Hacettepe Journal of Mathematics and Statistics
Volume 33 (2004), 23 – 30

A WAVELET-TYPE TRANSFORM GENERATED BY THE POISSON SEMIGROUP

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Received 11 : 05 : 2004 : Accepted 16 : 06 : 2004

Abstract

A wavelet-type transform generated with the aid of the Poisson Semigroup and a signed Borel measure is introduced. An analogue of the Calderón reproducing formula (in the framework of the L_2 and L_p-theory) is established.

Keywords: Wavelet transform, Poisson semigroup, Calderón's reproducing formula.

2000 AMS Classification: 65 R 10

1. Introduction

The Calderón reproducing formula is widely used in the theory of continuous wavelet transforms [3, 4], in fractional calculus and in integral geometry (see, e.g. [1, 2, 5, 6] and references therein). A version of the Calderón formula asserts that under certain conditions on $u(x)$, $(x \in \mathbb{R}^n)$

$$(1.1) \qquad \lim_{\substack{\varepsilon \to 0 \\ \rho \to \infty}} \int_{\varepsilon}^{\rho} \frac{f * u_t}{t} dt = c_u f, \quad f \in L_2(\mathbb{R}^n),$$

where $u_t(x) = t^{-n} u(x/t)$, $t > 0$, "$*$" is a convolution operator and the limit is taken with respect to the L_2-norm. The convolution $(W_u f)(x,t) = (f * u_t)(x)$ is called the *continuous wavelet transform, generated by the "wavelet function" u.*

A generalization of (1.1) has the form [6] :

$$(1.2) \qquad \int_0^{\infty} \frac{f * \mu_t}{t} dt \stackrel{def}{=} \lim_{\substack{\varepsilon \to 0 \\ \rho \to \infty}} \int_{\varepsilon}^{\rho} \frac{f * \mu_t}{t} dt = c_\mu f,$$

where μ is a suitable radial Borel measure, μ_t stands for the dilation of μ, and the limit is interpreted in the L_p-norm and in the pointwise (a.e.) sense.

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In this paper we introduce a new wavelet-type transform by making use of the Poisson kernel and finite Borel measure μ. The main purpose of the paper is to prove the relevant Calderón-type reproducing formula. The L_2 and L_p, $\quad (1 \le p \le \infty)$ cases are examined separately. The pointwise (a.e.) convergence of the corresponding "truncated integrals" $\int\limits_{\varepsilon}^{\rho}(\cdots)$ is also studied.

2. Preliminiaries

Let

$$P(x,t) = \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} . \frac{t}{(t^2+|x|^2)^{(n+1)/2}}, \qquad t>0 \ , \ x \in \mathbb{R}^n,$$

be the Poisson kernel which possess the following properties [7]:

$$\text{(2.1)} \qquad \Big[P(.,t)\Big]^{\wedge}(\xi) \equiv \int\limits_{\mathbb{R}^n} e^{-2\pi i x.\xi} P(x,t)\, dx = e^{-\pi t|\xi|} \,,$$

with $x.\xi = x_1\xi_1 + \cdots + x_n\xi_n$;

$$\text{(2.2)} \qquad \int\limits_{\mathbb{R}^n} P(x,t)\, dx = 1, \qquad \forall t>0\,;$$

$P(x,t)$ is homogeneous function of order $(-n)$, i.e

$$\text{(2.3)} \qquad P(\lambda x, \lambda t) = \lambda^{-n} P(x,t), \qquad \forall \lambda > 0\,;$$

$$\text{(2.4)} \qquad \int\limits_{\mathbb{R}^n} P(y,t)P(x-y,\tau)dy = P(x, t+\tau).$$

Given a function $f \in L_p(\mathbb{R}^n)$ with the norm $\|f\|_p = (\int\limits_{\mathbb{R}^n} |f(x)|^p dx)^{1/p}$ we denote by $P_t f(x),\ t>0$ the Poisson semigroup associated with f:

$$\text{(2.5)} \qquad P_t f(x) = \int\limits_{\mathbb{R}^n} f(x-z)P(z,t)dz,\ t>0; \quad P_0 f(x) = f(x).$$

