseteq (f \circ C_{\mathcal{S}}(x)) \cap (C_{\mathcal{S}}(x) \circ f)$. As f is a fuzzy quasi ideal of $\mathcal{S}$, thus we get $C_{\mathcal{S}}(x) \circ f \subseteq f$. Now by using Lemmas 2.4 and 3.7, f is a fuzzy two-sided ideal of $\mathcal{S}$. Thus by Theorem 3.8, f is a fuzzy $(1,2)$-ideal of $\mathcal{S}$. ∎

3.10. Theorem. *In a right regular LA-semigroup $\mathcal{S}$ with left identity, the following statements are equivalent.*

(i) f is a fuzzy bi-ideal of $\mathcal{S}$.
(ii) f is a fuzzy $(1,2)$-ideal of $\mathcal{S}$.

Proof. $(i) \Longrightarrow (ii)$: Assume that $\mathcal{S}$ is a right regular LA-semigroup with left identity and let $x, a, y, z \in \mathcal{S}$, then there exists $x' \in \mathcal{S}$ such that $x = x^2x'$. Let f be a fuzzy bi-ideal of $\mathcal{S}$, then by using $(3), (1)$ and (4), we have

$$\begin{aligned} f((xa)(yz)) &= f((zy)(ax)) = f(((ax)y)z) \\ &\geq f((ax)y) \wedge f(z) \geq f(ax) \wedge f(y) \wedge f(z) \\ &= f(a((xx)x')) \wedge f(y) \wedge f(z) \\ &= f((xx)(ax')) \wedge f(y) \wedge f(z) \\ &= f(((ax')x)x) \wedge f(y) \wedge f(z) \\ &= f(((ax')((xx)x'))x) \wedge f(y) \wedge f(z) \\ &= f(((ax')((xx)(ex')))x) \wedge f(y) \wedge f(z) \\ &= f(((ax')((x'e)(xx)))x) \wedge f(y) \wedge f(z) \\ &= f(((ax')(x((x'e)x)))x) \wedge f(y) \wedge f(z) \\ &= f(x((ax')((x'e)x))x) \wedge f(y) \wedge f(z) \\ &\geq f(x) \wedge f(x) \wedge f(y) \wedge f(z) \\ &= f(x) \wedge f(y) \wedge f(z). \end{aligned}$$

Which shows that f is a fuzzy $(1,2)$-ideal of $\mathcal{S}$.

$(ii) \Longrightarrow (i)$: Again let $\mathcal{S}$ be a right regular LA-semigroup with left identity, then for any a, b, x and $y \in \mathcal{S}$ there exist a', b', x' and $y' \in \mathcal{S}$ such that $a = a^2a'$, $b = b^2b'$, $x = x^2x'$ and $y = y^2y'$. Let f be a fuzzy $(1,2)$-ideal of $\mathcal{S}$, then by using (4) and (1), we have

$$\begin{aligned} f((xa)y) &= f(xa)((yy)y') = (yy)((xa)y') = (y'(xa))(yy) \\ &= (x(y'a))(yy) \geq f(x) \wedge f(y) \wedge f(y) \geq f(x) \wedge f(y). \end{aligned}$$

Which shows that f is a fuzzy bi-ideal of $\mathcal{S}$. ∎

3.11. Theorem. *In a right regular LA-semigroup $\mathcal{S}$ with left identity, the following statements are equivalent.*

(i) f is a fuzzy $(1,2)$-ideal of $\mathcal{S}$.
(ii) f is a fuzzy interior ideal of $\mathcal{S}$.

Proof. $(i) \Longrightarrow (ii)$: Let $\mathcal{S}$ be a right regular LA-semigroup with left identity and let $x, a, y \in \mathcal{S}$. Then for a there exists $u \in \mathcal{S}$ such that $a = a^2u$. Let f be a fuzzy $(1,2)$-ideal of $\mathcal{S}$. Then by using $(4), (1)$, (2) and (3), we have

$$\begin{aligned}
(xa)y &= (x(a^2u))y = (a^2(xu))y = (y(xu)(aa) = (y(xu)((a^2u)(a^2u)) \\
&= (y(xu)((a^2a^2)(uu)) = (y(xu)((uu)(a^2a^2)) = (y(xu)(a^2(u^2a^2)) \\
&= a^2((y(xu)(u^2a^2)) = ((u^2a^2)(y(xu))a^2 = ((a^2u^2)(y(xu))a^2 \\
&= (((y(xu)u^2)a^2)a^2 = (((y(xu)u^2)(aa))(aa) = ((a((y(xu)u^2)a))(aa) \\
&= (av)(aa), \text{ where } v = (y(xu)u^2)a.
\end{aligned}$$

Therefore $f((xa)y) = f((av)(aa)) \geq f(a) \wedge f(a) \wedge f(a)$. Which shows that f is a fuzzy interior ideal of $\mathcal{S}$.

