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Generalizations of prime submodules

S. Moradi * and A. Azizi †

Abstract

Let R be a commutative ring with identity and M a unitary R-module, and $n > 1$ an integer number. As a generalization of the concept of prime submodules, a proper submodule N of M will be called n-almost prime, if for $r \in R$ and $x \in M$ with $rx \in N \setminus (N : M)^{n-1}N$, either $x \in N$ or $r \in (N : M)$. We study n-almost prime submodules, in this paper.

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1. Introduction

Throughout this paper all rings are commutative with identity and all modules are unitary. Also we consider R to be a commutative ring with identity, M an R-module, $n > 1$ a positive integer and $\mathbb{N}$ the set of positive integers.

Let N be a submodule of an $R-$module M. The set $\{r \in R | rM \subseteq N\}$ is denoted by $(N : M)$ and particularly we denote $\{r \in R | rN = 0\}$ by $ann(N)$.

Let N a proper submodule of M. It is said that N is a *prime submodule* of M, if for $r \in R$ and $x \in M$ with $rx \in N$, either $x \in N$ or $rM \subseteq N$. In this case, if $P = (N : M)$, then P is a prime ideal. The concept of prime submodules has been studied in many papers in recent years (see, for example, [3, 8]).

*Department of Mathematics, College of Sciences, Shiraz University, Shiraz, 71457-44776, IRAN
Email: smoradi@shirazu.ac.ir
† Email:aazizi@shirazu.ac.ir

2. n-Almost Prime Submodules

According to [1] an ideal I of R is called an *n-almost prime* ideal if for $a, b \in R$ with $ab \in I \setminus I^n$, either $a \in I$ or $b \in I$. The case $n = 2$ is called an *almost prime* ideal and it is due to [5]. We will generalize this definition to modules as follows:

Definition. Let $n > 1$ be an integer number. A proper submodule N of M will be called n-almost prime, if for $r \in R$ and $x \in M$ with $rx \in N \setminus (N : M)^{n-1}N$, either $x \in N$ or $r \in (N : M)$. A 2-almost prime submodule will be called an almost prime submodule.

Evidently every prime submodule is an n-almost prime submodule, for any integer $n > 1$.

The following remark is an evident consequence of the definition of being almost prime submodules.

Remark.

(i) The zero submodule is an almost prime submodule.
(ii) Let N be a proper submodule of M such that $(N : M)^{n-1}N = N$. Then N is n-almost prime.
(iii) Let N be a proper submodule of a torsion-free divisible module M. Then N is prime if and only if N is n-almost prime.
(iv) Every n-almost prime submodule of an R-module M is $m-$almost prime, where $3 \leq n$ and $1 < m \leq n$.

2.1. Lemma. Let M be an R-module, and I an ideal of R.

(i) If $n \in \mathbb{N}$, then $(IM : M)^n M = I^n M$.
(ii) If K is a submodule of M such that $(K : M)$ is a maximal ideal, then K is a prime submodule.
(iii) If $1 < n \in \mathbb{N}$ such that $M \neq IM = I^n M$, then IM is an n-almost prime submodule.
(iv) Let F be a free R-module. Then I is an n-almost prime ideal of R if and only if IF is an n-almost prime submodule of F.
(v) Consider the R-module $F = \oplus_{i \in \mathbb{N}} R$ and let $N = I \oplus (\oplus_{1<i \in \mathbb{N}} R)$. Then the following are equivalent:
  (a) N is a prime submodule of F;
  (b) N is an n-almost prime submodule of F;
  (c) I is a prime ideal of R.

Proof. The proofs of (i),(ii) and (iii) are clear.

(iv) Consider $F = \oplus_{i \in \alpha} R$. It is easy to see that $(IF : F) = I$, for any ideal I of R. Then I is a proper ideal of R if and only if IF is a proper submodule of F. Also $(IF : F)^{n-1} IF = I^n F$.

Suppose I is a proper ideal of R, which is not n-almost prime. Then there exist $a, b \in R \setminus I$ such that $ab \in I \setminus I^n$. So $a(b, 0, 0, 0, \cdots) \in IF \setminus I^n F$, but $a \notin I = (IF : F)$, also $(b, 0, 0, 0, \cdots) \notin IF$, that is IF is not an n-almost prime submodule.

