GRINDING FROM THE
MATHEMATICAL POINT OF VIEW

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Abstract

A rigorous mathematical description of grinding processes used in powder technologies is developed. A grinding equation, an operator equation, connecting the final particle size distribution function to the particle size distribution function before the grinding process is studied. The mathematical model introduced here can be used to predict the results of grinding, to construct grinding systems with desired properties, and to improve the particle size measurement.

Keywords: Grinding equation, Operator equation, Particle size distribution function.

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1. Introduction

Powder technologies have many industrial applications in powder coating [5] and pharmaceutics [6], for example. The powder production is based on the use of special grinding systems [3]. The aim of this work is to develop a rigorous mathematical description of grinding processes.

Any grinding system contains grinders and classifiers. The grinder is responsible for the particle size reduction and the classifier separates small particles and takes them out from the grinder. These two principal elements of grinding systems can be described in terms of operators defined in spaces of particle size distribution functions. This allows to derive a grinding equation, an operator equation, connecting the grinding system 'output', the final particle size distribution, to the 'input', the particle size distribution before the grinding process. This mathematical model can be used to predict the results of grinding, to construct grinding systems with desired properties, and to improve the particle size measurement.

The paper is organized as follows. In the second section an informal outline of the approach is presented. The third section is devoted to geometric partition models and

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moment analysis. The partition operator and the classifier are studied in the fourth section. The fifth section deals with the grinding equation. Finally, the last section contains concluding remarks.

2. Informal Outline of the Approach

Consider a set of particles. Let $\nu(V)dV$ be the number of particles with the volumes in the interval $[V, V + dV]$. The density function f for the random value V (the particle volume) is given by

$$(1) \qquad f(V) = \frac{\nu(V)}{\int_0^\infty \nu(V)dV}.$$

2.1. Partition operator. Suppose that a particle with the volume V is divided into n parts with the volumes $\xi_k V$, $k = \overline{1, n}$, where

$$\bar{\xi} = (\xi_1, \ldots, \xi_n) \in \Xi = \{(\xi_1, \ldots, \xi_n) \mid \xi_k \geq 0, \; \xi_1 + \ldots + \xi_n = 1\}.$$

Let the probability of getting particles with the volumes belonging to the intervals $[\xi_k V, (\xi_k + d\xi_k)V]$ be $\phi(\bar{\xi})d\bar{\xi}$, where $\phi : \Xi \to \mathbb{R}$ is a symmetric density function. Denote by $\mathcal{P}(f)$ the density function for the particle volume after the partition. Obviously

$$(2) \qquad \mathcal{P}(f)(V)dV = \frac{\mathcal{P}(\nu)(V)dV}{\int_0^\infty \mathcal{P}(\nu)(V)dV},$$

where $\mathcal{P}(\nu)(V)dV$ stands for the number of particles with the volumes in the interval $[V, V + dV]$ after the partition. Observe that the number $\mathcal{P}(\nu)(V)dV$ can be written as a superficial integral

$$\mathcal{P}(\nu)(V)dV = \int_\Xi \sum_{k=1}^n \nu\left(\frac{V}{\xi_k}\right) d\left(\frac{V}{\xi_k}\right) \phi(\bar{\xi})dS_{\bar{\xi}}.$$

Dividing this equality by

$$\int_0^\infty \mathcal{P}(\nu)(V)dV = n \int_0^\infty \nu(V)dV$$

and invoking (1) and (2), we get

$$\mathcal{P}(f)(V) = \frac{1}{n} \int_\Xi \sum_{k=1}^n f\left(\frac{V}{\xi_k}\right) d\left(\frac{V}{\xi_k}\right) \phi(\bar{\xi})dS_{\bar{\xi}}.$$

Since the function ϕ is symmetric, we obtain

$$(3) \qquad \mathcal{P}(f)(V) = \int_0^1 f\left(\frac{V}{\eta}\right) \psi(\eta)\, d\eta,$$

where

$$\psi(\eta) = \frac{1}{\eta} \int_{\Xi_\eta} \phi(\eta, \xi_2, \ldots, \xi_n)dS_{\tilde{\xi}},$$

and

$$\Xi_\eta = \{\tilde{\xi} = (\xi_2, \ldots, \xi_n) \mid \xi_k \geq 0, \; \eta + \xi_2 + \ldots + \xi_n = 1\}.$$

Obviously

$$(4) \qquad \int_0^1 \eta\psi(\eta)\, d\eta = \int_\Xi \phi(\bar{\xi})dS_{\bar{\xi}} = 1.$$

Thus equation (3) gives a general form of the partition operator. The function ψ satisfying (4) can be found experimentally or derived theoretically from partition models.

2.2. Two types of grinding. Classifier. The grinding process can be modelled as a successive application of the partition operator given by (3):

$$f_{\text{out}} = \mathcal{P}^N(f_{\text{in}}),$$

where f_{out} and f_{in} stand for the the final particle size density function and the initial particle size density function, respectively. During this process the particles do not leave the grinder. If, for example, $\psi(\eta) = 2\delta(\eta - 1/2)$, then all particles are divided into two equal parts independently on their shapes. Consider the input density function

$$f_{\text{in}}(V) = \begin{cases} (aV^{a-1}/b^a)e^{-(x/b)^a}, & x \geq 0, \\ 0, & x < 0, \end{cases}$$

known as the Rosin-Rammler density function [1]. Then we have

$$f_{\text{out}}(V) = 2^N f_{\text{in}}\left(2^N V\right),$$

that is, the output is again a Rosin-Rammler density function with a new parameter b. Obviously, this density function can be used to describe particle sizes only in this special case $\psi(\eta) = 2\delta(\eta - 1/2)$.