$(ii) \Longrightarrow (i)$: Again let $\mathcal{S}$ be a right regular LA-semigroup with left identity and let $x, a, y, z \in \mathcal{S}$, then there exist x' and $z' \in \mathcal{S}$ such that $x = x^2x'$ and $z = z^2z'$. Now by using (3), we have

$$f((xa)(yz)) = f((zy)(ax)) \geq f(y).$$

Now by using (1) and (3), we have

$$\begin{aligned}
f((xa)(yz)) &= f((((xx)x')a)(yz)) = f(((ax')(xx))(yz)) \\
&= f(((xx)(x'a))(yz)) = f((((x'a)x)x)(yz)) \geq f(x).
\end{aligned}$$

Now by using (4), we have

$$\begin{aligned}
f((xa)(yz)) &= f((xa)(y(((zz)z')))) = f((xa)((zz)(yz'))) \\
&= f((zz)((xa)(yz'))) \geq f(z).
\end{aligned}$$

Thus we get that $f((xa)(yz)) \geq f(x) \wedge f(y) \wedge f(z)$.

Let a,$b \in \mathcal{S}$ then there exist a',$b' \in \mathcal{S}$ such that $a = a^2a'$ and $b = b^2b'$. Now by using (1), (3) and (4), we have

$$f(ab) = f(((aa)a')b) = f((ba')(aa)) = f((aa)(a'b)) \geq f(a)$$

and

$$f(ab) = f(a((bb)b')) = f((bb)(ab')) \geq f(b).$$

Thus f is a fuzzy $(1,2)$-ideal of $\mathcal{S}$. ∎

3.12. Corollary. *Fuzzy two-sided ideals, fuzzy bi-ideals, fuzzy generalized bi-ideals, fuzzy $(1,2)$-ideals, fuzzy interior ideals and fuzzy quasi-ideals coincide in a right regular LA-semigroup with left identity.*

3.13. Lemma. *In a right regular LA-semigroup $\mathcal{S}$ with left identity, the following statements are equivalent.*

(i) f is a fuzzy quasi ideal of $\mathcal{S}$.
(ii) $(f \circ C_{\mathcal{S}}(x)) \cap (C_{\mathcal{S}}(x) \circ f) = f$.

Proof. $(i) \Longrightarrow (ii)$ is followed by Lemma 3.5 and Theorem 3.9.
$(ii) \Longrightarrow (i)$ is obvious. ∎

3.14. Theorem. *In a right regular LA-semigroup $\mathcal{S}$ with left identity, the following statements are equivalent.*

(i) f is a fuzzy bi-(generalized bi-) ideal of $\mathcal{S}$.
(ii) $(f \circ C_{\mathcal{S}}(x)) \circ f = f$ and $f \circ f = f$.

Proof. $(i) \Longrightarrow (ii)$: Assume that f is a fuzzy bi-ideal of a right regular LA-semigroup $\mathcal{S}$ with left identity and let $a \in \mathcal{S}$, then there exists $x \in \mathcal{S}$ such that $a = a^2x$. Now by using (1), (4) and (3), we have

$$\begin{aligned} a &= (aa)x = (xa)a = (x((aa)x))a = ((aa)(xx))a \\ &= ((xx)(aa))a = (((aa)x)x)a = (((xa)a)x)a \\ &= (((x((aa)x))a)x)a = ((((aa)(xx))a)x)a \\ &= ((((xx)(aa))a)x)a = (((a(x^2a))a)x)a. \end{aligned}$$

Therefore

$$\begin{aligned} ((f \circ C_{\mathcal{S}}(x)) \circ f)(a) &= \bigvee_{a=(((a(x^2a))a)x)a} \{(f \circ C_{\mathcal{S}}(x))(((a(x^2a))a)x) \wedge f(a)\} \\ &\geq \bigvee_{((a(x^2a))a)x=((a(x^2a))a)x} \{f((a(x^2a))a) \wedge C_{\mathcal{S}}(x)(x)\} \wedge f(a) \\ &\geq f((a(x^2a))a) \wedge 1 \wedge f(a) \\ &\geq f(a) \wedge f(a) \wedge f(a) = f(a). \end{aligned}$$

Now by using (1), (4) and (3), we have

$$\begin{aligned} a &= (aa)x = (xa)a = (x((aa)x))a = ((aa)(xx))a \\ &= ((xx)(aa))a = (a(x^2a))a. \end{aligned}$$

Therefore

$$\begin{aligned} ((f \circ C_{\mathcal{S}}(x)) \circ f)(a) &= \bigvee_{a=(a(x^2a))a} \{(f \circ C_{\mathcal{S}}(x))((a(x^2a))) \wedge f(a)\} \\ &= \bigvee_{a=(a(x^2a))a} \left(\bigvee_{a(x^2a)=a(x^2a)} \{f(a) \wedge C_{\mathcal{S}}(x)(x^2a)\} \right) \wedge f(a) \\ &= \bigvee_{a=(a(x^2a))a} \left(\bigvee_{a(x^2a)=a(x^2a)} \{f(a) \wedge 1\} \right) \wedge f(a) \\ &= \bigvee_{a=(a(x^2a))a} \left(\bigvee_{a(x^2a)=a(x^2a)} f(a) \right) \wedge f(a) \\ &= \bigvee_{a=(a(x^2a))a} \{f(a) \wedge f(a)\} = f(a). \end{aligned}$$

Thus $(f \circ C_{\mathcal{S}}(x)) \circ f = f$.
Again by using (1), (4) and (3), we have

$$\begin{aligned} a &= (aa)x = (xa)a = (x((aa)x))a = ((aa)(xx))a \\ &= (((xx)a)a)a = (((xx)((aa)x))a)a \\ &= (((xx)((xa)a))a)a = (((xx)((ae)(ax)))a)a \\ &= (((xx)(a((ae)x)))a)a = ((a((xx)((ae)x)))a)a \end{aligned}$$