For the converse, suppose I is an n-almost prime ideal of R. We consider the following two cases:

Case 1. $F = R \oplus R$, that is $rank\ F = 2$.

Let $r(a, b) \in IF \setminus I^n F$, where $r \in R \setminus (IF : F) = I$ and $a, b \in R$. Then $ra, rb \in I$, and ra or rb is not in I^n. Without loss of generality, we may assume $ra \notin I^n$. Then $ra \in I \setminus I^n$ and as $r \notin I$, $a \in I$. Similarly if $rb \notin I^n$, then $b \in I$ and so $(a, b) \in IF$.

Now let $rb \in I^n$. Then $r(a + b) \in I$, and $ra \notin I^n$, and so $r(a + b) \in I \setminus I^n$, and $r \notin I$, hence $a + b \in I$. Also $a \in I$, therefore $b \in I$, that is $(a, b) \in IF$.

Case 2. F is a free module of arbitrary rank.

If $a \in F$, then $a \in \oplus_{i=1}^{n} Ra_i$, where $a_1, a_2, \cdots, a_n \in F$ for some integer n. Now by using case 1, we get the results.

(v) The proofs of (a) $\Longrightarrow$ (b) and (c) $\Longrightarrow$ (a) are straightforward.

(b) $\Longrightarrow$ (c) It is easy to see that I is an n-almost prime ideal of R. Now if I is not a prime ideal, then there exists $a, b \in R \setminus I$ such that $ab \in I$. Since I is an n-almost prime ideal, $ab \in I^n$. Therefore $a(b, 1, 1, 1, \cdots) \in N \setminus (N : M)^{n-1}N$, however $a \notin I = (N : M)$ and $(b, 1, 1, 1, \cdots) \notin N$, which is a contradiction. ■

Examples.

(1) If I is an ideal of R generated by idempotents, then by Lemma 2.1(iii), IM is an almost prime submodule, or $IM = M$, for any R-module M. For a specific example, let R' be an arbitrary ring, and consider $R = \prod_{n=1}^{\infty} R'$ and $I = \oplus_{n=1}^{\infty} R'$, particularly I is an almost prime ideal.

(2) Let $R = K[[X^3, X^4, X^5]]$, where K is a field, and $I = \langle X^3, X^4 \rangle$. By [1, Example 11], I is an almost prime ideal, which is not a 3-almost prime ideal.

Let F be a free R-module. By Lemma 2.1(iv), the submodule IF is an almost prime submodule, which is not a 3-almost prime submodule.

(3) Let R be an Artinian ring. Then for any ideal I of R, there exists an $n \in \mathbb{N}$ such that $I^n = I^{n+1}$. So the ideal $J = I^n$ is an almost prime ideal, and by Lemma 2.1(iv), for any free R-module F, the submodule JF is an almost prime submodule.

Let M, M' be two R-modules. For a projective resolution

$\cdots \xrightarrow{f_3} P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{\varepsilon} M \longrightarrow 0,$ of M, consider the complexes

$\cdots \xrightarrow{f_3} P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0=0} 0,$ and $\cdots \xrightarrow{f_3 \otimes 1} P_2 \otimes M' \xrightarrow{f_2 \otimes 1} P_1 \otimes M' \xrightarrow{f_1 \otimes 1} P_0 \otimes M' \xrightarrow{f_0 \otimes 1} 0.$

Now recall that $Tor_n(M, M')$ is defined to be $Tor_n(M, M') = \frac{Ker(f_n \otimes 1)}{Im(f_{n+1} \otimes 1)}$.

2.2. Proposition. Let M be an R-module, and suppose that I is an ideal of R with $IM \neq M$. If $Tor_1(\frac{R}{I}, \frac{M}{IM}) = 0$, then IM is an n-almost prime submodule for each $1 < n \in \mathbb{N}$.