It is well known that (see, e.g. [7, p. 8-16])

$$\text{(2.6)} \qquad \|P_t f\|_p \le \|f\|_p, \quad (1 \le p \le \infty), \quad \forall t \ge 0\,;$$

$$\text{(2.7)} \qquad P_t(P_\tau f)(x) = P_{t+\tau} f(x), \quad (t,\tau \ge 0)\,;$$

$$\text{(2.8)} \qquad \lim_{t \to 0^+} P_t f(x) = f(x)\,,$$

with the limit being understood in the $L_p, (1 \le p < \infty)$– norm or pointwise for almost all $x \in \mathbb{R}^n$. If $f \in C^0$ (the space of continuous functions vanishing at infinity), then convergence is uniform. Furthermore,

$$\text{(2.9)} \qquad \sup_{t>0} |P_t f(x)| \le M_f(x),$$

with the well known Hardy–Littlewood maximal function

$$\text{(2.10)} \qquad M_f(x) = \sup_{r>0} \frac{1}{|B_r|} \int\limits_{B_r} |f(x-z)|dz,\ B_r = \{x : |x| < r\}.$$

2.1. Definition. Let μ be a signed Borel measure on $\mathbb{R}^1$ such that

$$\mathrm{supp}\mu \subset [0,\infty); \quad |\mu|(\mathbb{R}^1) < \infty, \quad \mu(\{0\}) = 0, \text{ and}$$

(2.11)
$$\mu(\mathbb{R}^1) \equiv \int_{\mathbb{R}^1} d\mu(t) = 0.$$

In addition let $P(y,t)$ be the Poisson kernel extended to $t \le 0$ by zero. We define a wavelet transform of $f : \mathbb{R}^n \to \mathbb{C}$ as

$$W_\mu f(x,\eta) = \int_{\mathbb{R}^{n+1}} P(y,t) f(x-\eta y) dy\, d\mu(t)$$

$$\stackrel{(2.11)}{=} \int_{\mathbb{R}^n \times (0,\infty)} P(y,t) f(x-\eta y) dy\, d\mu(t). \tag{2.12}$$

By setting ty instead of y and using (2.3) we have

$$W_\mu f(x,\eta) = \int_{\mathbb{R}^n \times (0,\infty)} P(y,1) f(x-\eta t y) dy\, d\mu(t). \tag{2.13}$$

2.2. Remark. For any fixed $\eta > 0$ the operator W_μ is $L_p \to L_p$ bounded. Indeed, by the Minkowski inequality,

$$\|W_\mu f(\,.\,,\eta)\|_p \le \|f\|_p \int_{\mathbb{R}^n \times (0,\infty)} P(y,t) dy\, d|\mu|(t) \stackrel{(2.2)}{=} \|f\|_p \|\mu\| < \infty$$

where

$$\|\mu\| = \int_{(0,\infty)} d|\mu|(t) < \infty.$$

2.3. Remark. For $f \in L_p(\mathbb{R}^n)$, due to the Fubini theorem, we get

$$W_\mu f(x,\eta) = \int_{\mathbb{R}^n} f(x-\eta y) \Big(\int_{(0,\infty)} P(y,t) d\mu(t) \Big) dy.$$

Setting $w(y) = \int_{(0,\infty)} P(y,t) d\mu(t)$, by the Fubini theorem we have

$$\int_{\mathbb{R}^n} w(y) dy \stackrel{(2.2)}{=} \int_{(0,\infty)} d\mu(t) \stackrel{(2.11)}{=} 0.$$

That is, the function $w(y)$ is a usual wavelet function. Further,

$$W_\mu f(x,\eta) = \int_{\mathbb{R}^n} f(x-\eta y) w(y) dy = \frac{1}{\eta^n} \int_{\mathbb{R}^n} f(y) w(\frac{x-y}{\eta}) dy.$$

Therefore, $W_\mu f(x,\eta)$ is a continuous wavelet transform generated by the wavelet function $w(y) = \int_{(0,\infty)} P(y,t) d\mu(t)$.

2.4. Remark. In the following we will use the convention $\int_a^b \varphi(t) d\mu(t) = \int_{[a,b)} \varphi(t) d\mu(t)$.

In the case where $\lim_{t\to 0+} \varphi(t) = \infty$ we assume that $\mu(0) = 0$ and $\int_0^b \varphi(t) d\mu(t) = \int_{(0,b)} \varphi(t) d\mu(t)$.

We will need the following lemmas.

2.5. Lemma. [5, p.189] *Let μ be a Borel measure satisfying the conditions* (2.11) *and* $\int_0^\infty |\log t|\, d|\mu|(t) < \infty$. *Set* $k(s) = \frac{1}{s}\int_0^s d\mu(t)$. *Then*

$$k(s) \in L_1(0,\infty) \text{ and } \int_0^\infty k(s)ds = \int_0^\infty \log\frac{1}{s}d\mu(s)$$

.