Therefore

$$\begin{aligned} (f \circ f)(a) &= \bigvee_{a=((a((xx)((ae)x)))a)a} \{f((a((xx)((ae)x)))a) \wedge f(a)\} \\ &\geq f((a((xx)((ae)x)))a) \wedge f(a) \\ &\geq f(a) \wedge f(a) \wedge f(a) = f(a), \end{aligned}$$

therefore by using Lemma 2.4, $f \circ f = f$.
$(ii) \Longrightarrow (i)$: Let f be a fuzzy subset of a right regular LA-semigroup $\mathcal{S}$, then

$$\begin{aligned} f((xy)z) &= ((f \circ C_{\mathcal{S}}(x)) \circ f)((xy)z) \\ &= \bigvee_{(xy)z=(xy)z} \{(f \circ C_{\mathcal{S}}(x))(xy) \wedge f(z)\} \\ &\geq \bigvee_{xy=xy} \{f(x) \wedge C_{\mathcal{S}}(x)(y)\} \wedge f(z) \\ &\geq f(x) \wedge 1 \wedge f(z) = f(x) \wedge f(z). \end{aligned}$$

Since $f \circ f = f$, therefore by Lemma 2.4, f is a fuzzy LA-subsemigroup of $\mathcal{S}$. This shows that f is a fuzzy bi ideal of $\mathcal{S}$. ∎

3.15. Theorem. *In a right regular LA-semigroup $\mathcal{S}$ with left identity, the following statements are equivalent.*

(i) f is a fuzzy interior ideal of $\mathcal{S}$.
(ii) $(C_{\mathcal{S}}(x) \circ f) \circ C_{\mathcal{S}}(x) = f$.

Proof. It is simple. ∎

3.16. Theorem. *In a right regular LA-semigroup $\mathcal{S}$ with left identity, the following statements are equivalent.*

(i) f is a fuzzy $(1, 2)$-ideal of $\mathcal{S}$.
(ii) $(f \circ C_{\mathcal{S}}(x)) \circ (f \circ f) = f$ and $f \circ f = f$.

Proof. $(i) \Longrightarrow (ii)$: Let f be a fuzzy $(1, 2)$-ideal of a right regular LA-semigroup $\mathcal{S}$ with left identity and let $a \in \mathcal{S}$, then there exists $x \in \mathcal{S}$ such that $a = a^2x$. Now by using (1) and (4), we have

$$\begin{aligned} a &= (aa)x = (xa)a = (xa)((aa)x) = (aa)((xa)x) \\ &= (a((aa)x))((xa)x) = ((aa)(ax))((xa)x) \\ &= (((xa)x)(ax))(aa) = (a(((xa)x)x))(aa). \end{aligned}$$

Therefore

$$((f \circ C_{\mathcal{S}}(x)) \circ (f \circ f))(a) = \bigvee_{a=(a(((xa)x)x))(aa)} \{(f \circ C_{\mathcal{S}}(x))(a(((xa)x)x)) \wedge (f \circ f)(aa)\}.$$

Now

$$\begin{aligned} (f \circ C_{\mathcal{S}}(x))(a(((xa)x)x)) &= \bigvee_{a(((xa)x)x)=a(((xa)x)x)} \{f(a) \wedge C_{\mathcal{S}}(x)(((xa)x)x)\} \\ &\geq f(a) \wedge C_{\mathcal{S}}(x)(((xa)x)x) = f(a) \end{aligned}$$

and

$$(f \circ f)(aa) = \bigvee_{aa=aa} \{f(a) \wedge f(a)\} \geq f(a).$$

Thus we get

$$((f \circ C_{\mathcal{S}}(x)) \circ (f \circ f))(a) \geq f(a).$$

Now by using (4), (1) and (3), we have

$$\begin{aligned} a &= (aa)x = (((aa)x)((aa)x))x = ((aa)(((aa)x)x))x \\ &= ((aa)((xx)(aa)))x = ((aa)(x^2(aa)))x \\ &= (x(x^2(aa)))(aa) = (x(a(x^2a)))(aa) \\ &= (a(x(x^2a)))(aa) = (a(x(x^2((aa)x))))(aa) \\ &= (a(x((aa)x^3)))(aa). \end{aligned}$$

Therefore

$$((f \circ C_{\mathbb{S}}(x)) \circ (f \circ f))(a) = \bigvee_{a=(a(x((aa)x^3)))(aa)} \{(f \circ C_{\mathbb{S}}(x))(a(x((aa)x^3))) \wedge (f \circ f)(aa)\}.$$

Now

$$\begin{aligned} (f \circ C_{\mathbb{S}}(x))(a(x((aa)x^3))) &= \bigvee_{a(x((aa)x^3))=a(x((aa)x^3))} \{f(a) \wedge C_{\mathbb{S}}(x)(x((aa)x^3))\} \\ &= \bigvee_{a(x((aa)x^3))=a(x((aa)x^3))} f(a) \end{aligned}$$

and

$$(f \circ f)(aa) = \bigvee_{aa=aa} \{f(a) \wedge f(a)\} = \bigvee_{aa=aa} f(a).$$