Proof. Put $K = IM$. By the short exact sequence $0 \to K \to M \to \frac{M}{K} \to 0$, and according to [7, Theorem 6.26], there is an exact sequence

$$0 = Tor_1(\frac{R}{I}, \frac{M}{K}) \xrightarrow{f} \frac{R}{I} \otimes_R K \xrightarrow{g} \frac{R}{I} \otimes_R M.$$

The natural homomorphism $h : K \longrightarrow \frac{M}{IM}$ induces a homomorphism $\bar{h} : \frac{K}{IK} \longrightarrow \frac{M}{IM}$. Also note that there is an isomorphism $\theta_L : \frac{R}{I} \otimes_R L \longrightarrow \frac{L}{IL}$, for each R-module L.

In the following diagram the rows are exact and it is easy to see that the rectangle is commutative:

$$\begin{array}{ccccccc}
 & & 0 = Tor_1(\frac{R}{I}, \frac{M}{K}) & \xrightarrow{f} & \frac{R}{I} \otimes_R K & \xrightarrow{g} & \frac{R}{I} \otimes_R M \\
 & & & & \downarrow \theta_K & & \downarrow \theta_M \\
0 & \longrightarrow & Ker\bar{h} & \longrightarrow & \frac{K}{IK} & \xrightarrow{\bar{h}} & \frac{M}{IM}
\end{array}$$

It follows that $Ker\bar{h} \cong Kerg = Imf = 0$. On the other hand, $Ker\bar{h} = \frac{IM}{IK}$, hence $K = IM = IK$. Therefore by Lemma 2.1, $(K : M)K = (IM : M)IM = I(IM : M)M = I(IM) = IK = K$, and evidently $(K : M)K = K$ implies that K is an n-almost prime submodule for each $1 < n \in \mathbb{N}$. ■

2.3. Corollary. Let M be an R-module and I an ideal of R with $IM \neq M$. Then IM is an n-almost prime submodule of M, for each $1 < n \in \mathbb{N}$, if one of the following holds:

(i) $\frac{M}{IM}$ is a flat $\frac{R}{Ann\ M}$-module.

(ii) $\frac{R}{I}$ is a flat R-module.

Proof. Put $K = IM$. Note that $\frac{(K:_R M)}{Ann\ M} = (K :_{\frac{R}{Ann\ M}} M)$, thus K is an n-almost prime R-submodule of M, if and only if it is an n-almost prime $\frac{R}{Ann\ M}$-submodule of M. Therefore we can replace $\frac{R}{Ann\ M}$ with R for simplification.

We know that if $\frac{R}{I}$ or $\frac{M}{K}$ is a flat R-module, then $Tor_1(\frac{R}{I}, \frac{M}{K}) = 0$ (see for example [7, Theorem 7.2]). Now the proof follows from Proposition 2.2. ■

The following example shows that the converse of Corollary 2.3 is not necessarily true.

Example. Let $M = R = \mathbb{Z}$, and $I = 2\mathbb{Z}$. Then evidently $2\mathbb{Z}$ is a prime ideal [resp. submodule] of R [resp. the R-module M], with $Ann\ M = 0$. However $\frac{R}{I}$ is not a flat R-module, since it is not torsion-free.

Recall that a ring R is called a Von Neumann regular ring, if for any $a \in R$, $Ra = Ra^2$. By [7, Corollary 4.10], every semi-simple ring is a Von Neumann regular ring.

2.4. Corollary. Let M be an R-module, where R is a Von Neumann regular ring and suppose I is an ideal of R. If $IM \neq M$, then IM is an n-almost prime submodule for each $1 < n \in \mathbb{N}$.

Proof. According to [7, Theorem 4.9], every module over a Von Neumann regular ring is flat. So the proof is given by Corollary 2.3. ■

2.5. Lemma. Let N be an n-almost prime submodule of M.

(i) If there exist $x \in M \setminus N$ and $r \in R \setminus (N : M)$ with $rx \in N$, then $rN \cup (N : M)x \subseteq (N : M)^{n-1}N$.

(ii) If $0 \neq x + N \in \frac{M}{N}$, where $x \in M$, then $(ann(x + N))N \subseteq (N : M)N$.

(iii) $(N : M)N = (\bigcup_{x\in M\setminus N} ann(x + N))N$.

Proof. (i) As N is n-almost prime, $rx \in (N : M)^{n-1}N$. Let y be an arbitrary element of N. Then $y + x \notin N$ and $r(y + x) = ry + rx \in N$ and since N is n-almost prime, $r(y + x) \in (N : M)^{n-1}N$. Therefore $ry \in (N : M)^{n-1}N$, and so $rN \subseteq (N : M)^{n-1}N$.