Another type of grinding includes a separation process. A special device, known as classifier, separates small particles and takes them out from the grinder. To model this separation introduce a classifier operator $\mathcal{C}_\alpha$, $\alpha > 0$, defined by

$$\mathcal{C}_\alpha(f)(V) = c_\alpha(V)f(V),$$

where

$$c_\alpha(V) = \begin{cases} 0, & V < \alpha, \\ 1, & V \geq \alpha. \end{cases}$$

The physical meaning of the parameter α is very simple. If $V < \alpha$, then the particle leaves the grinder. The grinding process with classification can be represented in the following form

$$f_{\text{out}} = \sum_{k=0}^{\infty} (\mathcal{I} - \mathcal{C}_\alpha)(\mathcal{P} \circ \mathcal{C}_\alpha)^k(f_{\text{in}}),$$

where $\mathcal{I}$ is the identity operator.

2.3. Grinding equation. Equivalently the output can be written as

(5) $\qquad f_{\text{out}} = (\mathcal{I} - \mathcal{C}_\alpha)(g),$

where

(6) $\qquad g = \sum_{k=0}^{\infty} (\mathcal{P} \circ \mathcal{C}_\alpha)^k(f_{\text{in}}).$

Applying (formally) the operator $(\mathcal{I} - \mathcal{P} \circ \mathcal{C}_\alpha)$ to (6), we obtain the grinding equation

(7) $\qquad g = \mathcal{P} \circ \mathcal{C}_\alpha(g) + f_{\text{in}}.$

Using (3) the grinding equation (7) can be written in the integral form

(8) $\qquad g(V) = \int_0^1 c_\alpha\left(\frac{V}{\eta}\right) g\left(\frac{V}{\eta}\right) \psi(\eta)\, d\eta + f_{\text{in}}(V) = \int_0^{V/\alpha} g\left(\frac{V}{\eta}\right) \psi(\eta)\, d\eta + f_{\text{in}}(V).$

From (5) we have

$$f_{\text{out}}(V) = g(V), \quad V < \alpha.$$

Therefore (8) implies

$$(9) \qquad f_{\text{out}}(V) = \int_0^{V/\alpha} g\left(\frac{V}{\eta}\right) \psi(\eta)\, d\eta + f_{\text{in}}(V), \quad V < \alpha.$$

It is easy to see that the integral in (9) depends only on the values $g(V)$, with $V \in [\alpha, \infty[$. Hence to find the output from (9) it suffices to solve integral equation (8) in the interval $[\alpha, \infty[$, that is, to solve the equation

$$(10) \qquad g(V) = \int_0^1 g\left(\frac{V}{\eta}\right) \psi(\eta)\, d\eta + f_{\text{in}}(V), \quad V \geq \alpha.$$

2.4. Connection with the integro-differential batch grinding equation. It turns out that the grinding equation can be easily obtained from existing comminution models. The fundamental equation of fragmentation, known as the batch grinding equation, has the form

$$(11) \qquad \frac{\partial f(V,t)}{\partial t} = -s(V)f(V,t) + \int_V^\infty b(V,W)s(W)f(W,t)dW,$$

where $f(V,t)$ is the size density function at the moment t, $b(V,W)$ is the breakage function, giving the fraction of particles with volumes in the range $[V, V+dV]$ obtained by breakage of a particle of volume W, and $s(V)$ is the breakage rate of particles of volume V (see [8], for example). This is a simple balance law similar to that of population dynamics. Equation (11) is in a good agreement with the experimental data [7].

Integrating equation (11) and setting

$$g(V) = s(V) \int_0^\infty f(V,t)dt,$$

we get

$$\lim_{t \to \infty} f(V,t) + g(V) = \int_V^\infty b(V,W)g(W)dW + f(V,0).$$

All known breakage functions have the form

$$b(V,W) = \psi\left(\frac{V}{W}\right)\frac{V}{W^2}.$$

(This structure of b was confirmed by numerous experiments [7].) After the change of variables $\eta = V/W$ in the integral we obtain

$$(12) \qquad \lim_{t \to \infty} f(V,t) + g(V) = \int_0^1 g\left(\frac{V}{\eta}\right) \psi(\eta)\, d\eta + f(V,0).$$

Assume that $s(V) = 0$, whenever $V < \alpha$. Then we have

$$(13) \qquad \lim_{t \to \infty} f(V,t) = \int_0^{V/\alpha} g\left(\frac{V}{\eta}\right) \psi(\eta)\, d\eta + f(V,0), \quad V < \alpha.$$

Since $\lim_{t \to \infty} f(V,t) = 0$ whenever $V \geq \alpha$, equation (12) implies

$$(14) \qquad g(V) = \int_0^1 g\left(\frac{V}{\eta}\right) \psi(\eta)\, d\eta + f(V,0), \quad V \geq \alpha.$$

Equalities (13) and (14) coincide with (9) and (10), respectively. This is, probably, the easiest way to derive the grinding equation, although the structure of the partition operator must be postulated.

The batch grinding integro-differential equation describes distribution at any stage of the grinding process and contains the breakage rate function $s(V)$. Theoretical or experimental determination of this function causes serious difficulties. Equations (13) and (14) do not contain $s(V)$ and are more suitable to predict the final distribution, for

any given feed distribution. Since system (13) and (14) is a consequence of (11), it does not contradict the experimental data.