Therefore

$$\begin{aligned} (f \circ C_{\mathbb{S}}(x))(a(x((aa)x^3))) \wedge (f \circ f)(aa) &= \bigvee_{a(x((aa)x^3))=a(x((aa)x^3))} f(a) \wedge \bigvee_{aa=aa} f(a) \\ &= \bigvee_{a(x((aa)x^3))=a(x((aa)x^3))} \{f(a) \wedge f(a)\}. \end{aligned}$$

Thus from above, we get

$$\begin{aligned} ((f \circ C_{\mathbb{S}}(x)) \circ (f \circ f))(a) &= \bigvee_{a=(a(x((aa)x^3)))(aa)} \left(\bigvee_{a(x((aa)x^3))=a(x((aa)x^3))} \{f(a) \wedge f(a)\} \right) \\ &= \bigvee_{a=(a(x((aa)x^3)))(aa)} \{f(a) \wedge f(a) \wedge f(a)\} \\ &\leq \bigvee_{a=(a(x((aa)x^3)))(aa)} f((a(x((aa)x^3)))(aa)) = f(a). \end{aligned}$$

Therefore $(f \circ C_{\mathbb{S}}(x)) \circ (f \circ f) = f$.
Now by using (1) and (4), we have

$$\begin{aligned} a &= (aa)x = (xa)a = (x((aa)x))a = ((aa)(xx))a = ((a((aa)x))x^2)a \\ &= (((aa)(ax))x^2)a = ((x^2(ax))(aa))a = ((ax^3)(aa))a. \end{aligned}$$

Thus

$$\begin{aligned} (f \circ f)(a) &= \bigvee_{a=((ax^3)(aa))a} \{f(((ax^3)(aa))) \wedge f(a)\} \\ &\geq f(a) \wedge f(a) \wedge f(a) = f(a). \end{aligned}$$

Now by using Lemma 2.4, $f \circ f = f$.

$(ii) \Longrightarrow (i)$: Let f be a fuzzy subset of a right regular LA-semigroup $\mathcal{S}$. Now since $f \circ f = f$, therefore by Lemma 2.4, f is a fuzzy LA-subsemigroup of $\mathcal{S}$

$$\begin{aligned}
f((xa)(yz)) &= ((f \circ C_{\mathcal{S}}(x)) \circ (f \circ f))((xa)(yz)) \\
&= ((f \circ C_{\mathcal{S}}(x)) \circ f)((xa)(yz)) \\
&= \bigvee_{(xa)(yz)=(xa)(yz)} \{(f \circ C_{\mathcal{S}}(x))(xa) \wedge f(yz)\}. \\
&\geq (f \circ C_{\mathcal{S}}(x))(xa) \wedge f(yz) \\
&= \bigvee_{(xa)=(xa)} \{f(x) \wedge C_{\mathcal{S}}(x)(a)\} \wedge f(yz) \\
&\geq f(x) \wedge 1 \wedge f(y) \wedge f(z) \\
&= f(x) \wedge f(y) \wedge f(z).
\end{aligned}$$

This shows that $f((xa)(yz)) \geq f(x) \wedge f(y) \wedge f(z)$, therefore f is a fuzzy $(1,2)$-ideal of $\mathcal{S}$. ∎

3.17. Theorem. *Let $\mathcal{S}$ be an LA-semigroup with left identity, then the following conditions are equivalent.*

(i) $\mathcal{S}$ is right regular.
(ii) Every fuzzy left ideal of $\mathcal{S}$ is idempotent.

Proof. $(i) \Longrightarrow (ii)$: Let $\mathcal{S}$ be an LA-semigroup with left identity. Let $a \in \mathcal{S}$, then since $\mathcal{S}$ is right regular so by using (1),

$$a = (aa)x = (xa)a.$$

Let f be a fuzzy left ideal of $\mathcal{S}$, then clearly $f \circ f \subseteq f$ and therefore

$$(f \circ f)(a) = \bigvee_{a=(xa)a} \{f((xa)a) \wedge f(a)\} \geq f(a) \wedge f(a) = f(a).$$

Thus f is idempotent.

$(ii) \Longrightarrow (i)$: Assume that every fuzzy left ideal of an LA-semigroup $\mathcal{S}$ with left identity is idempotent. Since $\mathcal{S}a$ is a left ideal of $\mathcal{S}$, therefore by Lemma 2.6, its characteristic function $C_{\mathcal{S}a}$ is a fuzzy left ideal of $\mathcal{S}$. Since $a \in \mathcal{S}a$, therefore $C_{\mathcal{S}a}(a) = 1$. Now by using the given assumption and Lemma 2.5, we have

$$C_{\mathcal{S}a} \circ C_{\mathcal{S}a} = C_{\mathcal{S}a} \text{ and } C_{\mathcal{S}a} \circ C_{\mathcal{S}a} = C_{(\mathcal{S}a)^2}.$$

Thus we have $(C_{(\mathcal{S}a)^2})(a) = (C_{\mathcal{S}a})(a) = 1$, which implies that $a \in (\mathcal{S}a)^2$. Now by using (3) and (2), we have

$$a \in (\mathcal{S}a)^2 = (\mathcal{S}a)(\mathcal{S}a) = (a\mathcal{S})(a\mathcal{S}) = a^2\mathcal{S}.$$

This shows that $\mathcal{S}$ is right regular. ∎

Note that if an LA-semigroup has a left identity then $C_{\mathcal{S}}(x) \circ C_{\mathcal{S}}(x) = C_{\mathcal{S}}(x)$.