2.6. Lemma. [7, p.60] *Let $T_\varepsilon, \varepsilon > 0$ be a family of linear operators, mapping $L^p(\mathbb{R}^n)$, $1 \le p \le \infty$ into the space of measurable functions on $\mathbb{R}^n$. Define T^*f by setting*

$$(T^*f)(x) = \sup_{\varepsilon>0} |(T_\varepsilon f)(x)|, \ x \in \mathbb{R}^n.$$

Suppose that there exists a constant $c > 0$ and a real number $q \ge 1$ such that

$$\text{meas}\{\, x : |(T^*f)(x)| > t\} \le (c\|f\|_{L^p} t^{-1})^q$$

for all $t > 0$ and $f \in L^p(\mathbb{R}^n)$. If there exists a dense subset $\mathcal{D}$ of $L^p(\mathbb{R}^n)$ such that $\lim_{\varepsilon\to 0}(T_\varepsilon g)(x)$ exists and is finite almost everywhere (a.e.) whenever $g \in \mathcal{D}$, then for each $f \in L^p(\mathbb{R}^n)$, $\lim_{\varepsilon\to 0}(T_\varepsilon f)(x)$ exists and is finite a.e.

3. A Calderon-type reproducing formula associated with the wavelet-type transform $W_\mu f$

We will examine the L_2 and L_p, $(1 \le p \le \infty)$ cases separately. In the L_2-case the conditions on μ are expressed in terms of the Laplace transform of μ, and in the general case – in terms of μ itself.

3.1. Theorem. *Let μ satisfy the conditions in (2.11). Suppose that $\tilde{\mu}(t) = \int_0^\infty e^{-st}d\mu(s)$ is the Laplace transform of μ and the integral $\tilde{c}_\mu = \int_0^\infty \tilde{\mu}(t)dt/t$ is finite. Then,*

$$\int_0^\infty W_\mu f(x,\eta)\frac{d\eta}{\eta} \equiv \lim_{\substack{\varepsilon\to 0\\ \rho\to\infty}} \int_\varepsilon^\rho W_\mu f(x,\eta)\frac{d\eta}{\eta} = \tilde{c}_\mu f(x), \quad \forall f \in L_2(\mathbb{R}^n),$$

where the limit is interpreted in the L_2-norm.

Proof. Let

$$f_{\varepsilon,\rho}(x) = \int_\varepsilon^\rho W_\mu f(x,\eta)\frac{d\eta}{\eta}, \qquad 0 < \varepsilon < \rho < \infty; \quad f \in L_1 \cap L_2.$$

By employing the Fourier transform and the Fubini theorem, from (2.13) we have

$$f^{\wedge}_{\varepsilon,\rho}(y) = \int\limits_{\varepsilon}^{\rho} \frac{d\eta}{\eta} \int\limits_{\mathbb{R}^n\times(0,\infty)} P(z,1)\Big(\int\limits_{\mathbb{R}^n} e^{-2\pi i x.y} f(x-\eta t z)dx\Big)dz d\mu(t)$$

(we replace x with $x+\eta tz$)

$$= f^{\wedge}(y)\int\limits_{\varepsilon}^{\rho} \frac{d\eta}{\eta} \int\limits_{\mathbb{R}^n\times(0,\infty)} P(z,1)e^{-2\pi i(z.y)\eta t} dz d\mu(t)$$

$$\overset{(2.1)}{=} f^{\wedge}(y)\int\limits_{\varepsilon}^{\rho} \frac{d\eta}{\eta}\int\limits_{0}^{\infty} e^{-\pi\eta t|y|}d\mu(t) \quad (\text{put } \eta = s/\pi|y|)$$

$$= f^{\wedge}(y)\int\limits_{\varepsilon\pi|y|}^{\rho\pi|y|} \frac{ds}{s}\int\limits_{0}^{\infty} e^{-st}d\mu(t) = f^{\wedge}(y)\int\limits_{\varepsilon\pi|y|}^{\rho\pi|y|} \tilde{\mu}(s)\frac{ds}{s}.$$

Setting $k_{\varepsilon,\rho}(y) = \int\limits_{\varepsilon\pi|y|}^{\rho\pi|y|} \tilde{\mu}(s)\frac{ds}{s}$, we have

$$f^{\wedge}_{\varepsilon,\rho}(y) = f^{\wedge}(y)k_{\varepsilon,\rho}(y). \tag{3.1}$$