2.5. Geometric partition models. The fragmentation of a particle in the grinder obviously depends on the particle shape. For the sake of simplicity assume that there exists only a finite number M of shapes and any particle of the shape $m = \overline{1, M}$ is divided into at most two particles of shapes $m' = \overline{1, M}$ and $m'' = \overline{1, M}$. Such a partition model can be obtained using some approximation rules. For example, if we have only spherical particles (M=1), then any particle is divided into two particles (obviously non-spherical) with volumes V_1 and V_2. To form a one-shape partition model we have to approximate the new particles by spheres with volumes V_1 and V_2. The partition model can be completely artificial or based on a physical hypothesis. The shape set should be chosen to be rather simple, a finite number of ellipsoids or polyhedrons, for example. Consider one possible partition model. The dust is formed by ellipsoids of shapes $m = \overline{1, M}$. If the particle is sufficiently small, the geometry of the grinder is not important. Any fragmentation can be seen as a result of collision of a particle with an infinite rigid plane Π_0. An ellipsoid-shaped particle E is divided into two particles by a plane Π containing the normal vector to the plane Π_0 at the point of collision $\Pi_0 \cap E$, and such that the area of the ellipse $E \cap \Pi$ is minimal. This hypothesis is quite natural: in this case the energy needed to divide the particle is the minimal one. Each of the new particles is approximated by ellipsoids E' and E'' of shapes $m' = \overline{1, M}$ and $m'' = \overline{1, M}$, respectively. If V is the volume of the ellipsoid E, then the ellipsoids E' and E'' have volumes V' and V'' satisfying $V = V' + V''$. To model the grinding process it suffices to consider only a finite number of possible orientations $l = \overline{1, L}$, of the particles with respect to the plane Π_0. For example, the normal vector to the plane Π_0 at the point of collision $\Pi_0 \cap E$ is parallel to one of the ellipsoid axes.

A geometric partition model can be described by a finite number of rules

$$\mathcal{R}(m, l) = (m'(m, l), m''(m, l), \gamma), \ m = \overline{1, M}, \ l = \overline{1, L},$$

which establish a correspondence between a pair (m, l) (shape and orientation) and a pair of new shapes m' and m'', and the ratio of the volumes $\gamma = V'/V''$, $0 \leq V' \leq V''$. Obviously

$$(15) \qquad V' = \frac{\gamma}{1 + \gamma} V \text{ and } V'' = \frac{1}{1 + \gamma} V.$$

If a particle of the shape m with the orientation l is not divided, we use the rule $\mathcal{R}(m, l) = (0, m, 0)$. Three simple illustrative examples of geometric partition models are considered in the next section. If the number of shapes M and of orientations L are big enough, one can get a partition model close to reality.

Let $\nu_m(V)dV$ be the number of particles of the shape m with the volumes in the interval $[V, V + dV]$. Consider the functions f_m, $m = \overline{1, M}$, given by

$$f_m(V) = \frac{\nu_m(V)}{\sum_{n=1}^{M} \int_0^\infty \nu_n(V)dV}.$$

Put

$$\mathcal{P}(f_m)(V) = \frac{\mathcal{P}(\nu_m)(V)}{\sum_{n=1}^{M} \int_0^\infty \mathcal{P}(\nu_m)(V)dV},$$

where $\mathcal{P}(\nu_m)(V)dV$ stands for the number of particles with the shape m and the volumes in the interval $[V, V + dV]$, after the partition. Assume that all orientations of the particles with respect to the plane Π_0 are equally likely. This is a natural assumption in the case of non-isotropic materials. (If materials with a crystalline structure are considered, then

it is necessary to introduce corresponding probabilities of the orientations.) Using (15) we have

$$(16) \qquad \mathcal{P}(\nu_k)(V)dV = \frac{1}{L}\sum_{l=1}^{L}\sum_{(m,\theta)\in N(k,l)}\nu_m(\theta V)d(\theta V),$$

where

$$N(k,l) = \{(m,\theta) \mid \mathcal{R}(m,l) = (k,n,1/(\theta-1)) \ \text{ or } \ \mathcal{R}(m,l) = (n,k,\theta-1)\}.$$

Set

$$\tau(m,l) = \begin{cases} 1, & m'(m,l) = 0, \\ 2, & m'(m,l) \neq 0. \end{cases}$$

Assume that their exists a number τ satisfying

$$(17) \qquad \tau = \frac{1}{L}\sum_{l=1}^{L}\tau(m,l), \quad m = \overline{1,M}.$$

For example, if all particles are divided into two parts, then this condition is satisfied and $\tau = 2$. Since

$$\sum_{n=1}^{M}\int_0^\infty \mathcal{P}(\nu_n)(V)dV = \tau\sum_{n=1}^{M}\int_0^\infty \nu_n(V)dV,$$

dividing (16) by

$$\sum_{n=1}^{M}\int_0^\infty \mathcal{P}(\nu_n)(W)dW\,dV,$$

we get

$$(18) \qquad \mathcal{P}(f_k)(V) = \frac{1}{\tau L}\sum_{l=1}^{L}\sum_{(m,\theta)\in N(k,l)}\theta f_m(\theta V).$$

The density function f can be represented in the form

$$f(V) = \sum_{m=1}^{M} f_m(V).$$

The partition operator now takes the form

$$(19) \qquad \mathcal{P}(f) = \sum_{m=1}^{M}\mathcal{P}(f_m),$$

where $\mathcal{P}(f_m)$, $m = \overline{1,M}$, are given by (18).

