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On universal central extensions of Hom-Lie algebras

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Abstract

We develop a theory of universal central extensions of Hom-Lie algebras. Classical results of universal central extensions of Lie algebras cannot be completely extended to Hom-Lie algebras setting, because of the composition of two central extensions is not central. This fact leads to introduce the notion of universal α-central extension. Classical results as the existence of a universal central extension of a perfect Hom-Lie algebra remains true, but others as the central extensions of the middle term of a universal central extension is split only holds for α-central extensions. A homological characterization of universal (α)-central extensions is given.

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1. Introduction

The Hom-Lie algebra structure was initially introduced in [4] motivated by examples of deformed Lie algebras coming from twisted discretizations of vector fields. Hom-Lie algebras are $\mathbb{K}$-vector spaces endowed with a bilinear skew-symmetric bracket satisfying a Jacobi identity twisted by a map. When this map is the identity map, then the definition of Lie algebra is recovered.

The study of this algebraic structure was the subject of several papers [4, 7, 8, 9, 11]. In particular, a (co)homology theory for Hom-Lie algebras, which generalizes the Chevalley-Eilenberg (co)homology for a Lie algebra, was the subject of [1, 2, 3, 10, 12].

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In the classical setting, homology theory is closely related with universal central extensions. Namely, the second homology with trivial coefficients group is the kernel of the universal central extension and universal central extensions are characterized by means of the first and second homologies with trivial coefficients.

Our goal in the present paper is to investigate if the homology for Hom-Lie algebras introduced in [10, 12] allows the characterization of universal central extensions of Hom-Lie algebras in terms of Hom-Lie homologies. A similar study for Hom-Leibniz algebras homology can be seen in [3]. But when we try to generalize the classical results of universal central extensions theory of Lie algebras to Hom-Lie algebras an important problem occurs, namely the composition of central extensions is not central in general, as Example 4.9 shows. This fact doesn't allow a complete generalization of classical results, however requires the introduction of a new concept of centrality for Hom-Lie algebra extensions.

To show our results, we organize the paper as follows: in Section 2 we recall some basic needed material on Hom-Lie algebras, the notions of center, commutator and module. In order to have examples, we include the classification of two-dimensional complex Hom-Lie algebras. In section 3 we recall the chain complex given in [12] and we prove its well-definition by means of the Generalized Cartan's formulas; the interpretation of low-dimensional homologies is given. In section 4 we present our main results on universal central extensions, namely we extend classical results and present a counter-example showing that the composition of two central extension is not a central extension (see Example 4.9). This fact lead us to define α-central extensions as extensions for which the image by the twisting endomorphism α of the kernel is included in the center of the middle Hom-Lie algebra. We can extend classical results as: a Hom-Lie algebra is perfect if and only if admits a universal central extension and the kernel of the universal central extension is the second homology with trivial coefficients of the Hom-Lie algebra. Nevertheless, other result as: if a central extension $0 \to (M, \alpha_M) \stackrel{i}{\to} (K, \alpha_K) \stackrel{\pi}{\to} (L, \alpha_L) \to 0$ is universal, then (K, α_K) is perfect and every central extension of (K, α_K) is split only holds for universal α-central extensions, which means that only lifts on α-central extensions. Other relevant result, which cannot be extended in the usual way, is: if $0 \to (M, \alpha_M) \stackrel{i}{\to} (K, \alpha_K) \stackrel{\pi}{\to} (L, \alpha_L) \to 0$ is a universal α-central extension, then $H_1^{\alpha}(K) = H_2^{\alpha}(K) = 0$. Of course, when the twisting endomorphism is the identity morphism, then all the new notions and all the new results coincide with the classical ones.

2. Hom-Lie algebras

2.1. Definition. [4] A Hom-Lie algebra is a triple $(L, [-,-], \alpha_L)$ consisting of a $\mathbb{K}$-vector space L, a bilinear map $[-,-] : L \times L \to L$ and a $\mathbb{K}$-linear map $\alpha_L : L \to L$ satisfying:

a) $[x, y] = -[y, x]$ (skew-symmetry)
b) $[\alpha_L(x), [y, z]] + [\alpha_L(z), [x, y]] + [\alpha_L(y), [z, x]] = 0$ (Hom-Jacobi identity)

for all $x, y, z \in L$.

In terms of the adjoint representation $ad_x : L \to L, ad_x(y) = [x, y]$, the Hom-Jacobi identity can be written as follows [7]:

$$ad_{\alpha_L(z)} \circ ad_y = ad_{\alpha_L(y)} \circ ad_z + ad_{[z,y]} \circ \alpha_L$$

2.2. Definition. [10] A Hom-Lie algebra $(L, [-,-], \alpha_L)$ is said to be multiplicative if the linear map α_L preserves the bracket.

2.3. Example.

a) Taking $\alpha_L = Id$ in Definition 2.1 we obtain the definition of a Lie algebra. Hence Hom-Lie algebras include Lie algebras as a subcategory, thereby motivating the name "Hom-Lie algebras" as a deformation of Lie algebras twisted by an endomorphism. Moreover it is a multiplicative Hom-Lie algebra.
b) Let (A, μ_A, α_A) be a multiplicative Hom-associative algebra [7]. Then $HLie(A) = (A, [-,-], \alpha_A)$ is a multiplicative Hom-Lie algebra in which $[x,y] = \mu_A(x,y) - \mu_A(y,x)$, for all $x, y \in A$ [7, 10].
c) Let $(L, [-,-])$ be a Lie algebra and $\alpha : L \to L$ be a Lie algebra endomorphism. Define $[-,-]_\alpha : L \otimes L \to L$ by $[x,y]_\alpha = \alpha[x,y]$, for all $x, y \in L$. Then $(L, [-,-]_\alpha, \alpha)$ is a multiplicative Hom-Lie algebra [10, Th. 5.3].
d) Abelian or commutative Hom-Lie algebras are $\mathbb{K}$-vector spaces V with trivial bracket and any linear map $\alpha : V \to V$ [4].
e) The Jackson Hom-Lie algebra $\mathfrak{sl}_2(\mathbb{K})$ is a Hom-Lie deformation of the classical Lie algebra $\mathfrak{sl}_2(\mathbb{K})$ defined by $[h,f] = -2f, [h,e] = 2e, [e,f] = h$. The Jackson $\mathfrak{sl}_2(\mathbb{K})$ is related to derivations. As a $\mathbb{K}$-vector space is generated by e, f, h with multiplication given by $[h,j]_t = -2f - 2tf, [h,e]_t = 2e, [e,f]_t = h + \frac{t}{2}h$ and the linear map α_t is defined by $\alpha_t(e) = \frac{2+t}{2(1+t)}e = e + \sum\limits_{k=0}^{\infty} \frac{(-1)^k}{2} t^k e, \alpha_t(h) = h, \alpha_t(f) = f + \frac{t}{2} f$ [8].
f) For examples coming from deformations we refer to [10].