3.18. Theorem. *For an LA-semigroup $\mathcal{S}$ with left identity, the following conditions are equivalent.*

(i) $\mathcal{S}$ is right regular.
(ii) $f = (C_{\mathcal{S}}(x) \circ f)^2$, where f is any fuzzy left ideal of $\mathcal{S}$.

Proof. $(i) \Longrightarrow (ii)$: Assume that $\mathcal{S}$ is a right regular LA-semigroup and let f be any fuzzy left ideal of $\mathcal{S}$, then clearly $C_{\mathcal{S}}(x) \circ f$ is a fuzzy left ideal of $\mathcal{S}$. Now by using Theorem 3.17, $C_{\mathcal{S}}(x) \circ f$ is idempotent and, therefore, we have

$$(C_{\mathcal{S}}(x) \circ f)^2 = C_{\mathcal{S}}(x) \circ f \subseteq f.$$

Now let $a \in \mathcal{S}$, since $\mathcal{S}$ is right regular, therefore there exists $x \in \mathcal{S}$ such that $a = a^2x$ and by using (1), we have

$$a = (aa)x = (xa)a = (xa)((aa)x) = (xa)((xa)a)$$

Therefore

$$\begin{aligned}(C_{\mathcal{S}}(x) \circ f)^2(a) &= \bigvee_{a=(xa)((xa)a)} \{(C_{\mathcal{S}}(x) \circ f)(xa) \wedge (C_{\mathcal{S}}(x) \circ f)((xa)a)\} \\ &\geq (C_{\mathcal{S}}(x) \circ f)(xa) \wedge (C_{\mathcal{S}}(x) \circ f)((xa)a) \\ &= \bigvee_{xa=xa} \{C_{\mathcal{S}}(x)(x) \wedge f(a)\} \wedge \bigvee_{(xa)a=(xa)a} \{C_{\mathcal{S}}(x)(xa) \wedge f(a)\} \\ &\geq C_{\mathcal{S}}(x)(x) \wedge f(a) \wedge C_{\mathcal{S}}(x)(xa) \wedge f(a) = f(a).\end{aligned}$$

Thus we obtain $f = (C_{\mathcal{S}}(x) \circ f)^2$.

$(ii) \Longrightarrow (i)$: Let $f = (C_{\mathcal{S}}(x) \circ f)^2$ holds for any fuzzy left ideal f of $\mathcal{S}$, then by given assumption, we have

$$f = (C_{\mathcal{S}}(x) \circ f)^2 \subseteq f^2 = f \circ f \subseteq C_{\mathcal{S}}(x) \circ f \subseteq f.$$

Thus by using Theorem 3.17, $\mathcal{S}$ is right regular. ∎

An LA-semigroup $\mathcal{S}$ is called a left (right) duo if every left (right) ideal of $\mathcal{S}$ is a two-sided ideal of $\mathcal{S}$ and is called a duo if it is both a left and a right duo.

Consider an LA-semigroup $\mathcal{S}$ in Example 3.1, the right ideals of $\mathcal{S}$ are $\{a, b, c, e\}$ and $\{a, e\}$ which are also two-sided ideals of $\mathcal{S}$. Thus $\mathcal{S}$ is a right duo. On the other hand, the left ideals of $\mathcal{S}$ are $\{a, b, e\}$, $\{a, c, e\}$, $\{a, b, c, e\}$ and $\{a, e\}$. Note that $\mathcal{S}$ is not a left duo because $\{a, b, e\}$ and $\{a, c, e\}$ are not the right ideals of $\mathcal{S}$.

An LA-semigroups considered in Examples 2.1 and 3.2 are duo because in both examples, the only right (left, two-sided) ideals of $\mathcal{S}$ are $\{a, b\}$.

An LA-semigroup $\mathcal{S}$ is called a fuzzy left (right) duo if every fuzzy left (right) ideal of $\mathcal{S}$ is a fuzzy two-sided ideal of $\mathcal{S}$ and is called a fuzzy duo if it is both a fuzzy left and a fuzzy right duo.

By Lemma 3.7, every right regular LA-semigroup $\mathcal{S}$ with left identity is a fuzzy left (right) duo.

3.19. Theorem. *A right regular LA-semigroup $\mathcal{S}$ with left identity is a left (right) duo if and only if it is a fuzzy left (right) duo.*

Proof. Let a right regular LA-semigroup $\mathcal{S}$ be a left duo and assume that f is any fuzzy left ideal of $\mathcal{S}$. Let $a, b \in \mathcal{S}$, then $a \in (aa)\mathcal{S}$. Now $\mathcal{S}a$ is a left ideal of $\mathcal{S}$, therefore by hypothesis, $\mathcal{S}a$ is a two sided ideal of $\mathcal{S}$. Therefore by using (1), we have

$$ab \in ((aa)\mathcal{S})b = ((\mathcal{S}a)a)b \subseteq ((\mathcal{S}a)\mathcal{S})\mathcal{S} \subseteq (\mathcal{S}a).$$

Thus $ab = ca$ for some $c \in \mathcal{S}$. Now $f(ab) = f(ca) \geq f(a)$, implies that f is a fuzzy right ideal of $\mathcal{S}$ and therefore $\mathcal{S}$ is a fuzzy left duo.