2.6. Moments. The partition operator given by (19) cannot be reduced to form (3). The study of moments helps to understand the relation between (19) and (3) and to develop a rigorous mathematical theory of grinding. Set

$$\mu_s^{(m)} = \int_0^\infty V^s f_m(V)dV, \text{ and}$$

$$\mathcal{P}(\mu_s^{(m)}) = \int_0^\infty V^s \mathcal{P}(f_m(V))dV, \ m = \overline{1,M}, \ s = 0,1,\ldots.$$

From (18) we have

$$(20) \qquad \mathcal{P}(\mu_s^{(k)}) = \frac{1}{\tau L} \sum_{l=1}^{L} \sum_{(m,\theta) \in N(k,l)} \theta^{-s} \mu_s^{(m)}, \quad k = \overline{1,M}, \quad s = 0,1,\ldots.$$

Introducing column-vectors $\bar{\mu}_s$ and $\mathcal{P}(\bar{\mu}_s)$ with the components $\mu_s^{(m)}$ and $\mathcal{P}(\mu_s^{(m)})$, $m = \overline{1,M}$, respectively, equalities (20) can be written as

$$(21) \qquad \mathcal{P}(\bar{\mu}_s) = P_s \bar{\mu}_s, \quad s = 0,1,\ldots,$$

where P_s is a $M \times M$ matrix with the elements

$$(P_s)_{km} = \frac{1}{\tau L} \sum_{l=1}^{L} \sum_{\{\theta | (m,\theta) \in N(k,l)\}} \theta^{-s}.$$

Note that (17) implies

$$\sum_{k=1}^{M} (P_0)_{km} = \frac{1}{\tau L} \sum_{l=1}^{L} \tau(m,l) = 1, \quad m = \overline{1,M},$$

that is the matrix P_0 is stochastic.

Consider the moments of the density functions f and $\mathcal{P}(f)$:

$$\mu_s = \int_0^\infty V^s f(V) dV = \sum_{m=1}^{M} \mu_s^{(m)}, \quad s = 0,1,\ldots,$$

and

$$\mathcal{P}(\mu_s) = \int_0^\infty V^s \mathcal{P}(f)(V) dV, \quad s = 0,1,\ldots.$$

If $\mathcal{P}(f)$ is given by (3), then we have

$$(22) \qquad \mathcal{P}(\mu_s) = \nu_{s+1} \mu_s, \quad s = 0,1,\ldots,$$

where

$$\nu_s = \int_0^\infty \eta^s \psi(\eta) \, d\eta, \ s = 0,1,\ldots.$$

On the other hand, if $\mathcal{P}(f)$ is given by (19), then we obtain

$$(23) \qquad \mathcal{P}(\mu_s) = \sum_{m=1}^{M} \mathcal{P}(\mu_s^{(m)}), \ s = 0,1,\ldots,$$

where $\mathcal{P}(\mu_s^{(m)})$, $m = \overline{1,M}$, are defined by (20) or, equivalently, by (21). Moment transformation (22) is a special case of (23). Indeed, if $P_s \bar{\mu}_s = \nu_{s+1} \bar{\mu}_s$, $s = 0,1,\ldots$, then (22) and (23) coincide.

2.7. Main objectives. The first issue we address in this paper is the construction of geometric partition models and the study of the moment sequences generated by the dominant eigenvalues λ_s of the corresponding matrices P_s, $s = 0,1,\ldots$.

We show that under some natural conditions the sequences $\lambda_s^{-N} P_s^N \bar{\mu}_s$ tend to eigenvectors $\hat{\bar{\mu}}_s$ of the matrices P_s as N goes to infinity. This implies that

$$\sum_{m=1}^{M} \mathcal{P}^N(\mu_s^{(m)}) \approx \lambda_s^N \mu_s,$$

whenever N is big enough, that is, after many partitions the transformation of the moments is described (approximately) by (22) and $\nu_{s+1} = \lambda_s$. This observation allows us to

obtain the partition operator representation (3) from geometrical partition models, but formula (3) should be understood in some generalized sense.

A rigorous theory of the grinding equation is the second main objective of this work. We show that the equation has a unique solution in an appropriate space.

3. Geometric Partition Models

We shall consider geometric partition models satisfying the following conditions:

(C1): The matrices P_s, $s = 0, 1, \ldots$, have the form

$$P_s = pI + \eta^s(D + \hat{P}_s),$$

where $p \in [0, 1[$, $\eta \in]0, 1]$, I is the identity matrix, D is a diagonal matrix with the elements $d_m \geq 0$ such that $\max_{m=\overline{1,M}} d_m = d_{\hat{m}} > d_m$, $m \neq \hat{m}$, $\hat{P}_s = \hat{P}(\eta_1^s, \ldots, \eta_K^s)$, $\eta_k \in]0, 1[$, $k = \overline{1, K}$, and $\hat{P}(z_1, \ldots, z_K)$ is a matrix with the elements

$$(\hat{P})_{ij} = \sum_{k=1}^{K} a_{ijk} z_k, \quad a_{ijk} \geq 0, \quad i, j = \overline{1, M}, \quad k = \overline{1, K}.$$

(C2): There exists n such that the matrices $(D + \hat{P}(z_1, \ldots, z_K))^n$ are positive, whenever $z_k > 0$, $k = \overline{1, K}$.

3.1. Examples. Here we present three simple examples of geometric partition models. All these models satisfy conditions (C1) and (C2).

Example 1 Consider a two-dimensional dust composed of ellipses

$$E = \left\{ (x, y) \in \mathbb{R}^2 \ \middle| \ \frac{x^2}{a^2} + \frac{b^2}{b^2} \leq 1 \right\}$$

of two types: $a = b$ and $a = 2b$. Assume that there are three possible orientations of the ellipses with respect to the plane (in this case the line) Π_0: the normal vector to Π_0 is parallel to the axis $0x$, to a line between the axes $0x$ and $0y$, and to the axis $0y$. The corresponding geometric partition model is described by the following rules:

$$\mathcal{R}(1, l) = (2, 2, 1), \ l = \overline{1, 3},$$

$$\mathcal{R}(2, l) = \begin{cases} (1, 1, 1), & l = 1, \\ (2, 2, \gamma), & l = 2, \\ (2, 2, 1), & l = 3, \end{cases}$$

where $\gamma \in]0, 1[$. Transformation (18) of the functions f_1 and f_2 is given by

$$\mathcal{P}(f_1)(V) = \frac{1}{6}(4f_2(2V)),$$

$$\mathcal{P}(f_2)(V) = \frac{1}{6}(12f_1(2V) + 4f_2(2V) + (1 + 1/\gamma)f_2((1 + 1/\gamma)V) +$$
$$+ (1 + \gamma)f_2((1 + \gamma)V)).$$