2.4. Definition. A homomorphism of Hom-Lie algebras $f : (L, [-,-], \alpha_L) \to (L', [-,-]', \alpha_{L'})$ is a $\mathbb{K}$-linear map $f : L \to L'$ such that

a) $f([x,y]) = [f(x), f(y)]'$
b) $f \circ \alpha_L(x) = \alpha_{L'} \circ f(x)$

for all $x, y \in L$.

The Hom-Lie algebras $(L, [-,-], \alpha_L)$ and $(L', [-,-]', \alpha_{L'})$ are isomorphic if there is a Hom-Lie algebras homomorphism $f : (L, [-,-], \alpha_L) \to (L', [-,-]', \alpha_{L'})$ such that $f : L \to L'$ is bijective.

A homomorphism of multiplicative Hom-Lie algebras is a homomorphism of the underlying Hom-Lie algebras.

So we have defined the category $\mathsf{Hom-Lie}$ (respectively, $\mathsf{Hom-Lie}_{\mathrm{mult}}$) whose objects are Hom-Lie (respectively, multiplicative Hom-Lie) algebras and whose morphisms are the homomorphisms of Hom-Lie (respectively, multiplicative Hom-Lie) algebras. There is an obvious inclusion functor $inc : \mathsf{Hom-Lie}_{\mathrm{mult}} \to \mathsf{Hom-Lie}$. This functor has as left adjoint the multiplicative functor $(-)_{\mathrm{mult}} : \mathsf{Hom-Lie} \to \mathsf{Hom-Lie}_{\mathrm{mult}}$ which assigns to a Hom-Lie algebra $(L, [-,-], \alpha_L)$ the Hom-Lie multiplicative algebra $(L/I, [-,-], \overline{\alpha})$, where I is the ideal of L spanned by the elements $\alpha_L[x,y] - [\alpha_L(x), \alpha_L(y)]$, for all $x, y \in L$ and $\overline{\alpha}$ is induced by α.

In the sequel we refer Hom-Lie algebra to a multiplicative Hom-Lie algebra and we will use the short notation (L, α_L) when there is not confusion with the bracket.

Let $(L, [-,-], \alpha_L)$ be an n-dimensional Hom-Lie algebra with basis $\{a_1, a_2, \ldots, a_n\}$ and endomorphism α_L represented by the matrix $A = (\alpha_{ij})$ with respect to the given basis. To determine its algebraic structure is enough to know its structure constants, i.e. the scalars c_{ij}^k such that $[a_i, a_j] = \sum\limits_{k=1}^{n} c_{ij}^k a_k$, and the entries α_{ij} corresponding to the matrix A. These terms are related according to the following

2.5. Proposition. (see also [7]) Let $(L, [-,-], \alpha_L)$ be a Hom-Lie algebra with basis $\{a_1, a_2, \ldots, a_n\}$. Let $c_{ij}^k, 1 \le i,j,k \le n$ be the structure constants relative to this basis and $\alpha_{ij}, 1 \le i,j \le n$ the entries of the matrix A associated to the endomorphism α_L with respect to the given basis. Then $(L, [-,-], \alpha_L)$ is a Hom-Lie algebra if and only if the structure constants and the entries α_{ij} satisfy the following properties:

a) $c_{ij}^k + c_{ji}^k = 0, 1 \le i,j,k \le n; \quad c_{ii}^k = 0, 1 \le i,k \le n$, char$(\mathbb{K}) \ne 2$.

b) $\sum\limits_{p=1}^{n} \alpha_{pi} \left(\sum\limits_{q=1}^{n} c_{jk}^q c_{pq}^l \right) + \sum\limits_{p=1}^{n} \alpha_{pk} \left(\sum\limits_{q=1}^{n} c_{ij}^q c_{pq}^l \right) + \sum\limits_{p=1}^{n} \alpha_{pj} \left(\sum\limits_{q=1}^{n} c_{ki}^q c_{pq}^l \right) = 0,$
$1 \le i,j,k,l, \le n.$

Proof.

a) There is not difference with Lie-algebras case [6].

b) Applying Hom-Jacobi identity 2.1 b):

$$[\alpha(a_i), [a_j, a_k]] + [\alpha(a_k), [a_i, a_j]] + [\alpha(a_j), [a_k, a_i]] = 0$$

$$[\sum_{l=1}^{n} \alpha_{li} a_l, \sum_{m=1}^{n} c_{jk}^m a_m] + [\sum_{p=1}^{n} \alpha_{pk} a_p, \sum_{q=1}^{n} c_{ij}^q a_q] + [\sum_{r=1}^{n} \alpha_{rj} a_r, \sum_{s=1}^{n} c_{ki}^s a_s] = 0$$

$$\sum_{p=1}^{n} \alpha_{pi} \left(\sum_{q=1}^{n} c_{jk}^q [a_p, a_q] \right) + \sum_{p=1}^{n} \alpha_{pk} \left(\sum_{q=1}^{n} c_{ij}^q [a_p, a_q] \right) + \sum_{p=1}^{n} \alpha_{pj} \left(\sum_{q=1}^{n} c_{ki}^q [a_p, a_q] \right) = 0$$

$$\sum_{l=1}^{n} \left\{ \sum_{p=1}^{n} \alpha_{pi} \left(\sum_{q=1}^{n} c_{jk}^q c_{pq}^l \right) + \sum_{p=1}^{n} \alpha_{pk} \left(\sum_{q=1}^{n} c_{ij}^q c_{pq}^l \right) + \sum_{p=1}^{n} \alpha_{pj} \left(\sum_{q=1}^{n} c_{ki}^q c_{pq}^l \right) \right\} a_l = 0$$

□

2.6. Lemma. The Hom-Lie algebras $(L, [-,-], \alpha_L)$ and $(L, [-,-]', \alpha_{L'})$ with same underlying $\mathbb{K}$-vector space are isomorphic if and only if there exists a regular matrix P such that $A' = P^{-1}.A.P$ and $P.[a_i, a_j] = [P.a_i, P.a_j]'$, where A, A' and P denote the corresponding matrices representing $\alpha_L, \alpha_{L'}$ and f with respect to the basis $\{a_1, \ldots, a_n\}$, respectively.