Conversely, assume that $\mathcal{S}$ is a fuzzy left duo and let L be any left ideal of $\mathcal{S}$. Now by Lemma 2.6, the characteristic function C_L of L is a fuzzy left ideal of $\mathcal{S}$. Thus by hypothesis C_L is a fuzzy two-sided ideal of $\mathcal{S}$ and by using Lemma 2.6, L is a two sided ideal of $\mathcal{S}$. Thus $\mathcal{S}$ is a left duo.

Now again let $\mathcal{S}$ be a right regular LA-semigroup such that $\mathcal{S}$ is a right duo and assume that f is any fuzzy right ideal of $\mathcal{S}$. Let $a, b \in \mathcal{S}$, then there exists $x \in \mathcal{S}$ such that $b = b^2x$. Now clearly $b^2 \in b^2\mathcal{S}$ and since $b^2\mathcal{S}$ is a right ideal of $\mathcal{S}$, therefore

$$b = b^2x \in (b^2\mathcal{S})\mathcal{S} \subseteq b^2\mathcal{S}.$$

As $b^2\mathcal{S}$ is a right ideal of $\mathcal{S}$, therefore by hypothesis $b^2\mathcal{S}$ is a two sided ideal of $\mathcal{S}$. Now by using (1), we have

$$ab \in a(b^2\mathcal{S}) \subseteq \mathcal{S}(b^2\mathcal{S}) \subseteq b^2\mathcal{S}.$$

Thus $ab = (bb)c$ for some $c \in \mathcal{S}$. Now $f(ab) = f((bb)c) \geq f(b)$, implies that f is a fuzzy left ideal of $\mathcal{S}$ and therefore $\mathcal{S}$ is a fuzzy right duo.

The Converse is simple. ∎

3.20. Theorem. *Let $\mathcal{S}$ be a right regular LA-semigroup with left identity, then the following statements are equivalent.*

(i) f is a fuzzy left ideal of $\mathcal{S}$.
(ii) f is a fuzzy right ideal of $\mathcal{S}$.
(iii) f is a fuzzy two-sided ideal of $\mathcal{S}$.
(iv) f is a fuzzy bi-ideal of $\mathcal{S}$.
(v) f is a fuzzy generalized bi-ideal of $\mathcal{S}$.
(vi) f is a fuzzy $(1, 2)$-ideal of $\mathcal{S}$.
(vii) f is a fuzzy interior ideal of $\mathcal{S}$.
($viii$) f is a fuzzy quasi ideal of $\mathcal{S}$.
(ix) $f \circ C_{\mathcal{S}}(x) = f$ and $C_{\mathcal{S}}(x) \circ f = f$.

Proof. $(i) \Longrightarrow (ix)$: Let f be a fuzzy left ideal of a right regular LA-semigroup $\mathcal{S}$. Let $a \in \mathcal{S}$, then there exists $a' \in \mathcal{S}$ such that $a = a^2a'$. Now by using (1) and (3), we have

$$a = (aa)a' = (a'a)a \text{ and } a = (aa)a' = (aa)(ea') = (a'e)(aa).$$

Therefore

$$(f \circ C_{\mathcal{S}}(x))(a) = \bigvee_{a=(a'a)a} \left\{f(a'a) \wedge C_{\mathcal{S}}(x)(a)\right\} \geq f(a'a) \wedge 1 \geq f(a)$$

and

$$(C_{\mathcal{S}}(x) \circ f)(a) = \bigvee_{a=(a'e)(aa)} \left\{C_{\mathcal{S}}(x)(a'e) \wedge f(aa)\right\} \geq 1 \wedge f(aa) \geq f(a).$$

Now by using Lemmas 3.7 and 2.4, we get that $f \circ C_{\mathcal{S}}(x) = f$ and $C_{\mathcal{S}}(x) \circ f = f$.

$(ix) \Longrightarrow (viii)$ is obvious.

$(viii) \Longrightarrow (vii)$: Let f be a fuzzy quasi ideal of a right regular LA-semigroup $\mathcal{S}$. Now for $a \in \mathcal{S}$ there exists $a' \in \mathcal{S}$ such that $a = a^2a'$ and therefore by using (3) and (4), we have

$$(xa)y = (xa)(ey) = (ye)(ax) = a((ye)x)$$

also

$$\begin{aligned}(xa)y &= (x((aa)a'))y = ((aa)(xa'))y = ((a'x)(aa))y \\ &= (a((a'x)a))y = (y((a'x)a))a.\end{aligned}$$

Since f is a fuzzy quasi ideal of $\mathcal{S}$, therefore by Lemma 3.13, we have

$$f((xa)y) = ((f \circ C_{\mathcal{S}}(x)) \cap (C_{\mathcal{S}}(x) \circ f))((xa)y) = (f \circ C_{\mathcal{S}}(x))\,((xa)y) \wedge (C_{\mathcal{S}}(x) \circ f)\,((xa)y).$$

Now

$$(f \circ C_{\mathcal{S}}(x))\,((xa)y) = \bigvee_{(xa)y=a((ye)x)} \{f(a) \wedge C_{\mathcal{S}}(x)((ye)x)\} \geq f(a)$$

and

$$\left(C_{\mathcal{S}}(x) \circ f\right)((xa)y) = \bigvee_{(xa)y=(y((a^{'}x)a))a} \left\{C_{\mathcal{S}}(x)(y((a^{'}x)a)) \wedge f(a)\right\} \geq f(a).$$