Formula (21) takes the form

$$\mathcal{P}\begin{pmatrix} \mu_s^{(1)} \\ \mu_s^{(2)} \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 0 & 1/2^{s-1} \\ 3/2^{s-1} & 1/2^{s-1} + (\gamma/(1+\gamma))^s + 1/(1+\gamma)^s \end{pmatrix}\begin{pmatrix} \mu_s^{(1)} \\ \mu_s^{(2)} \end{pmatrix}$$

Condition (C1) is satisfied with $p = 0$, $\eta = 1/(1 + \gamma)$, and

$$D = \begin{pmatrix} 0 & 0 \\ 0 & 1/6 \end{pmatrix}$$

Obviously condition (C2) is satisfied with $n = 2$.

Example 2 Consider a three-dimensional dust composed of ellipsoids

$$E = \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle|\; \frac{x^2}{a^2} + \frac{b^2}{b^2} + \frac{z^2}{c^2} \leq 1 \right\}$$

of three types: $a = b = c$, $a = b = 2c$, and $a = 2b = 2c$. Assume that there are seven possible orientations of the ellipsoids with respect to the plane Π_0: the normal vector to the plane is parallel to the lines generated by the vectors $(0, 0, 1)$, $(0, 1, 0)$, $(1, 0, 0)$, (ξ, ζ, η), $(-\xi, \zeta, \eta)$, $(\xi, -\zeta, \eta)$, and $(-\xi, -\zeta, \eta)$. The corresponding geometric partition model is described by the following rules:

$$\mathcal{R}(1, l) = (2, 2, 1), \; l = \overline{1, 7},$$

$$\mathcal{R}(2, l) = (3, 3, 1), \; l = \overline{1, 7},$$

$$\mathcal{R}(3, l) = \begin{cases} (1, 1, 1), & l = 1, 2 \\ (3, 3, 1), & l = 3, \\ (2, 3, \gamma), & l = \overline{4, 7}, \end{cases}$$

where $\gamma \in\,]0, 1[$. Transformation (18) of the functions f_1, f_2, and f_3 is given by

$$\mathcal{P}(f_1)(V) = \frac{1}{14} (8 f_3(2V)),$$

$$\mathcal{P}(f_2)(V) = \frac{1}{14} (28 f_1(2V) + 4(1 + 1/\gamma) f_3((1 + 1/\gamma)V)),$$

$$\mathcal{P}(f_3)(V) = \frac{1}{14} (28 f_2(2V) + 4 f_3(2V) + 4(1 + \gamma) f_3((1 + \gamma)V)).$$

Formula (21) takes the form

$$\mathcal{P} \begin{pmatrix} \mu_s^{(1)} \\ \mu_s^{(2)} \\ \mu_s^{(3)} \end{pmatrix} = \frac{1}{14} \begin{pmatrix} 0 & 0 & 1/2^{s-2} \\ 7/2^{s-1} & 0 & 4\, (\gamma/(1 + \gamma))^s \\ 0 & 7/2^{s-1} & 1/2^{s-1} + 4/(1 + \gamma)^s \end{pmatrix} \begin{pmatrix} \mu_s^{(1)} \\ \mu_s^{(2)} \\ \mu_s^{(3)} \end{pmatrix}$$

In this case condition (C1) is satisfied with $p = 0$, $\eta = 1/(1 + \gamma)$, and

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2/7 \end{pmatrix}$$

It is easy to verify that condition (C2) is satisfied and $n = 4$.

Example 3 Consider a two-dimensional dust composed of triangles with the angles

$$(\pi/3, \pi/3, \pi/3), \; (\pi/2, \pi/3, \pi/6), \; \text{and} \; (\pi/6, \pi/6, 2\pi/3).$$

The first and the second vertices of the triangles belong to the axis $0x$. The third vertex is in the upper half-plane. Assume that there are twelve possible orientations of the triangles with respect to the plane (line) Π_0: the angle between the axes $0x$ and the

normal vector to Π_0 is equal to $(l-1)\pi/6$, $l = \overline{1, 12}$. The corresponding geometric partition model is described by the following rules:

$$\mathcal{R}(1, l) = \begin{cases} (0, 1, 0), & l \neq 4, 8, 12 \\ (2, 2, 1), & l = 4, 8, 12, \end{cases}$$

$$\mathcal{R}(2, l) = \begin{cases} (0, 2, 0), & l \neq 8, 9, 12 \\ (2, 2, 1/3), & l = 8, \\ (1, 3, 1), & l = 9, \\ (2, 3, 1/2), & l = 12, \end{cases}$$

$$\mathcal{R}(3, l) = \begin{cases} (0, 3, 0), & l \neq 3, 4, 5 \\ (3, 2, 1/2), & l = 3, 5, \\ (2, 2, 1), & l = 4. \end{cases}$$

The equality $m'(m, l) = 0$ in these rules implies that the particle with the shape m and the orientation l is not divided. In this model the parameter τ is equal to $5/4$. Transformation (18) of the functions f_1, f_2, and f_3 is given by

$$\mathcal{P}(f_1)(V) = \frac{1}{15}(9f_1(V) + 2f_2(2V)),$$

$$\mathcal{P}(f_2)(V) = \frac{1}{15}(12f_1(2V) + 9f_2(V) + 4f_2(4V) + (4/3)f_2((4/3)V) + 3f_2(3V) + \\ + 4f_3(2V) + 3f_3((3/2)V)),$$

$$\mathcal{P}(f_3)(V) = \frac{1}{15}(2f_2(2V) + (3/2)f_2((3/2)V) + 9f_3(V) + 6f_3(3V)).$$