Proof. The fact comes directly from Definition 2.4. □

2.7. Proposition. The 2-dimensional complex multiplicative Hom-Lie algebras with basis $\{a_1, a_2\}$ are isomorphic to one in the following isomorphism classes:

a) Abelian.

b) $[a_1, a_2] = -[a_2, a_1] = a_1$ and α_l is represented by the matrix $\begin{pmatrix} 0 & \alpha_{12} \\ 0 & \alpha_{22} \end{pmatrix}$.

c) $[a_1, a_2] = -[a_2, a_1] = a_1$ and α_L is represented by the matrix $\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ 0 & 1 \end{pmatrix}$, with $\alpha_{11} \ne 0$.

Proof. From the skew-symmetry condition we have that $[a_1, a_1] = [a_2, a_2] = 0$ and $[a_1, a_2] = -[a_2, a_1] = x.a_1 + y.a_2$. The Hom-Jacobi identity 2.1 b) is satisfied independently of the homomorphism α_L. So we only have restrictions coming from the fact that the $\mathbb{C}$-linear map $\alpha_L : L \to L$ represented by the matrix $\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}$ must preserve the bracket.

First at all, we apply the change of basis given by the equations $\left\{ \begin{array}{l} a_1' = x.a_1 + y.a_2 \\ a_2' = \frac{1}{x}.a_2 \end{array} \right.$, if $x \ne 0$, and $\left\{ \begin{array}{l} a_1' = a_2 \\ a_2' = -\frac{1}{y}.a_1 \end{array} \right.$, if $x = 0$ and $y \ne 0$, to normalize the bracket, obtaining the bracket $[a_1', a_1'] = [a_2', a_2'] = 0, [a_1', a_2'] = -[a_2', a_1'] = p.a_1'$, for $p = 0, 1$.

From the fact that $\alpha_L : L \to L$ preserves the bracket, we derive the following equations:

$$\left.\begin{array}{rcl} (\alpha_{11}\alpha_{22} - \alpha_{21}\alpha_{12}).p & = & p.\alpha_{11} \\ p.\alpha_{21} & = & 0 \end{array}\right\}$$

which reduces to the following system:

$$\left.\begin{array}{rcl} p.\alpha_{11}.(\alpha_{22} - 1) & = & 0 \\ p.\alpha_{21} & = & 0 \end{array}\right\}$$

Hence, for $p = 0$ the system is trivially satisfied. All the matrices representing α_L are valid and the bracket is trivial, so $(L, [-,-], \alpha_L)$ is an abelian Hom-Lie algebra. In case $p = 1$, we derive the matrices corresponding to the cases b) and c).

The different classes obtained are not pairwise isomorphic thanks to Lemma 2.6. □

2.8. Remark.

a) Two algebras of the class b) in Proposition 2.7, with endomorphisms given by the matrices $\begin{pmatrix} 0 & \alpha_{12} \\ 0 & \alpha_{22} \end{pmatrix}$ and $\begin{pmatrix} 0 & \beta_{12} \\ 0 & \beta_{22} \end{pmatrix}$, are isomorphic if and only if $\alpha_{22} = \beta_{22}$ and $\beta_{12} = p.\alpha_{12} + q.\alpha_{22}, p, q \in \mathbb{C}, p \neq 0$.

b) Two algebras of the class c) in Proposition 2.7, with endomorphisms given by the matrices $\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} \beta_{11} & \beta_{12} \\ 0 & 1 \end{pmatrix}$, are isomorphic if and only if $\alpha_{11} = \beta_{11}$ and $\beta_{12} = p.\alpha_{12} - q.\alpha_{11} + q, p, q \in \mathbb{C}, p \neq 0$.

c) Obviously if $\Phi : (L, [-,-], \alpha_L) \to (L, [-,-]', \alpha_{L'})$ is an isomorphism of Hom-Lie algebras, then $det(\alpha_L) = det(\alpha'_L)$. Consequently, if $det(\alpha_L) \neq det(\alpha'_L)$, then the Hom-Lie algebras are not isomorphic.

d) The following table shows by means of its algebraic properties that the classes given in Proposition 2.7 are not pairwise isomorphic.

	Abelian	$det(\alpha)$
2.7 a)	Yes	
2.7 b)	Non	0
2.7 c)	Non	$\neq 0$

Complex two-dimensional Hom-Lie algebras

2.9. Definition. Let $(L, [-,-], \alpha_L)$ be a Hom-Lie algebra. A Hom-Lie subalgebra (H, α_H) of $(L, [-,-], \alpha_L)$ is a linear subspace H of L, which is closed for the bracket and invariant by α_L, that is,

a) $[x, y] \in H$, for all $x, y \in H$.
b) $\alpha_L(x) \in H$, for all $x \in H$ $(\alpha_H = \alpha_{L|})$.

A Hom-Lie subalgebra (H, α_H) of $(L, [-,-], \alpha_L)$ is said to be a Hom-ideal if $[x, y] \in H$ for all $x \in H, y \in L$.

If (H, α_H) is a Hom-ideal of $(L, [-,-], \alpha_L)$, then $(L/H, [-,-], \overline{\alpha_L})$ naturally inherits a structure of Hom-Lie algebra, which is said to be the quotient Hom-Lie algebra.

2.10. Definition. Let (H, α_H) and (K, α_K) be Hom-ideals of a Hom-Lie algebra $(L, [-,-], \alpha_L)$. The commutator Hom-Lie subalgebra of (H, α_H) and (K, α_K), denoted by $([H, K], \alpha_{[H,K]})$, is the Hom-subalgebra of $(L, [-,-], \alpha_L)$ spanned by the brackets $[h, k], h \in H, k \in K$.

2.11. Lemma. Let (H, α_H) and (K, α_K) be Hom-ideals of a Hom-Lie algebra $(L, [-,-], \alpha_L)$. The following statements hold:

a) $(H \cap K, \alpha_{H \cap K})$ and $(H + K, \alpha_{H+K})$ are Hom-ideals of (L, α_L).
b) $[H, K] \subseteq H \cap K$.
c) $([H, K], \alpha_{[H,K]})$ is a Hom-ideal of (L, α_L) when α_L is surjective.
d) $([H, K], \alpha_{[H,K]})$ is a Hom-ideal of (H, α_H) and (K, α_K), respectively.

f) If $H = K = L$, then $([L,L], \alpha_{[L,L]})$ is a Hom-ideal of (L, α_L).