Which implies that $f((xa)y) \geq f(a)$. Thus f is a fuzzy interior ideal of $\mathcal{S}$.
$(vii) \Longrightarrow (vi)$ is followed by Theorem 3.11.
$(vi) \Longrightarrow (v)$ is followed by Theorem 3.10.
$(v) \Longrightarrow (iv)$: It is simple.
$(iv) \Longrightarrow (iii)$ is followed by Theorems 3.10 and 3.8.
$(iii) \Longrightarrow (ii)$ and $(ii) \Longrightarrow (i)$ are easy consequences of Lemma 3.7. ∎

3.21. Theorem. *For an LA-semigroup $\mathcal{S}$ with left identity, the following conditions are equivalent.*

(i) $\mathcal{S}$ is right regular.
(ii) $f \cap g \subseteq f \circ g$, for every fuzzy right ideal f and g of $\mathcal{S}$, where f and g are fuzzy semiprime.
(iii) $f \cap g \subseteq f \circ g$, for every fuzzy left ideal f and g of $\mathcal{S}$, where f and g are fuzzy semiprime.

Proof. $(i) \Rightarrow (iii)$: Assume that f and g are fuzzy left ideals of $\mathcal{S}$ with left identity. Let a be any element in $\mathcal{S}$, since $\mathcal{S}$ is right regular, so exists x in $\mathcal{S}$, such that $a = a^2x$. Now by using (1), (4), (3) and (2), we have

$$\begin{aligned} a &= (aa)x = (xa)a = (xa)(a^2x) = a^2((xa)x) \\ &= (aa)((xa)x) = (x(xa))(aa) = (xa)((xa)a). \end{aligned}$$

Therefore

$$\begin{aligned} (f \circ g)(a) &= \bigvee_{a=(xa)((xa)a)} \{f(xa) \wedge g((xa)a)\} \geq f(xa) \wedge g((xa)a) \\ &\geq f(a) \wedge g(a) = (f \cap g)(a). \end{aligned}$$

$(iii) \Rightarrow (ii)$ can be followed from Lemma 2.3.
$(ii) \Rightarrow (i)$: Assume that f and g are any fuzzy left ideals of $\mathcal{S}$ with left identity and let R and $R^{'}$ be any right ideals of $\mathcal{S}$, then by Lemma 2.6, C_R and $C_{R^{'}}$ are fuzzy right ideals of $\mathcal{S}$. Let $a \in R \cap R^{'}$,therefore by Lemma 2.5 and given assumption, we have

$$1 = C_{R \cap R^{'}}(a) = (C_R \cap C_{R^{'}})(a) \subseteq (C_R \circ C_{R^{'}})(a) = (C_{RR^{'}})(a),$$

which implies that $R \cap R^{'} \subseteq RR^{'}$. Since f and g are fuzzy semiprime, so R and $R^{'}$ are fuzzy semiprime. As $a^2\mathcal{S}$ is a right ideal of $\mathcal{S}$ and clearly $a^2 \in a^2\mathcal{S}$, therefore $a \in a^2\mathcal{S}$. Now by using (4), we have

$$a \in a^2\mathcal{S} \cap a^2\mathcal{S} \subseteq (a^2\mathcal{S})(a^2\mathcal{S}) = a^2((a^2\mathcal{S})\mathcal{S}) \subseteq a^2\mathcal{S}.$$

This shows that $\mathcal{S}$ is right regular. ∎

A subset A of an LA-semigroup $\mathcal{S}$ is called semiprime if $a^2 \in A$ implies $a \in A$.

The subset $\{a, b\}$ of an LA-semigroup $\mathcal{S}$ in Example 2.1 is semiprime.

A fuzzy subset f of an LA-semigroup $\mathcal{S}$ is called a fuzzy semiprime if $f(a) \geq f(a^2)$ for all a in $\mathcal{S}$.

Let us define a fuzzy subset f of an LA-semigroup $\mathcal{S}$ in Example 3.1 as follows: $f(a) = 0.2, f(b) = 0.5, f(c) = 0.6, f(d) = 0.1$ and $f(e) = 0.4$, then f is a fuzzy semiprime.

3.22. Lemma. *For a right regular LA-semigroup $\mathcal{S}$, the following holds.*

(i) Every fuzzy right ideal of $\mathcal{S}$ is a fuzzy semiprime.
(ii) Every fuzzy left ideal of $\mathcal{S}$ is a fuzzy semiprime if $\mathcal{S}$ has a left identity.

Proof. (i) : It is simple.
(ii) : Let f be a fuzzy left ideal of a right regular LA-semigroup $\mathcal{S}$ and let $a \in \mathcal{S}$, then there exists $x \in \mathcal{S}$ such that $a = a^2x$. Now by using (3), we have

$$f(a) = f((aa)(ex)) = f((xe)a^2) \geq f(a^2).$$

This shows that f is a fuzzy semiprime. ∎

Right, left and two-sided ideals of an LA-semigroup $\mathcal{S}$ are semiprime if and only if their characteristic functions are fuzzy semiprime.

3.23. Lemma. *Let $\mathcal{S}$ be an LA-semigroup with left identity, then the following statements are equivalent.*

(i) $\mathcal{S}$ is right regular.
(ii) Every fuzzy right (left, two-sided) ideal of $\mathcal{S}$ is fuzzy semiprime.