Formula (21) takes the form

$$\mathcal{P}\begin{pmatrix} \mu_s^{(1)} \\ \mu_s^{(2)} \\ \mu_s^{(3)} \end{pmatrix} = \frac{1}{15}\begin{pmatrix} 9 & 1/2^s & 0 \\ 6/2^s & 9 + 1/4^s + (3/4)^s + 1/3^s & 2/2^s + 2(2/3)^s \\ 0 & 1/2^s + (2/3)^s & 9 + 2/3^s \end{pmatrix}\begin{pmatrix} \mu_s^{(1)} \\ \mu_s^{(2)} \\ \mu_s^{(3)} \end{pmatrix}$$

Condition (C1) is satisfied with $p = 3/5$, $\eta = 3/4$, and

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1/15 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Obviously condition (C2) is satisfied with $n = 2$.

3.2. Auxiliary results. Recall the following corollary of the Frobenius theorem [2].

3.1. Theorem. *Let A be a non-negative $M \times M$ matrix. Assume that there exists n such that the matrix A^n is positive. Then there exists a simple eigenvalue $\lambda > 0$ of A (called the dominant eigenvalue of A) corresponding to an eigenvector with positive coordinates and such that $\lambda > |\lambda'|$ for any eigenvalue $\lambda' \neq \lambda$ of A.*

Let $\bar{\mu} = (\mu^{(1)}, \ldots, \mu^{(M)}) \in C^M$ and $\bar{\nu} = (\nu^{(1)}, \ldots, \nu^{(M)}) \in C^M$ be complex vectors. The inner product is denoted by $\langle \bar{\mu}, \bar{\nu} \rangle$. The norm is defined as $|\bar{\mu}| = \sum_{m=1}^{M} |\mu^{(m)}|$. If A is a matrix, then the transposed matrix is denoted by A^T.

Set $\bar{z} = (z_1, \ldots, z_K) \in C^K$. From condition (C2) and Theorem 3.1 we see that the matrices $(D + \hat{P}(\bar{z}))$ and $(D + \hat{P}^T(\bar{z}))$ have a dominant eigenvalue $\mu(\bar{z})$, whenever $z_k > 0$, $k = \overline{1, K}$. Denote by $\hat{\mu}(\bar{z})$ and $\check{\mu}(\bar{z})$ the corresponding eigenvectors satisfying

$|\hat{\mu}(\bar{z})| = |\check{\mu}(\bar{z})| = 1$. If $\bar{z} = 0$, then we have $\hat{\mu}(0) = \check{\mu}(0) = \check{\mu}$, where the vector $\check{\mu}$ has the components $\check{\mu}^m = 0$, $m \neq \hat{m}$, and $\check{\mu}^{\hat{m}} = 1$ (see condition (C1)).

Suppose that $z_k > 0$, $k = \overline{1, K}$. Then the subspace $E(\bar{z}) = \{\bar{e} \mid \langle \check{\mu}(\bar{z}), \bar{e} \rangle = 0\}$ satisfies $(D + \hat{P}(\bar{z}))E(\bar{z}) \subset E(\bar{z})$, and its dimension is equal to $M - 1$. Consider a vector $\bar{v} = (v^{(1)}, \ldots, v^{(M)}) \in C^M$. Then the linear system

$$(24) \qquad \begin{cases} \alpha \hat{\mu}(\bar{z}) + \bar{e} = \bar{v}, \\ \langle \check{\mu}(\bar{z}), \bar{e} \rangle = 0 \end{cases}$$

has a unique solution $(\alpha(\bar{z}, \bar{v}), \bar{e}(\bar{z}, \bar{v}))$. The scalar $\alpha(\bar{z}, \bar{v})$ can be written as

$$(25) \qquad \alpha(\bar{z}, \bar{v}) = \sum_{m=1}^{M} \alpha^{(m)}(\bar{z}) v^{(m)}.$$

3.2. Proposition. *The functions $\mu(\bar{z})$, $\hat{\mu}(\bar{z})$, and $\alpha^{(m)}(\bar{z})$, $m = \overline{1, M}$, are analytic in a neighborhood of zero.*

Proof. 1. Consider the characteristic equation

$$\Delta(\mu, \bar{z}) = \det(D + \hat{P}(\bar{z}) - \mu I) = 0.$$

Obviously

$$\left. \frac{\partial \Delta(\mu, 0)}{\partial \mu} \right|_{\mu = d_{\hat{m}}} = - \prod_{m \neq \hat{m}} (d_m - d_{\hat{m}}) \neq 0.$$

By the Implicit Function theorem [4] the function $\mu(\bar{z})$ is analytic in a neighborhood of zero.

2. Denote by I_*, D_*, and $\hat{P}_*$ the $(M - 1) \times M$ matrices obtained after the elimination of the $\hat{m}$-th row from the matrices I, D, and $\hat{P}$, respectively. Consider the equation

$$\Phi(\bar{\mu}, \bar{z}) = \begin{pmatrix} (D_* + \hat{P}_*(\bar{z}) - \mu(\bar{z}) I_*) \bar{\mu} \\ \sum_{m=1}^{M} \mu^{(m)} - 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Since

$$\det \left. \frac{\partial \Phi(\bar{\mu}, 0)}{\partial \bar{\mu}} \right|_{\bar{\mu} = \check{\mu}} = \pm \prod_{m \neq \hat{m}} (d_m - d_{\hat{m}}) \neq 0,$$

by the Implicit Function theorem the function $\hat{\mu}(\bar{z})$ is analytic in a neighborhood of zero. Analogously one can prove that the function $\check{\mu}(\bar{z})$ is analytic in a neighborhood of zero.