Now let s be an arbitrary element of $(N : M)$. Clearly $r+s \notin (N : M)$ and $(r+s)x \in N$ and as N is n-almost prime, $(r + s)x \in (N : M)^{n-1}N$. Then since $rx \in (N : M)^{n-1}N$, $sx \in (N : M)^{n-1}N$. Hence $(N : M)x \subseteq (N : M)^{n-1}N$.

(ii) Let $r \in ann(x + N)$. Then $rx \in N$. If $r \in (N : M)$, then clearly $rN \subseteq (N : M)N$. If $r \notin (N : M)$, then in this case by part (i), $rN \subseteq (N : M)^{n-1}N \subseteq (N : M)N$.

(iii) Evidently $(N : M) \subseteq \bigcup_{x\in M\setminus N} ann(x + N)$. Then by part (ii) we have,

$(N : M)N \subseteq (\bigcup_{x\in M\setminus N} ann(x + N))N \subseteq \bigcup_{x\in M\setminus N}(ann(x + N)N) \subseteq (N : M)N$. ■

2.6. Proposition. Let I be an ideal of a ring R and N a submodule of an R-module M.

(i) If $IM \neq IN$, $IN \neq N$, then $K = IN$ is n-almost prime if and only if $K = (K : M)^{n-1}K$.

(ii) If for some positive integer $k > 1$, $I^{k-1}M \neq I^kM = K$, then K is n-almost prime if and only if $K = (K : M)^{n-1}K$. Consequently in this case K is almost prime if and only if K is n-almost prime, for any (or some) positive integer $n \geq 3$.

(iii) Let R be an integral domain and M a Noetherian module with $ann(N) = 0$. Then for every proper ideal I of R with $IM \neq IN$, IN is not n-almost prime.

Proof. (i) If $K = (K : M)^{n-1}K$, then clearly K is n-almost prime. Now assume K is n-almost prime. Evidently K is almost prime. If $K \neq (K : M)K$, then consider $a \in I$ and $x \in N$, where $ax \notin (K : M)K$. Then since $ax \in IN = K \setminus (K : M)K$, either $a \in (K : M)$ *or* $x \in K$. Let $a \in (K : M)$. As $K = IN \subset IM$, $I \not\subseteq (K : M)$ and so we can choose an element $r \in I \setminus (K : M)$. As $rx \in IN = K$, Lemma 2.5(i) implies that $(K : M)x \subseteq (K : M)K$, and so $ax \in (K : M)K$.

Now suppose that $x \in K$. By our assumption $N \not\subseteq K$, hence there exists $z \in N \setminus K$. Note that $az \in IN = K$. Again by Lemma 2.5(i), $aK \subseteq (K:M)K$. Then in this case $ax \in (K:M)K$.

Therefore $K = (K:M)K$, and consequently $K = (K:M)^{n-1}K$.

(ii) We have $IM \neq K$, otherwise $I^{k-1}M \subseteq IM = K = I^kM \subseteq I^{k-1}M$, which is impossible. Now apply part (i) for $N = I^{k-1}M$.

(iii) Clearly $N \neq IN$, otherwise by Nakayama's lemma, there exists $s \in I$ such that $(s+1)N = 0$ and since $ann(N) = 0$, $1 = -s \in I$, which is a contradiction with the fact that $I \neq R$.

Note that $(IN:M)N \subseteq IN$. If IN is n-almost prime, then IN is almost prime and so by part (i), $IN = (IN:M)IN = I(IN:M)N \subseteq I^2N \subseteq IN$, that is $IN = I^2N$. Again by Nakayama's lemma, for some $t \in I$, $(t+1)IN = 0$. As $ann(N) = 0$, $(t+1)I = 0$. So $1 = -t \in I$ or $I = 0$, which is a contradiction with the fact that $I \neq R$ and $IM \neq IN$. Consequently IN is not n-almost prime. ■

Recall that a ring R is said to be ZPI-ring, if every non-zero proper ideal of R can be written as a product of prime ideals of R (see [6, Chapters VI and IX]). According to [6, Theorem 9.10], every ZPI-ring is a Noetherian ring.