2.12. Lemma. Let (H, α_H) and (K, α_K) be Hom-ideals of a Hom-Lie algebra $(L, [-,-], \alpha_L)$ such that $H, K \subseteq \alpha_L(L)$, then $([H,K], \alpha_{[H,K]})$ is a Hom-ideal of $(\alpha_L(L), [-,-], \alpha_{L_|})$.

2.13. Definition. The center of a Hom-Lie algebra $(L, [-,-], \alpha_L)$ is the $\mathbb{K}$-vector subspace

$$Z(L) = \{x \in L \mid [x, y] = 0, \text{for all } y \in L\}$$

2.14. Remark. When $\alpha_L : L \to L$ is a surjective endomorphism, then $(Z(L), \alpha_{L|})$ is a Hom-ideal of $(L, [-,-], \alpha_L)$.

2.15. Definition. Let $(L, [-,-], \alpha_L)$ and $(M, [-,-], \alpha_M)$ be Hom-Lie algebras. A Hom-L-action from $(L, [-,-], \alpha_L)$ over $(M, [-,-], \alpha_M)$ consists in a bilinear map $\rho : L \otimes M \to M$, given by $\rho(x \otimes m) = x \centerdot m$, satisfying the following properties:

a) $[x, y] \centerdot \alpha_M(m) = \alpha_L(x) \centerdot (y \centerdot m) - \alpha_L(y) \centerdot (x \centerdot m)$
b) $\alpha_L(x) \centerdot [m, m'] = [x \centerdot m, \alpha_M(m')] + [\alpha_M(m), x \centerdot m']$
c) $\alpha_M(x \centerdot m) = \alpha_L(x) \centerdot \alpha_M(m)$

for all $x, y \in L$ and $m, m' \in M$.

Under these circumstances, we say that (L, α_L) Hom-acts over (M, α_M).

2.16. Remark. When (M, α_M) is an abelian Hom-Lie algebra, Definition 2.15 goes back to the definition of Hom-L-module in [10].

2.17. Example.

a) $(L, [-,-], \alpha_L)$ acts on itself by the action given by the bracket.
b) Let $\mathfrak{g}$ and $\mathfrak{m}$ be Lie algebras with a Lie action from $\mathfrak{g}$ over $\mathfrak{m}$. Then $(\mathfrak{g}, Id_{\mathfrak{g}})$ Hom-acts over $(\mathfrak{m}, Id_{\mathfrak{m}})$.
c) Let $\mathfrak{g}$ be a Lie algebra, $\alpha : \mathfrak{g} \to \mathfrak{g}$ an endomorphism and M a $\mathfrak{g}$-module in the usual sense, such that the action from $\mathfrak{g}$ over M satisfies the condition $\alpha(x) \centerdot m = x \centerdot m$, for all $x \in \mathfrak{g}$ and $m \in M$. Then (M, Id) is a Hom-$\mathfrak{g}$-module.

An example of this situation is given by the 2-dimensional Lie algebra L generated by $\{e, f\}$ with bracket $[e, f] = -[f, e] = e$ and endomorphism α represented by the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, where M is the ideal spanned by $\{e\}$.
d) An abelian sequence of Hom-Lie algebras is an exact sequence of Hom-Lie algebras $0 \to (M, \alpha_M) \xrightarrow{i} (K, \alpha_K) \xrightarrow{\pi} (L, \alpha_L) \to 0$, where (M, α_M) is an abelian Hom-Lie algebra, $\alpha_K \circ i = i \circ \alpha_M$ and $\pi \circ \alpha_K = \alpha_L \circ \pi$.

The abelian sequence induces a Hom-L-module structure on (M, α_M) by means of the action given by $\rho : L \otimes M \to M, \rho(l, m) = [k, m], \pi(k) = l$.
e) For other examples we refer to Example 6.2 in [10].

3. Homology

Following [10, 12], for a Hom-Lie algebra (L, α_L) and a (left) Hom-L-module (M, α_M), one denotes by

$$C_n^{\alpha}(L, M) := M \otimes \Lambda^n L, \ n \geqslant 0$$

the n-chain module of (L, α_L) with coefficients in (M, α_M).

For $n \geqslant 1$, one defines the $\mathbb{K}$-linear map,

$$d_n : C_n^{\alpha}(L, M) \longrightarrow C_{n-1}^{\alpha}(L, M)$$

by

$$d_n(m \otimes x_1 \wedge \dots \wedge x_n) = \sum_{i=1}^{n} (-1)^{i+1} x_i \centerdot m \otimes \alpha_L(x_1) \wedge \dots \wedge \widehat{\alpha_L(x_i)} \wedge \dots \wedge \alpha_L(x_n) +$$

$$\sum_{1\leqslant i<j\leqslant n} (-1)^{i+j}\, \alpha_M(m) \otimes [x_i, x_j] \wedge \alpha_L(x_1) \wedge \cdots \wedge \widehat{\alpha_L(x_i)} \wedge \cdots \wedge \widehat{\alpha_L(x_j)} \wedge \cdots \wedge \alpha_L(x_n)$$

Although in [10, 12] is proved that $(CL_n^\alpha(L,M), d_n)$ is a well-defined chain complex, we present an alternative proof by means of a generalization of Cartan's formulas. Firstly, we define for all $y \in L$ and $n \in \mathbb{N}$, two linear maps,

$$\theta_n(y) : C_n^\alpha(L,M) \longrightarrow C_n^\alpha(L,M)$$

by

$$\theta_n(y)(m \otimes x_1 \wedge \cdots \wedge x_n) = y \centerdot m \otimes \alpha_L(x_1) \wedge \cdots \wedge \alpha_L(x_n) +$$

$$\sum_{i=1}^{n} (-1)^i\, \alpha_M(m) \otimes [x_i, y] \wedge \alpha_L(x_1) \wedge \cdots \wedge \widehat{\alpha_L(x_i)} \wedge \cdots \wedge \alpha_L(x_n)$$

and

$$i_n(\alpha_L(y)) : C_n^\alpha(L,M) \longrightarrow C_{n+1}^\alpha(L,M)$$

by

$$i_n(\alpha_L(y))(m \otimes x_1 \wedge \cdots \wedge x_n) = (-1)^n\, m \otimes x_1 \wedge \cdots \wedge x_n \wedge y$$

3.1. Proposition. (Generalized Cartan's formulas)
The following identities hold:

a) $d_{n+1} \circ i_n(\alpha_L(y)) + i_{n-1}\left(\alpha_L^2(y)\right) \circ d_n = -\theta_n(y)$, for all $n \geqslant 1$.
b) $\theta_n(\alpha_L(x)) \circ \theta_n(y) - \theta_n(\alpha_L(y)) \circ \theta_n(x) = \theta_n([x,y]) \circ (\alpha_M \otimes \alpha_L^{\wedge n})$, for all $n \geq 0$.
c) $\theta_n(x) \circ i_{n-1}(\alpha_L(y)) - i_{n-1}\left(\alpha_L^2(y)\right) \circ \theta_{n-1}(x) = i_{n-1}(\alpha_L[x,y]) \circ \left(\alpha_M \otimes \alpha_L^{\wedge(n-1)}\right)$, for all $n \geq 1$.
d) $\theta_{n-1}(\alpha_L(y)) \circ d_n = d_n \circ \theta_n(y)$, for all $n \geq 1$.
e) $d_n \circ d_{n+1} = 0$, for all $n \geq 1$.