Proof. (i) $\Longrightarrow$ (ii) : It follows from Lemma 3.22.
(ii) $\Longrightarrow$ (i) : Assume that $\mathcal{S}$ is an LA-semigroup with left identity and let every fuzzy right (left, two-sided) ideal of $\mathcal{S}$ be fuzzy semiprime. Since $a^2\mathcal{S}$ is a right and also a left ideal of $\mathcal{S}$, therefore, $a^2\mathcal{S}$ is semiprime. Now clearly $a^2 \in a^2\mathcal{S}$, therefore $a \in a^2\mathcal{S}$, which shows that $\mathcal{S}$ is right regular. ∎

3.24. Theorem. *The following statements are equivalent for an LA-semigroup $\mathcal{S}$ with left identity.*

(i) $\mathcal{S}$ is right regular.
(ii) Every fuzzy right ideal of $\mathcal{S}$ is fuzzy semiprime.
(iii) Every fuzzy left ideal of $\mathcal{S}$ is fuzzy semiprime.

Proof. (i) $\Longrightarrow$ (iii) and (ii) $\Longrightarrow$ (i) are followed by Lemma 3.23.
(iii) $\Longrightarrow$ (ii) : Assume that $\mathcal{S}$ is an LA-semigroup and let f be a fuzzy right ideal of $\mathcal{S}$, then by using Lemma 2.3, f is a fuzzy left ideal of $\mathcal{S}$ and therefore by given assumption f is a fuzzy semiprime. ∎

3.25. Theorem. *For an LA-semigroup $\mathcal{S}$ with left identity, the following conditions are equivalent.*

(i) $\mathcal{S}$ is right regular.
(ii) Every fuzzy two-sided ideal of $\mathcal{S}$ is fuzzy semiprime.
(iii) Every fuzzy bi-ideal of $\mathcal{S}$ is fuzzy semiprime.
(iv) $f(a) = f(a^2)$, for all fuzzy two sided ideal f of $\mathcal{S}$, for all $a \in \mathcal{S}$.
(v) $f(a) = f(a^2)$, for all fuzzy bi-ideal f of $\mathcal{S}$, for all $a \in \mathcal{S}$.

Proof. (i) $\Rightarrow$ (v) : Assume that f is any fuzzy bi-ideal of $\mathcal{S}$. Let a be any element of $\mathcal{S}$. Since $\mathcal{S}$ is right regular, so there exists x in $\mathcal{S}$, such that $a = a^2x$. Now by using (1), (4) and (3), we have

$$\begin{aligned}
f(a) &= f((aa)x) = f((xa)a) = f((x(a^2x)a) = f((a^2x^2)a) \\
&= f((ax^2)a^2) = f(((a^2x)x^2)a^2) = f(((x^2x)a^2)a^2) \\
&= f(((x^2x)(aa))a^2) = f(((aa)(x^2x))a^2) \\
&= f((a^2x^3)a^2) \geq f(a^2) \wedge f(a^2) = f(aa) \\
&= f((a^2x)a) = f(((aa)(ex))a) = f(((xe)(aa))a) \\
&= f((a((xe)a))a) \geq f(a) \wedge f(a) = f(a).
\end{aligned}$$

This shows that $f(a) = f(a^2)$ for all a in $\mathcal{S}$. Clearly $(v) \Rightarrow (iv)$.

$(iv) \Rightarrow (i)$: Since $a^2\mathcal{S}$ is a two sided ideal of $\mathcal{S}$ with left identity, therefore it is clear to see that $a^2 \in a^2\mathcal{S}$. Now by Lemma 2.6, $C_{a^2\mathcal{S}}$ is a fuzzy two sided ideal of $\mathcal{S}$ and by given assumption , we have $C_{a^2\mathcal{S}}(a) = C_{a^2\mathcal{S}}\left(a^2\right) = 1$. Therefore $a \in a^2\mathcal{S}$, which shows that $\mathcal{S}$ is a right regular.

It is easy to observe that $(ii) \iff (iv)$ and $(iii) \iff (v)$. ∎

3.26. Theorem. *For an LA-semigroup $\mathcal{S}$ with left identity, the following conditions are equivalent.*

(i) $\mathcal{S}$ is right regular.
(ii) Every right ideal of $\mathcal{S}$ is semiprime.
(iii) Every fuzzy right ideal of $\mathcal{S}$ is fuzzy semiprime.
(iv) $f(a) = f(a^2)$, for every fuzzy right ideal f of $\mathcal{S}$ and for all a in $\mathcal{S}$.
(v) $f(a) = f(a^2)$, for every fuzzy left ideal f of $\mathcal{S}$ and for all a in $\mathcal{S}$.

Proof. $(i) \Rightarrow (v)$: Assume that f is a fuzzy left ideal of $\mathcal{S}$. Let a be any element in $\mathcal{S}$, since $\mathcal{S}$ is right regular, so exists x in $\mathcal{S}$, such that $a = a^2x$. Now by using (3), we have

$$f(a^2x) = f((aa)(ex)) = f((xe)(aa)) \geq f(aa) \geq f(a),$$

therefore $f(a) = f(a^2)$ for all a in $\mathcal{S}$.

From Lemma 2.3, it is clear that $(v) \Rightarrow (iv)$ and $(iv) \Rightarrow (iii)$ are obvious.

Now $(iii) \Rightarrow (ii)$ and $(ii) \Rightarrow (i)$ are easy. ∎

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