2.7. Theorem. Let M be an R-module and I an ideal of R with $IM \neq M$.

(i) If R is a ZPI-ring and IM is an n-almost prime submodule, then $IM = I^nM$, or $IM = PM$, where P is a prime ideal of R.
(ii) If R is a Dedekind domain, then IM is an n-almost prime submodule if and only if $IM = I^nM$ or IM is a prime submodule of M.
(iii) If (R, m) is a local ZPI-ring and IM is finitely generated, then IM is n-almost prime if and only if $IM = 0$ *or* $IM = mM$.

Proof. (i) Let $I = P_1^{k_1}...P_m^{k_m}$, where $P_i's$ are distinct prime ideals of R and $k_i's$ are positive integers.

Assume that $IM \neq PM$ for each prime ideal P of R. Then $IM = P_1^{k_1}...P_m^{k_m}M$ and without loss of generality we may suppose that $IM \neq P_1^{k_1-1}P_2^{k_2}...P_m^{k_m}M$ and $(k_1-1)+k_2+k_3+\cdots+k_m > 0$.

Put $N = P_1^{k_1-1}P_2^{k_2}...P_m^{k_m}M$ and $K = IM$. Then $K = P_1N$ and $P_1M \neq K$ and $K \neq N$, then by Proposition 2.6(i), $K = (K:M)^{n-1}K$, that is $IM = (IM:M)^{n-1}(IM)$, and by Lemma 2.1(i), $(IM:M)^{n-1}(IM) = I(IM:M)^{n-1}M = I^nM$. Thus $IM = I^nM$.

(ii) Let R be a Dedekind domain and suppose IM is an n-almost prime submodule. By part (i), $IM = I^nM$, or $IM = PM$, where P is a prime ideal of R.

If $IM = PM$, where P is a prime ideal of R, then $P = 0$ or P is a maximal ideal of R. Evidently $P = 0$ implies that $I^nM = 0 = IM$. Now suppose P is a maximal ideal of R. As $P \subseteq (PM:M)$, we have $P = (PM:M)$ or $PM = M$. By our hypothesis $PM = IM \neq M$, then $(IM:M) = (PM:M) = P$ and so IM is a prime submodule of M, by Lemma 2.1(ii).

Now for the converse, suppose that $IM = I^nM$. Then by Lemma 2.1(iii), IM is n-almost prime.

(iii) If $IM = mM$, then by Lemma 2.1(ii), mM is a prime submodule. Also clearly 0 is an n-almost prime submodule.

Now assume that IM is an n-almost prime submodule of M. By [6, Theorem 9.10], R is a Noetherian ring. If $m = m^2$, by Nakayama's lemma, $m = 0$, then R is a field and so $IM = 0$.

Now let $m^2 \neq m$. Choose $x \in m \setminus m^2$. Then $m^2 \subset m^2 + Rx \subseteq m$. By [6, Theorem 9.10], there are no ideals of R strictly between m^2 and m. So $m^2 + Rx = m$ and by Nakayama's lemma, $m = Rx$.

Now let P be a non-zero prime ideal of R, and $0 \neq y \in P$. By the Krull Intersection Theorem, we have $\cap_{n=1}^{+\infty} m^n = 0$. Thus there is a positive integer k such that $y \in m^k$ and $y \notin m^{k+1}$. Since $y \in m^k = Rx^k$, there exists an element $u \in R$ such that $y = ux^k$, and since $y \notin m^{k+1}$, $u \notin m$. Then u is a unit element of R. Hence $x^k = u^{-1}y \in P$. We know that P is a prime ideal of R, so $x \in P$, that is $m = P$. Whence m is the only nonzero prime ideal of R. Now by part (i), $IM = I^nM$ or $IM = mM$

If $IM = mM$, then Lemma 2.1(ii) implies that IM is a prime submodule.

In case $IM = I^nM$, Nakayama's lemma implies that $IM = 0$. ■

The following result is an obvious consequence of the above theorem.

2.8. Corollary. Let R be a ZPI-ring and I a proper ideal of R.

(i) I is an n-almost prime ideal if and only if $I = I^n$ or I is a prime ideal.
(ii) If (R, m) is a local ring, then I is an n-almost prime ideal if and only if $I = 0$ *or* $I = m$.