Proof. The proof follows with a routine induction, so we omit it. □

In case $\alpha_L = Id_L, \alpha_M = Id_M$, the above formulas become Cartan's formulas for the Chevalley-Eilenberg homology [5].

Thanks to Proposition 3.1, $(C_\star^\alpha(L,M), d_\star)$ is a well-defined chain complex (an alternative proof can be seen in [12]). Its homology is said to be the homology of the Hom-Lie algebra (L, α_L) with coefficients in the Hom-L-module (M, α_M) and it is denoted by:

$$H_\star^\alpha(L,M) := H_\star(C_\star^\alpha(L,M), d_\star)$$

An easy computation in low-dimensional cycles and boundaries provides the following results:

$$H_0^\alpha(L,M) = \frac{Ker(d_0)}{\operatorname{Im}(d_1)} = \frac{M}{{}^L M}$$

where ${}^L M = \{l \centerdot m : m \in M, l \in L\}$.

Now let us consider M as a trivial Hom-L-module, i.e. $l \centerdot m = 0$, then

$$H_1^\alpha(L,M) = \frac{Ker(d_1)}{\operatorname{Im}(d_2)} = \frac{M \otimes L}{\alpha_M(M) \otimes [L,L]}$$

In particular, if $M = \mathbb{K}$, then $H_1^\alpha(L,\mathbb{K}) = \frac{L}{[L,L]}$.

4. Universal central extensions

Through this section we will deal with universal central extensions of Hom-Lie algebras. We will generalize classical results of universal central extensions theory of Lie algebras, but here an important problem appears, namely the composition of central extensions is not central in general, as Example 4.9 shows. This fact doesn't allow a complete generalization of classical results, however requires the introduction of a new concept of centrality for Hom-Lie algebra extensions.

4.1. Definition. A short exact sequence of Hom-Lie algebras $(K) : 0 \to (M, \alpha_M) \stackrel{i}{\to} (K, \alpha_K) \stackrel{\pi}{\to} (L, \alpha_L) \to 0$ is said to be central if $[M, K] = 0$. Equivalently, $M \subseteq Z(K)$.

The sequence (K) is said to be α-central if $[\alpha_M(M), K] = 0$. Equivalently, $\alpha_M(M) \subseteq Z(K)$.

4.2. Remark. Let us observe that both notions coincide when $\alpha_M = Id_M$. Obviously, every central extension is an α-central extension, but the converse doesn't hold as the following counter-example shows:

Consider the two-dimensional Hom-Lie algebra L with basis $\{a_1, a_2\}$, bracket given by

$$[a_1, a_2] = -[a_2, a_1] = a_1,$$

and endomorphism $\alpha_L = 0$.

Let K be the three-dimensional Hom-Lie algebra with basis $\{b_1, b_2, b_3\}$, bracket given by

$$[b_1, b_2] = -[b_2, b_1] = b_1, [b_1, b_3] = -[b_3, b_1] = b_1, [b_2, b_3] = -[b_3, b_2] = b_2,$$

and endomorphism $\alpha_K = 0$.

The surjective homomorphism $\pi : (K, 0) \to (L, 0)$ given by

$$\pi(b_1) = 0, \pi(b_2) = a_1, \pi(b_3) = a_2,$$

is an α-central extension, since Ker $(\pi) = \langle \{b_1\} \rangle$ and $[\alpha_K(Ker(\pi)), K] = 0$, but is not a central extension, since $[Ker(\pi), K] = \langle \{b_1\} \rangle$.

4.3. Definition. A central extension $(K) : 0 \to (M, \alpha_M) \stackrel{i}{\to} (K, \alpha_K) \stackrel{\pi}{\to} (L, \alpha_L) \to 0$ is said to be universal if for every central extension $(K') : 0 \to (M', \alpha_{M'}) \stackrel{i'}{\to} (K', \alpha_{K'}) \stackrel{\pi'}{\to} (L, \alpha_L) \to 0$ there exists a unique homomorphism of Hom-Lie algebras $h : (K, \alpha_K) \to (K', \alpha_{K'})$ such that $\pi' \circ h = \pi$.

A central extension $(K) : 0 \to (M, \alpha_M) \stackrel{i}{\to} (K, \alpha_K) \stackrel{\pi}{\to} (L, \alpha_L) \to 0$ is said to be universal α-central if for every α-central extension $(K') : 0 \to (M', \alpha_{M'}) \stackrel{i'}{\to} (K', \alpha_{K'}) \stackrel{\pi'}{\to} (L, \alpha_L) \to 0$ there exists a unique homomorphism of Hom-Lie algebras $h : (K, \alpha_K) \to (K', \alpha_{K'})$ such that $\pi' \circ h = \pi$.

4.4. Remark. Obviously, every universal α-central extension is a universal central extension. Let us observe that both notions coincide when $\alpha_M = Id_M$.

4.5. Definition. A Hom-Lie algebra (L, α_L) is said to be perfect if $L = [L, L]$.

4.6. Lemma. Let $\pi : (K, \alpha_K) \to (L, \alpha_L)$ be a surjective homomorphism of Hom-Lie algebras. If (K, α_K) is a perfect Hom-Lie algebra, then (L, α_L) is a perfect Hom-Lie algebra as well.

4.7. Lemma. Let $0 \to (M, \alpha_M) \stackrel{i}{\to} (K, \alpha_K) \stackrel{\pi}{\to} (L, \alpha_L) \to 0$ be a central extension and (K, α_K) a perfect Hom-Lie algebra. If there exists a homomorphism of Hom-Lie algebras $f : (K, \alpha_K) \to (A, \alpha_A)$ such that $\tau \circ f = \pi$, where $0 \to (N, \alpha_N) \stackrel{j}{\to} (A, \alpha_A) \stackrel{\tau}{\to} (L, \alpha_L) \to 0$ is a central extension, then f is unique.