3. Consider the equation

$$\Phi(\alpha, \bar{e}, \bar{z}, \bar{v}) = \begin{pmatrix} \alpha \hat{\mu}(\bar{z}) + \bar{e} - \bar{v}, \\ \langle \check{\mu}(\bar{z}), \bar{e} \rangle \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Let $(\alpha_0, \bar{e}_0)$ be a solution to system (24) with $\bar{z} = 0$. Since

$$\det \left. \frac{\partial \Phi(\alpha, \bar{e}, \bar{z}, \bar{v})}{\partial(\alpha, \bar{e})} \right|_{(\alpha, \bar{e}, \bar{z}, \bar{v}) = (\alpha_0, \bar{e}_0, 0, \bar{v})} = (-1)^{M+1} \neq 0,$$

and the functions $\hat{\mu}(\bar{z})$ and $\check{\mu}(\bar{z})$ are analytic in a neighborhood of zero, by the Implicit Function theorem the functions $\alpha^{(m)}(\bar{z})$ are analytic in a neighborhood of zero. $\qquad \square$

3.3. The Condition (S). In the sequel we shall consider numerical sequences $\{q_s\}_{s \geq 0}$ satisfying the following condition:

  (S): There exist an integer $\hat{s} \geq 0$ and numbers $1 > \xi_1 > \xi_2 > \ldots > 0$ such that

  $$q_s = \sum_{j \geq 0} q^{(j)} \xi_j^s, \; s \geq \hat{s},$$

  and the series converges absolutely.

Obviously the sum and the product of two sequences satisfying condition (S) satisfy condition (S).

3.3. Proposition. *Assume that conditions (C1) and (C2) are satisfied. Then sequences of the dominant eigenvalues λ_s of the matrices P_s and of the components of the corresponding eigenvectors $\hat{\bar{\mu}}_s$, $|\hat{\bar{\mu}}_s| = 1$, $s = 0, 1, \ldots$, satisfy condition (S). Moreover, if the components of vectors $\bar{v}_s$, $s = 0, 1, \ldots$, form sequences satisfying condition (S), then the sequence α_s of solutions to systems (24) with $\bar{v} = \bar{v}_s$, $s = 0, 1, \ldots$, satisfies condition (S).*

Proof. Since by Proposition 3.2 the functions $\mu(\bar{z})$, $\hat{\mu}^{(m)}(\bar{z})$, and $\alpha^{(m)}(\bar{z})$, $m = \overline{1, M}$, are analytic in a neighborhood of zero and

$$\lambda_s = p + \eta^s \mu(\eta_1^s, \ldots, \eta_K^s), \; \hat{\bar{\mu}}_s = \hat{\bar{\mu}}(\eta_1^s, \ldots, \eta_K^s),$$

and

$$\alpha_s = \sum_{m=1}^{M} \alpha^{(m)}(\eta_1^s, \ldots, \eta_K^s) v_s^{(m)}$$

(see (25)), we obtain the result. $\square$

Note that in the case of Examples 1-3, using the formulae for the roots of the corresponding characteristic polynomials of the matrices P_s, we can found explicitly the representation

$$\lambda_s = p + \eta^s \sum_{m_1, \ldots, m_K} a_{m_1, \ldots, m_K} (\eta_1^{m_1} \ldots \eta_K^{m_K})^s, \; s \geq \hat{s}.$$

4. The Main Operators

In this section we give a correct definition of the partition operator and the classifier. To this end we have to introduce appropriate spaces, which are a generalization of the space of functions of bounded variation. Since the sizes of the particles forming a dust are limited, it suffices to consider particles with the volumes $V \in [0, 1]$.

4.1. The Spaces. The sequence $\{\lambda_s\}_{s=0}^{\infty}$ of the dominant eigenvalues of the matrices P_s may not be a moment sequence of a function of bounded variation. But it can be associated with an element of a larger space, which we construct as a dual space to a subspace of the space of continuous functions. Denote by $C(0, 1)$ the space of continuous functions on the interval $[0, 1]$ and by $BV(0, 1)$ the space of functions of bounded variation on the interval $[0, 1]$. Consider the space

$$X(n) = \{h(x) = a_0 + a_1 x + \ldots + a_{n-1} x^{n-1} + x^n \tilde{h}(x) \mid \tilde{h} \in C(0, 1)\}$$

with the norm

$$|h|_n = |a_0| + \ldots + |a_{n-1}| + |\tilde{h}|_{C(0,1)}.$$

It is easy to see that $X(n) \cong \mathbb{R}^n \times C(0, 1)$ is a Banach space. Its conjugate space is

$$X^*(n) \cong \mathbb{R}^n \times BV(0, 1) = \{\Psi^* = (\mu_0, \ldots, \mu_{n-1}, \tilde{\Psi}^*) \mid \mu_k \in \mathbb{R}, \; \tilde{\Psi}^* \in BV(0, 1)\},$$

and the elements $\Psi^* \in X^*(n)$ and $h \in X(n)$ satisfy the identity

$$\langle \Psi^*, h \rangle = a_0 \mu_0 + \ldots + a_{n-1} \mu_{n-1} + \int_0^1 \tilde{h}(x) d\tilde{\Psi}^*(x).$$

For example, we have

$$\langle \Psi^*, x^k \rangle = \mu_k, \ k = \overline{0, n-1}, \ \text{and} \ \langle \Psi^*, x^k \rangle = \int_0^1 x^{k-n} \, d\tilde{\Psi}^*(x), \ k \geq n.$$

The norm of a functional $\Psi^* \in X^*(n)$ is given by

$$|\Psi^*| = \max \left\{ |\mu_0|, \ldots, |\mu_{n-1}|, \bigvee_0^1 (\tilde{\Psi}^*) \right\},$$

where $\bigvee_0^1 (\tilde{\Psi}^*)$ is the total variation of the function $\tilde{\Psi}^*$. Obviously $X(n+1) \subset X(n)$ and $X^*(n+1) \supset X^*(n)$.