2.9. Proposition. Let M be an R-module, and I an ideal which is a product of a finite number of maximal ideals of R. Then IM is an n-almost prime submodule if and only if IM is a prime submodule of M, or $IM = I^nM$.

Proof. For each maximal ideal P of R, we have $P \subseteq (PM : M)$, then by Lemma 2.1(ii), PM is a prime submodule or $PM = M$. Thus if IM is an n-almost prime submodule, which is not a prime submodule, then there exist maximal ideals P_i, $1 \leq i \leq m$ and positive numbers k_i, $1 \leq i \leq m$ such that $IM = P_1^{k_1}P_2^{k_2}\cdots P_m^{k_m}M$ and $IM \neq P_1^{k_1-1}P_2^{k_2}\cdots P_m^{k_m}M$. Therefore if we put $N = P_1^{k_1-1}P_2^{k_2}\cdots P_m^{k_m}M$ and $K = P_1N$, since K is not prime, we get $K \neq P_1M$, also $K = P_1N \neq N$, hence by Proposition 2.6(i), $K = (K : M)^{n-1}K$.

Consequently by Lemma 2.1(i), $K = IM = (IM : M)^{n-1}(IM) = I(IM : M)^{n-1}M = I^nM$.

For the converse suppose $IM = I^nM$. Then according to Lemma 2.1(iii), IM is n-almost prime. ■

Recall that a *multiplicatively closed subset* of a ring R is a subset S such that $0 \notin S$ and $1 \in S$ and $xy \in S$ for each $x, y \in S$.

The following result studies when the localization of an n-almost prime submodule is n-almost prime.

2.10. Proposition. Let N be an n-almost prime submodule of an R-module M, and S a multiplicatively closed subset of R.

(i) If $S \cap (N : M) = \emptyset$ and for some $x \in M \setminus N$, $S \cap ((N : M)^{n-1}N : x) = \emptyset$, then $S^{-1}N \neq S^{-1}M$.
(ii) If $S^{-1}N \neq S^{-1}M$, then $S^{-1}N$ is an n-almost prime submodule of $S^{-1}M$.

Proof. (i) Let $x \in M \setminus N$. If $S^{-1}N = S^{-1}M$, then there exists an element $s \in S$ such that $sx \in N$. Since $S \cap ((N : M)^{n-1}N : x) = \emptyset$, $sx \notin (N : M)^{n-1}N$. As N is an n-almost prime submodule and $x \notin N$, $s \in (N : M) \cap S$, which is a contradiction. Hence $S^{-1}N \neq S^{-1}M$.

(ii) Let for $\frac{r}{s} \in S^{-1}R$, $\frac{y}{t} \in S^{-1}M$, $\frac{r}{s}\frac{y}{t} \in S^{-1}N \setminus (S^{-1}N : S^{-1}M)^{n-1}S^{-1}N$. Then there exists an element $u \in S$ such that $ury \in N$. If $ury \in (N : M)^{n-1}N$, then $\frac{ry}{st} = \frac{ury}{ust} \in S^{-1}((N : M)^{n-1}N) \subseteq (S^{-1}N : S^{-1}M)^{n-1}S^{-1}N$, a contraction. Hence $ury \in N \setminus (N : M)^{n-1}N$. As N is almost prime, either $ur \in (N : M)$ or $y \in N$, so either $\frac{r}{s} = \frac{ur}{us} \in S^{-1}(N : M) \subseteq (S^{-1}N : S^{-1}M)$ or $\frac{y}{t} \in S^{-1}N$. ■

3. Essential multiplicatively closed subsets

Recall that an ideal I of a ring R is said to be *essential* if $I \cap J \neq 0$, for each non-zero ideal J of R (that is $J \nsubseteq 0$). In this section we introduce a similar notion for multiplicatively closed subsets of R, and we find some connections between this notion and n-almost primes.

Definition. Let S be a multiplicatively closed subset of R and P a prime ideal of R with $S \cap P = \emptyset$. Then S will be called P-essential, if $S \cap J \neq \emptyset$, for each ideal J with $J \nsubseteq P$.

Evidently $R \setminus P$ is a P-essential multiplicatively closed subset, for each prime ideal P of R.