The proofs of these two last Lemmas use classical arguments, so we omit them.

4.8. Lemma. If $0 \to (M,\alpha_M) \stackrel{i}{\to} (K,\alpha_K) \stackrel{\pi}{\to} (L,\alpha_L) \to 0$ is a universal central extension, then (K,α_K) and (L,α_L) are perfect Hom-Lie algebras.

Proof. Let us assume that (K,α_K) is not a perfect Hom-Lie algebra, then $[K,K] \subsetneq K$. Hence $(K/[K,K],\widetilde{\alpha})$, where $\widetilde{\alpha}$ is the induced homomorphism, is an abelian Hom-Lie algebra, consequently, it is a trivial Hom-L-module. Let us consider the central extension $0 \to (K/[K,K],\widetilde{\alpha}) \to (K/[K,K] \times L, \widetilde{\alpha} \times \alpha_L) \stackrel{pr}{\to} (L,\alpha_L) \to 0$. Then the homomorphisms of Hom-Lie algebras $\varphi,\psi : (K,\alpha_K) \to (K/[K,K] \times L, \widetilde{\alpha} \times \alpha_L)$ given by $\varphi(k) = (\overline{k},\pi(k))$ and $\psi(k) = (0,\pi(k)), k \in K$, verify that $pr \circ \phi = \pi = pr \circ \psi$, so $0 \to (M,\alpha_M) \stackrel{i}{\to} (K,\alpha_K) \stackrel{\pi}{\to} (L,\alpha_L) \to 0$ cannot be a universal central extension.
Lemma 4.6 ends the proof. □

Classical categories as groups, Lie algebras, Leibniz algebras and other similar ones share the following property: the composition of two central extensions is a central extension, which is absolutely necessary in order to obtain characterizations of the universal central extensions. Unfortunately this property doesn't remain for the category of Hom-Lie algebras as the following counter-example 4.9 shows. This problem lead us to introduce the notion of α-central extensions in Definition 4.1, whose properties relative to the composition are given in Lemma 4.10.

4.9. Example. Consider the four-dimensional Hom-Lie algebra (L,α_L) with basis $\{a_1,a_2,a_3,a_4\}$, bracket operation given by

$$\left\{ \begin{array}{ll} [a_1,a_3] = -[a_3,a_1] = a_4, & [a_1,a_4] = -[a_4,a_1] = a_3, \\ [a_2,a_3] = -[a_3,a_2] = a_1, & [a_2,a_4] = -[a_4,a_2] = a_2, \end{array} \right.$$

(the non-written brackets are equal to zero) and endomorphism $\alpha_L = 0$.

Let (K,α_K) be the five-dimensional Hom-Lie algebra with basis $\{b_1,b_2,b_3,b_4,b_5\}$, bracket operation given by

$$\left\{ \begin{array}{ll} [b_2,b_3] = -[b_3,b_2] = b_1, & [b_2,b_4] = -[b_4,b_2] = b_5, \\ [b_2,b_5] = -[b_5,b_2] = b_4, & [b_3,b_4] = -[b_4,b_3] = b_2, \\ [b_3,b_5] = -[b_5,b_3] = b_3, & \end{array} \right.$$

(the non-written brackets are equal to zero) and endomorphism $\alpha_K = 0$.

Obviously (K,α_K) is a perfect Hom-Lie algebra since $K = [K,K]$. On the other hand, $Z(K,\alpha_K) =< \{b_1\} >$.

The linear map $\pi : (K,\alpha_K) \to (L,\alpha_L)$ given by

$$\pi(b_1) = 0, \pi(b_2) = a_1, \pi(b_3) = a_2, \pi(b_4) = a_3, \pi(b_5) = a_4,$$

is a central extension since π is a surjective homomorphism of Hom-Lie algebras and $\mathrm{Ker}(\pi) =< \{b_1\} > \subseteq Z(K,\alpha_K)$.

Now let us consider the six-dimensional Hom-Lie algebra (F,α_F) with basis $\{e_1,e_2,e_3,e_4,e_5,e_6\}$, bracket operation given by

$$\left\{ \begin{array}{ll} [e_2,e_3] = -[e_3,e_2] = e_1, & [e_2,e_4] = -[e_4,e_2] = e_1, \\ [e_2,e_5] = -[e_5,e_2] = e_1, & [e_3,e_4] = -[e_4,e_3] = e_2, \\ [e_3,e_5] = -[e_5,e_3] = e_6, & [e_3,e_6] = -[e_6,e_3] = e_5, \\ [e_4,e_5] = -[e_5,e_4] = e_3, & [e_4,e_6] = -[e_6,e_4] = e_4, \\ [e_5,e_6] = -[e_6,e_5] = e_1, & \end{array} \right.$$

(the non-written brackets are equal to zero) and endomorphism $\alpha_F = 0$.

The linear map $\rho : (F,\alpha_F) \to (K,\alpha_K)$ given by

$$\rho(e_1) = 0, \rho(e_2) = b_1, \rho(e_3) = b_2, \rho(e_4) = b_3, \rho(e_5) = b_4, \rho(e_6) = b_5,$$

is a central extension since ρ is a surjective homomorphism of Hom-Lie algebras and $\text{Ker}(\rho) =< \{e_1\} >= Z(F, \alpha_F)$.

The composition $\pi \circ \rho : (F, \alpha_F) \to (L, \alpha_L)$ is given by

$$\begin{array}{lll} \pi \circ \rho(e_1) = \pi(0) = 0, & \pi \circ \rho(e_2) = \pi(b_1) = 0, & \pi \circ \rho(e_3) = \pi(b_2) = a_1, \\ \pi \circ \rho(e_4) = \pi(b_3) = a_2, & \pi \circ \rho(e_5) = \pi(b_4) = a_3, & \pi \circ \rho(e_6) = \pi(b_5) = a_4, \end{array}$$

Consequently, $\pi \circ \rho : (F, \alpha_F) \to (L, \alpha_L)$ is a surjective homomorphism, but is not a central extension, since $Z(F, \alpha_F) =< \{e_1\} >$ and $\text{Ker}(\pi \circ \rho) = < \{e_1, e_2\} >$, i. e. $\text{Ker}(\pi \circ \rho) \nsubseteq Z(F, \alpha_F)$.