Since $\tilde{c}_\mu = \int\limits_0^\infty \tilde{\mu}(s)\frac{ds}{s}$ is finite and the function $\int\limits_0^t \tilde{\mu}(s)\frac{ds}{s}$ continuous on $[0,\infty)$,

$$c \overset{def}{=} \sup_{t>0}\Big|\int\limits_0^t \tilde{\mu}(s)\frac{ds}{s}\Big|$$

is finite. Hence

$$|k_{\varepsilon,\rho}(y)| = \Big|\int\limits_0^{\rho\pi|y|} \tilde{\mu}(s)\frac{ds}{s} - \int\limits_0^{\varepsilon\pi|y|} \tilde{\mu}(s)\frac{ds}{s}\Big| \le 2c. \tag{3.2}$$

Now by the Plancherel and Lebesque Dominated Convergence theorems it follows that

$$\|f_{\varepsilon,\rho} - \tilde{c}_\mu f\|_2 = \|f^{\wedge}_{\varepsilon,\rho} - \tilde{c}_\mu f^{\wedge}\|_2 \overset{(3.1)}{=} \|f^{\wedge}(k_{\varepsilon,\rho} - \tilde{c}_\mu)\|_2 \to 0 \text{ as } \varepsilon\to 0,\ \rho\to\infty.$$

Hence, for any $f \in L_1 \cap L_2$,

$$\lim_{\substack{\varepsilon\to 0\\ \rho\to\infty}} \|f_{\varepsilon,\rho} - \tilde{c}_\mu f\|_2 = 0.$$

The statement for arbitrary $f \in L_2$ follows in a standard way by using uniform $L_2 \to L_2$ boundedness of the family of linear operators $A_{\varepsilon,\rho}f \equiv f_{\varepsilon,\rho}$:

$$\|A_{\varepsilon,\rho}f\|_2 = \|f_{\varepsilon,\rho}\|_2 = \|f^{\wedge}_{\varepsilon,\rho}\|_2 = \|f^{\wedge}k_{\varepsilon,\rho}\|_2 \overset{(3.2)}{\le} 2c\|f^{\wedge}\|_2 = 2c\|f\|_2,$$

that is $\|A_{\varepsilon,\rho}f\|_2 \le 2c\|f\|_2$, $\forall f \in L_1 \cap L_2$.

The General case follows by density. □

The following result gives a L_p-version of the Calderón-type reproducing formula for arbitrary $p \ge 1$.

3.2. Theorem. *Let $f \in L_p(\mathbb{R}^n)$, $\quad 1 \le p \le \infty$ ($L_\infty \equiv C^0$- the space of continuous functions vanishing at infinity). Let μ be a finite (signed) Borel measure on $\mathbb{R}^1$ such that*

$$\mu(\mathbb{R}^1) = 0,\ \mu(\{0\}) = 0,\ \operatorname{supp}\mu \subset [0,\infty) \quad \text{and} \quad \int_0^\infty |\log t| d|\mu|(t) < \infty,$$

then

$$\int_0^\infty W_\mu f(x,\eta)\frac{d\eta}{\eta} \equiv \lim_{\varepsilon\to 0}\int_\varepsilon^\infty W_\mu f(x,\eta)\frac{d\eta}{\eta} = c_\mu f(x)\,, \tag{3.3}$$

where

$$c_\mu = \int_0^\infty \log\frac{1}{t}\, d\mu(t),$$

the limit being with respect to the L_p-norm ($1 \le p < \infty$), or taken pointwise for almost all $x \in \mathbb{R}^n$. In the case $p = \infty$ it is assumed that $L_\infty = C^0$ and the limit is understood in the sup-norm.

Proof. We need the following modification of the wavelet-type transform $W_\mu f$.

$$W_\mu f(x,\eta) = \int_{\mathbb{R}^n\times(0,\infty)} P(y,t) f(x-\eta y)dy\, d\mu(t)$$

(we put $y = (1/\eta)z,\ dy = (1/\eta)^n dz$ and use (2.3))

$$= \int_0^\infty \Big(\int_{\mathbb{R}^n} P(z,\eta t) f(x-z)dz\Big) d\mu(t) \overset{(2.5)}{=} \int_0^\infty P_{t\eta} f(x)\, d\mu(t). \tag{3.4}$$