Recall that a multiplicatively closed subset S of R is said to be *saturated* if

$$xy \in S \Longleftrightarrow x, y \in S.$$

The following lemma is a well known result (see [2, p. 44, Exercise 7 (ii)]).

3.1. Lemma. Let S be a multiplicatively closed subset of R. Then

$$\bar{S} = R \setminus \cup\{P|P \text{ is a prime ideal with } P \cap S = \emptyset\}$$

is a saturated multiplicatively closed subset of R containing S and there is no saturated multiplicatively closed subset of R strictly between S and $\bar{S}$.

It is obvious that for each prime ideal P of R, the ring R_P is a local ring and the ideal P_P is a maximal ideal of R_P. The following result shows that $S^{-1}R$ being a local ring is indeed related to P-essentiality of S.

Let S be a multiplicatively closed subset of R. For any ideal J of $S^{-1}R$, we consider $J^c = \{r \in R \mid r/1 \in J\}$.

3.2. Proposition. Let S be a multiplicatively closed subset of R and P a prime ideal of R with $S \cap P = \emptyset$. Then the following are equivalent:

(i) S is P-essential;
(ii) $S^{-1}R = R_P$;
(iii) $\bar{S} = R \setminus P$;
(iv) $S^{-1}P$ is the only maximal ideal of $S^{-1}R$.

Proof. $(i) \Rightarrow (ii)$ Clearly $S^{-1}R \subseteq R_P$, since $S \subseteq R \setminus P$. Now suppose that $\frac{y}{t} \in R_P$. Hence as $t \in R \setminus P$, for some $r \in R$, we have $rt \in S \subseteq R \setminus P$, and so $r \in R \setminus P$. Then $\frac{y}{t} = \frac{ry}{rt} \in S^{-1}R$ and hence $S^{-1}R = R_P$.

$(ii) \Rightarrow (iii)$ Since $P \cap S = \emptyset$, by Lemma 3.1, $\bar{S} \subseteq R \setminus P$. Now let $r \in R \setminus P$. We have $1/r \in R_P = S^{-1}R$, then there exists $s \in S$, $x \in R$ with $1/r = x/s$. Thus for some $s' \in S$ we have $s'rx = ss' \in S \subseteq \bar{S}$, and so $r \in \bar{S}$, because $\bar{S}$ is saturated.

$(iii) \Rightarrow (iv)$ Let m be a maximal ideal of $S^{-1}R$. Then m^c is a prime ideal of R with $m^c \cap S = \emptyset$. Note that $R \setminus (P \cup m^c)$ is a saturated multiplicatively closed subset of R and since $S \subseteq R \setminus (P \cup m^c) \subseteq (R \setminus P) = \bar{S}$, Lemma 3.1 implies that $R \setminus (P \cup m^c) = (R \setminus P) = \bar{S}$. Hence $m^c \subseteq (P \cup m^c) = P$, and thus $m = S^{-1}(m^c) \subseteq S^{-1}P$, and so $m = S^{-1}P$, because of maximality of m.

$(iv) \Rightarrow (i)$ Let J be an ideal of R such that $J \nsubseteq P$. If $J \cap S = \emptyset$, since $S^{-1}P$ is the only maximal ideal of $S^{-1}R$, we have $S^{-1}J \subseteq S^{-1}P$. So $J \subseteq P$, which is impossible. Consequently S is P-essential. ■

3.3. Theorem. Let N be an n-almost prime submodule of an R-module M with $I = (N : M)$. Then $S = [(R \setminus I) \cup (I^{n-1}N : M)] \setminus P$ is P-essential, for each prime ideal P of R.

Proof. First to prove that S is multiplicatively closed, let $r,s \in S$. If $rs \notin S$, then $rs \in I$. Also $rs \notin P$, because if $rs \in P$, then $r \in P$ or $s \in P$, although $S \cap P = \emptyset$. Thus $rs \in I \setminus P$.

If $r \in (I^{n-1}N : M)$ or $s \in (I^{n-1}N : M)$, then $rs \in (I^{n-1}N : M) \setminus P$, and so $rs \in S$.

Now on the contrary suppose $r, s, rs \notin (I^{n-1}N : M)$. Hence there exists $m \in M$ such that $rsm \notin I^{n-1}N$, and we know that $rs \in I = (N : M)$, therefore $rsm \in N \setminus I^{n-1}N$.