4.10. Lemma. Let $0 \to (M, \alpha_M) \stackrel{i}{\to} (K, \alpha_K) \stackrel{\pi}{\to} (L, \alpha_L) \to 0$ and $0 \to (N, \alpha_N) \stackrel{j}{\to} (F, \alpha_F) \stackrel{\rho}{\to} (K, \alpha_K) \to 0$ be central extensions with (K, α_K) a perfect Hom-Lie algebra. Then the composition extension $0 \to (P, \alpha_P) = \text{Ker}\ (\pi \circ \rho) \to (F, \alpha_F) \stackrel{\pi\circ\rho}{\to} (L, \alpha_L) \to 0$ is an α-central extension.

Moreover, if $0 \to (M, \alpha_M) \stackrel{i}{\to} (K, \alpha_K) \stackrel{\pi}{\to} (L, \alpha_L) \to 0$ is a universal α-central extension, then $0 \to (N, \alpha_N) \stackrel{j}{\to} (F, \alpha_F) \stackrel{\rho}{\to} (K, \alpha_K) \to 0$ is split, that is, there exists a Hom-Lie algebra homomorphism $\sigma : (K, \alpha_K) \to (F, \alpha_F)$ such that $\rho \circ \sigma = Id_K$.

Proof. We must prove that $[\alpha_P(P), F] = 0$.

Since (K, α_K) is a perfect Hom-Lie algebra, then every element $f \in F$ can be written as $f = \sum\limits_i \lambda_i [f_{i_1}, f_{i_2}] + n, n \in N, \lambda_i \in \mathbb{K}, f_{i_j} \in F, j = 1, 2$ since $\rho(f) \in K$, then $\rho(f) = \sum\limits_i \lambda_i [k_{i_1}, k_{i_2}] = \sum\limits_i \lambda_i [\rho(f_{i1}), \rho(f_{i_2})] = \rho\left(\sum\limits_i \lambda_i [f_{i_1}, f_{i_2}]\right)$, hence $f - \sum\limits_i \lambda_i [f_{i_1}, f_{i_2}] \in$ Ker (ρ).

So, for all $p \in P, f \in F$ we have that

$$[\alpha_P(p), f] = \sum_i \lambda_i \left([[p, f_{i_1}], \alpha_F(f_{i_2})] + [[f_{i_2}, p], \alpha_F(f_{i_1})]\right) + [\alpha_P(p), n] = 0$$

since $[p, f_{i_j}] \in$ Ker $(\rho) \subseteq Z(F)$.

For the second statement, if $0 \to (M, \alpha_M) \stackrel{i}{\to} (K, \alpha_K) \stackrel{\pi}{\to} (L, \alpha_L) \to 0$ is a universal α-central extension, then by the first statement, $0 \to (P, \alpha_P) = \text{Ker}\ (\pi \circ \rho) \to (F, \alpha_F) \stackrel{\pi\circ\rho}{\to} (L, \alpha_L) \to 0$ is an α-central extension, then there exists a unique homomorphism of Hom-Lie algebras $\sigma : (K, \alpha_K) \to (F, \alpha_F)$ such that $\pi \circ \rho \circ \sigma = \pi$. On the other hand, $\pi \circ \rho \circ \sigma = \pi = \pi \circ Id$ and (K, α_K) is perfect, then Lemma 4.7 implies that $\rho \circ \sigma = Id$. □

4.11. Theorem.

a) If a central extension $0 \to (M, \alpha_M) \stackrel{i}{\to} (K, \alpha_K) \stackrel{\pi}{\to} (L, \alpha_L) \to 0$ is a universal α-central extension, then (K, α_K) is a perfect Hom-Lie algebra and every central extension of (K, α_K) is split.

b) Let $0 \to (M, \alpha_M) \stackrel{i}{\to} (K, \alpha_K) \stackrel{\pi}{\to} (L, \alpha_L) \to 0$ be a central extension.
If (K, α_K) is a perfect Hom-Lie algebra and every central extension of (K, α_K) is split, then $0 \to (M, \alpha_M) \stackrel{i}{\to} (K, \alpha_K) \stackrel{\pi}{\to} (L, \alpha_L) \to 0$ is a universal central extension.

c) A Hom-Lie algebra (L, α_L) admits a universal central extension if and only if (L, α_L) is perfect.

d) The kernel of the universal central extension is canonically isomorphic to $H_2^\alpha(L)$.

Proof.

a) If $0 \to (M, \alpha_M) \stackrel{i}{\to} (K, \alpha_K) \stackrel{\pi}{\to} (L, \alpha_L) \to 0$ is a universal α-central extension, then it is a universal central extension by Remark 4.4, so (K, α_K) is a perfect Hom-Lie algebra by Lemma 4.8 and every central extension of (K, α_K) is split by Lemma 4.10.

b) Consider a central extension $0 \to (N, \alpha_N) \stackrel{j}{\to} (A, \alpha_A) \stackrel{\tau}{\to} (L, \alpha_L) \to 0$. Construct the pull-back extension $0 \to (N, \alpha_N) \stackrel{\chi}{\to} (P, \alpha_P) \stackrel{\overline{\tau}}{\to} (K, \alpha_K) \to 0$, where $P = \{(a, k) \in A \times K \mid \tau(a) = \pi(k)\}$ and $\alpha_P(a, k) = (\alpha_A(a), \alpha_K(k))$, which is central, consequently is split, i.e. there exists a homomorphism $\sigma : (K, \alpha_K) \to (P, \alpha_P)$ such that $\overline{\tau} \circ \sigma = Id$.

Then $\overline{\pi} \circ \sigma$, where $\overline{\pi} : (P, \alpha_P) \to (A, \alpha_A)$ is induced by the pull-back construction, satisfies $\tau \circ \overline{\pi} \circ \sigma = \pi$. Lemma 4.8 ends the proof.

c) and d) For a Hom-Lie algebra (L, α_L) consider the homology chain complex $C_\star^\alpha(L)$, which is $C_\star^\alpha(L, \mathbb{K})$ where $\mathbb{K}$ is endowed with the trivial Hom-L-module structure.

As $\mathbb{K}$-vector spaces, let I_L be the subspace of $L \wedge L$ spanned by the elements of the form $-[x_1, x_2] \wedge \alpha_L(x_3) + [x_1, x_3] \wedge \alpha_L(x_2) - [x_2, x_3] \wedge \alpha_L(x_1), x_1, x_2, x_3 \in L$. That is, $I_L = \text{Im }(d_3 : C_3^\alpha(L) \to C_2^\alpha(L))$.