Let

$$V_\varepsilon f(x) = \int_\varepsilon^\infty W_\mu f(x,\eta)\frac{d\eta}{\eta}, \quad \varepsilon > 0. \tag{3.5}$$

Then, by using (3.4) and the Fubini theorem, we have

$$V_\varepsilon f(x) = \int_\varepsilon^\infty \Big(\int_0^\infty P_{t\eta} f(x)\, d\mu(t)\Big)\frac{d\eta}{\eta} = \int_0^\infty \Big(\int_\varepsilon^\infty P_{t\eta} f(x)\,\frac{d\eta}{\eta}\Big) d\mu(t)$$

$$= \int_0^\infty \Big(\int_{\varepsilon t}^\infty P_s f(x)\,\frac{ds}{s}\Big) d\mu(t) = \int_0^\infty \Big(\frac{1}{s}\int_0^{s/\varepsilon} d\mu(t)\Big) P_s f(x)\, ds. \tag{3.6}$$

Setting $k(s) = \frac{1}{s}\int_0^s d\mu(t)$ and $k_\tau(s) = \frac{1}{\tau}k(s/\tau)$, we have $k_\tau(s) = \frac{1}{s}\int_0^{s/\tau} d\mu(t)$ and therefore, $\frac{1}{s}\int_0^{s/\varepsilon} d\mu(t) = k_\varepsilon(s)$. Making use of this in (3.6) we have

$$V_\varepsilon f(x) = \int_0^\infty k_\varepsilon(s) P_s f(x) ds. \tag{3.7}$$

By setting $\tilde{c}_\mu = \int\limits_0^\infty k(s)\, ds$ (which is finite and equal to $c_\mu \equiv \int\limits_0^\infty \log \frac{1}{\tau} d\mu(\tau)$ by Lemma 2.5), and using Minkowski inequality we have

$$\begin{aligned}\|V_\varepsilon f(x) - \tilde{c}_\mu f(x)\|_p &= \Big\| \int\limits_0^\infty k_\varepsilon(s) P_s f(x) ds - \int\limits_0^\infty k(s) f(x) ds \Big\|_p \\ &= \Big\| \int\limits_0^\infty k(s) P_{s\varepsilon} f(x) ds - \int\limits_0^\infty k(s) f(x) ds \Big\|_p \\ &\le \int\limits_0^\infty |k(s)| \|P_{s\varepsilon} f(x) - f(x)\|_p ds.\end{aligned}$$

From (2.6), (2.8) and Lebesgue's convergence theorem it follows that the last expression tends to zero as $\varepsilon \to 0$. For similar reasons the convergence is uniform for $f \in C^0$.

It remains to show the pointwise (a.e) convergence in (3.3). For $f \in L_p$, $(1 \le p < \infty)$, we have

$$|V_\varepsilon f(x)| \le \int\limits_0^\infty |k_\varepsilon(s)| |P_s f(x)| ds$$

$$\le \sup_{s>0} |P_s f(x)| \int\limits_0^\infty |k_\varepsilon(s)| ds = c \cdot \sup_{s>0} |P_s f(x)|, \tag{3.8}$$

where $c = \int\limits_0^\infty |k_\varepsilon(s)| ds = \int\limits_0^\infty |k(s)| ds < \infty$ by Lemma 2.5. From (3.8) and (2.9) it follows that for any $\lambda > 0$

$$\text{meas}\{\, x \in \mathbb{R}^n : \sup_{\varepsilon>0} |V_\varepsilon f(x)| > \lambda\} \le c_1.\text{meas}\{x \in \mathbb{R}^n : M_f(x)| > \lambda\} \le \Big(c_2 \frac{\|f\|_p}{\lambda}\Big)^p.$$

Thus the maximal operator $\sup\limits_{\varepsilon>0} |V_\varepsilon f(x)|$ is of weak (p,p)-type. Now by employing Lemma 2.6 and keeping in mind that $V_\varepsilon f(x) \to \tilde{c}_\mu f(x)$ pointwise as $\varepsilon \to 0$ for any $f \in C^0$ (this class of functions is dense in L_p, $(1 \le p < \infty)$), we obtain for any $f \in L_p$ that $V_\varepsilon f(x) \to \tilde{c}_\mu f(x)$ a.e. as $\varepsilon \to 0$. To complete the proof of the theorem it remains only to recall that

$$\tilde{c}_\mu = c_\mu \equiv \int\limits_0^\infty \log \frac{1}{s} d\mu(s) \quad \text{(see Lemma 2.5).}$$

□

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