As $r, s \in S \subseteq [(R \setminus I) \cup (I^{n-1}N : M)]$ and $r, s \notin (I^{n-1}N : M)$, we have $r, s \notin I = (N : M)$. Note that $rsm \in N \setminus I^{n-1}N$ and $r, s \notin (N : M)$ and N is n-almost prime, thus $m \in N$.

Now consider $m' \in M \setminus N$. If $rsm' \notin I^{n-1}N$, the above argument shows that $m' \in N$, which is impossible.

Then we may assume $rsm' \in I^{n-1}N$. Thus for $x = m+m'$, we have $rsx \in N \setminus I^{n-1}N$. Now since $m \in N$ and $m' \notin N$, we have $x \notin N$, consequently $r \in (N : M) = I$ or $s \in (N : M) = I$, which is a contradiction.

Next we will prove that S is P-essential. Let J be an ideal of R such that $J \nsubseteq P$. If $I \subseteq P$, then $S = R \setminus P$, and obviously S is P-essential. So suppose that $I \nsubseteq P$.

If $J \cap S = \emptyset$, it is easy to see $J \cap [(R \setminus I) \cup (I^{n-1}N : M)] \subseteq P$ and so $J \subseteq I \cup P$. Therefore $J \subseteq I$

Note that $I = (N : M)$, so $I^nM = I^{n-1}(N : M)M \subseteq I^{n-1}N$, that is $I^n \subseteq (I^{n-1}N : M)$. Hence $J^n \subseteq J \cap I^n \subseteq J \cap (I^{n-1}N : M) \subseteq P$, which is impossible. Consequently $J \cap S \neq \emptyset$ and so S is P-essential. ■

3.4. Corollary. Let I be an ideal of R such that $N(R) \subseteq I^n$ and consider $S_P = [(R \setminus I) \cup I^n] \setminus P$. Then the following are equivalent:

(i) I is n-almost prime;
(ii) S_P is multiplicatively closed for any prime ideal P;
(iii) S_P is multiplicatively closed for any minimal prime ideal P.

Proof. $(i) \Rightarrow (ii)$ The proof is given by Theorem 3.3.

$(ii) \Rightarrow (iii)$ The proof is evident.

$(iii) \Rightarrow (i)$ Let $ab \in I \setminus I^n$. So $ab \notin N(R)$ and there exists a minimal prime ideal P of R such that $ab \notin P$. Then $ab \notin S_P$. Hence $a \notin S_P$ or $b \notin S_P$. Therefore $a \in I$ or $b \in I$. ■

3.5. Corollary. Let R be an integral domain and M an R-module.

(i) If N is an n-almost prime submodule of M with $(N : M) = I$, then $S = [(R \setminus I) \cup (IN : M)] \setminus \{0\}$ is a multiplicatively closed subset of R and $S^{-1}R$ is a field.
(ii) An ideal I of R is n-almost prime if and only if $S = [(R \setminus I) \cup I^n] \setminus \{0\}$ is a multiplicatively closed subset of R. When this is the case, $S^{-1}R$ is a field.

Proof. (i) The proof is given by Theorem 3.3 and Proposition 3.2.

(ii) The proof of the first part is given by Corollary 3.4. By part (i), $S^{-1}R$ is a field, if I is n-almost prime. ■

The following remark studies the converse of the above corollary.

Remark. Let S be a saturated multiplicatively closed subset of R such that $S^{-1}R$ is a field. Then there exist prime ideals I and P of R such that $S = [(R \setminus P) \cup I^2] \setminus P$.

Proof. Since S is a multiplicatively closed subset of R, there exists a prime ideal P of R with $P \cap S = \emptyset$. Then $S^{-1}P$ is a proper ideal of $S^{-1}R$ and $S^{-1}R$ is a field, so $S^{-1}P$ is the only maximal ideal of $S^{-1}R$. Hence by Proposition 3.2, $\bar{S} = R \setminus P$. Note that S is a saturated multiplicatively closed subset of R, then by Lemma 3.1, $S = \bar{S} = R \setminus P$. Thus it is enough to consider $I = P$. ■

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