Now we denote the quotient $\mathbb{K}$-vector space $\frac{L \wedge L}{I_L}$ by $\mathfrak{uce}(L)$. Every class $x_1 \wedge x_2 + I_L$ is denoted by $\{x_1, x_2\}$, for all $x_1, x_2 \in L$.

By construction, the following identity holds:

$$\{[x_1, x_2], \alpha_L(x_3)\} + \{[x_2, x_3], \alpha_L(x_1)\} + \{[x_3, x_1], \alpha_L(x_2)\} = 0 \tag{4.1}$$

for all $x_1, x_2, x_3 \in L$.

Now $d_2(I_L) = 0$, so it induces a $\mathbb{K}$-linear map $u_L : \mathfrak{uce}(L) \to L$, given by $u_L(\{x_1, x_2\}) = [x_1, x_2]$. Moreover $(\mathfrak{uce}(L), \widetilde{\alpha})$, where $\widetilde{\alpha} : \mathfrak{uce}(L) \to \mathfrak{uce}(L)$ is defined by $\widetilde{\alpha}(\{x_1, x_2\}) = \{\alpha_L(x_1), \alpha_L(x_2)\}$, is a Hom-Lie algebra with respect to the bracket $[\{x_1, x_2\}, \{y_1, y_2\}] = \{[x_1, x_2], [y_1, y_2]\}$ and $u_L : (\mathfrak{uce}(L), \widetilde{\alpha}) \to (L, \alpha_L)$ is a homomorphism of Hom-Lie algebras. Actually, Im $u_L = [L, L]$, but (L, α_L) is a perfect Hom-Lie algebra, so u_L is a surjective homomorphism.

From the construction, it follows that $\text{Ker}(u_L) = H_2^\alpha(L)$, so we have the extension

$$0 \to (H_2^\alpha(L), \widetilde{\alpha}_|) \to (\mathfrak{uce}(L), \widetilde{\alpha}) \stackrel{u_L}{\to} (L, \alpha_L) \to 0$$

which is central, since $[\text{Ker}(u_L), \mathfrak{uce}(L)] = 0$, and universal, since for any central extension $0 \to (M, \alpha_M) \to (K, \alpha_K) \stackrel{\pi}{\to} (L, \alpha_L) \to 0$ there exists the homomorphism of Hom-Lie algebras $\varphi : (\mathfrak{uce}(L), \widetilde{\alpha}) \to (K, \alpha_K)$ given by $\varphi(\{x_1, x_2\}) = [k_1, k_2], \pi(k_i) = x_i, i = 1, 2$, such that $\pi \circ \varphi = u_L$. Moreover, $(\mathfrak{uce}(L), \widetilde{\alpha})$ is a perfect Hom-Lie algebra, so by Lemma 4.7, φ is unique. □

4.12. Corollary.

a) Let $0 \to (M, \alpha_M) \stackrel{i}{\to} (K, \alpha_K) \stackrel{\pi}{\to} (L, \alpha_L) \to 0$ be a universal α-central extension, then $H_1^\alpha(K) = H_2^\alpha(K) = 0$.

b) Let $0 \to (M, \alpha_M) \stackrel{i}{\to} (K, \alpha_K) \stackrel{\pi}{\to} (L, \alpha_L) \to 0$ be a central extension such that $H_1^\alpha(K) = H_2^\alpha(K) = 0$, then $0 \to (M, \alpha_M) \stackrel{i}{\to} (K, \alpha_K) \stackrel{\pi}{\to} (L, \alpha_L) \to 0$ is a universal central extension.

Proof.

a) If $0 \to (M, \alpha_M) \stackrel{i}{\to} (K, \alpha_K) \stackrel{\pi}{\to} (L, \alpha_L) \to 0$ is a universal α-central extension, then (K, α_K) is perfect by Remark 4.4 and Lemma 4.8, so $H_1^\alpha(K) = 0$. By Lemma 4.10 and Theorem 4.11 c) and d) the universal central extension corresponding to (K, α_K) is split, so $H_2^\alpha(K) = 0$.

b) $H_1^\alpha(K) = 0$ implies that (K, α_K) is a perfect Hom-Lie algebra.

$H_2^\alpha(K) = 0$ implies that $(\mathfrak{uce}(K), \widetilde{\alpha}) \stackrel{\sim}{\to} (K, \alpha_K)$. Theorem 4.11 b) ends the proof. □

4.13. Definition. An α-central extension $0 \to (M, \alpha_M) \stackrel{i}{\to} (K, \alpha_K) \stackrel{\pi}{\to} (L, \alpha_L) \to 0$ is said to be universal if for every central extension $0 \to (R, \alpha_R) \stackrel{j}{\to} (A, \alpha_A) \stackrel{\tau}{\to} (L, \alpha_L) \to 0$ there exists a unique homomorphism $\varphi : (K, \alpha_K) \to (A, \alpha_A)$ such that $\tau \circ \varphi = \pi$.

4.14. Proposition. Let $0 \to (M, \alpha_M) \stackrel{i}{\to} (K, \alpha_K) \stackrel{\pi}{\to} (L, \alpha_L) \to 0$ and $0 \to (N, \alpha_N) \stackrel{j}{\to} (F, \alpha_F) \stackrel{\rho}{\to} (K, \alpha_K) \to 0$ be central extensions. If $0 \to (N, \alpha_N) \stackrel{j}{\to} (F, \alpha_F) \stackrel{\rho}{\to} (K, \alpha_K) \to 0$ is a universal α-central extension, then $0 \to (P, \alpha_P) = \mathrm{Ker}(\pi \circ \rho) \stackrel{\chi}{\to} (F, \alpha_F) \stackrel{\pi \circ \rho}{\to} (L, \alpha_L) \to 0$ is an α-central extension which is universal in the sense of Definition 4.13.

Proof. $0 \to (P, \alpha_P) = \mathrm{Ker}(\pi \circ \rho) \stackrel{\chi}{\to} (F, \alpha_F) \stackrel{\pi \circ \rho}{\to} (L, \alpha_L) \to 0$ is an α-central extension by Lemma 4.10.

In order to obtain the universality, for any central extension $0 \to (S, \alpha_S) \to (A, \alpha_A) \stackrel{\sigma}{\to} (L, \alpha_L) \to 0$ construct the pull-back extension corresponding to σ and $\pi \circ \rho$, then Theorem 4.11 and Lemma 4.7 end the proof. □

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