4.1. Proposition. *Assume that the sequence $\{q_s\}_{s \geq 0}$ satisfies condition (S). Let $k \geq 0$ be an integer. Then there exists $\Psi^* \in X^*(\hat{s} + k)$ such that $q_s = \langle \Psi^*, x^{s+k} \rangle$, $s = 0, 1, \ldots$.*

Proof. The function $\tilde{\Psi}^*$ given by

$$(26) \qquad \tilde{\Psi}^*(x) = \begin{cases} \sum_{\{j \geq 1 | \xi_j < x\}} q^{(j)} \xi_j^{\hat{s}+k}, & x \in]0, 1[, \\ \sum_{j \geq 0} q^{(j)}, & x = 1, \end{cases}$$

belongs to the space $BV(0,1)$ (the series converges absolutely). Setting

$$\Psi^* = (q_{-k}, \ldots, q_{-1}, q_0, \ldots, q_{\hat{s}-1}, \tilde{\Psi}^*),$$

where q_{-s}, $s = \overline{1, k}$, are arbitrary numbers, we get the result. $\qquad \square$

4.2. The Partition Operator. Suppose that functions $\Psi^*, F^* \in BV(0, 1)$ are continuously differentiable, $(\Psi^*)' = \psi^*$ and $(F^*)' = f^*$. Then for any $h \in C(0, 1)$ from (3) we have

$$\langle \mathcal{P}(f^*), h \rangle = \int_0^1 \int_0^1 f^* \left(\frac{V}{\eta} \right) \psi^*(\eta) \, d\eta h(V) \, dV.$$

Since $\operatorname{supp} f^* \subset [0, 1]$, this can be rewritten as

$$\int_0^1 \int_V^1 f^* \left(\frac{V}{\eta} \right) \psi^*(\eta) \, d\eta h(V) dV = \int_0^1 \int_0^\eta f^* \left(\frac{V}{\eta} \right) h(V) dV \, \psi^*(\eta) \, d\eta.$$

After the change of variables $V = \eta W$ we obtain

$$\langle \mathcal{P}(f^*), h \rangle = \int_0^1 \int_0^1 f^*(W) \eta h(\eta W) dW \psi^*(\eta) \, d\eta = \langle \Psi^*, \phi_{F^*, h} \rangle,$$

where $\phi_{F^*, h}(\eta) = \eta \langle F^*, h_\eta \rangle$ and $h_\eta(w) = h(\eta W)$. This observation allows to give a correct definition of the partition operator $\mathcal{P}_{\Psi^*} : X^*(n) \to X^*(n)$, associated to a functional $\Psi^* \in X^*(n+1)$.

4.2. Lemma. *Consider $F^* \in X^*(n)$ and $h \in X(n)$. Then $\phi_{F^*, h} \in X(n+1)$.*

Proof. Let $F^* = (\mu_0, \ldots, \mu_{n-1}, \tilde{F}^*)$ and $h(x) = a_0 + a_1 x + \ldots + a_{n-1} x^{n-1} + x^n \tilde{h}(x)$. Then we have

$$(27) \qquad \phi_{F^*, h}(\eta) = \eta \langle F^*, h_\eta \rangle = \eta a_0 \mu_0 + \ldots + \eta^n a_{n-1} \mu_{n-1} + \eta^{n+1} \int_0^1 \tilde{h}(\eta x) \, d\tilde{F}^*(x).$$

It suffice to prove that the function $\int_0^1 \tilde{h}(\eta x)\, d\tilde{F}^*(x)$ is continuous. Since the function $\tilde{h}: [0,1] \to \mathbb{R}$ is continuous, for any $\epsilon > 0$ there exists $\delta > 0$ such that $|\tilde{h}(x) - \tilde{h}(y)| < \epsilon$, whenever $|x - y| < \delta$. Hence if $|\eta - \xi| < \delta$, we get

$$\left| \int_0^1 (\tilde{h}(\eta x) - \tilde{h}(\xi x))\, d\tilde{F}^*(x) \right| \leq \sup_{x \in [0,1]} |\tilde{h}(\eta x) - \tilde{h}(\xi x)| \bigvee_0^1 (\tilde{F}^*) < \epsilon \bigvee_0^1 (\tilde{F}^*).$$

$\square$

Let $\Psi^* \in X^*(n+1)$, $F^* \in X^*(n)$, and $h \in X(n)$. Set

$$\langle \mathcal{P}_{\Psi^*}(F^*), h \rangle = \langle \Psi^*, \phi_{F^*, h} \rangle,$$

where $\phi_{F^*, h}(\eta) = \eta \langle F^*, h_\eta \rangle$ and $h_\eta(w) = h(\eta W)$. From Lemma 4.2 we see that this equality defines a functional $\mathcal{P}_{\Psi^*}(F^*) \in X^*(n)$.

4.3. Proposition. *The operator $\mathcal{P}_{\Psi^*}: X^*(n) \to X^*(n)$ is linear and continuous. Its norm does not exceed $|\Psi^*|_{X^*(n+1)}$.*

Proof. The linearity of this operator is obvious. Let $\Psi^* = (\lambda, \lambda_0, \ldots, \lambda_{n-1}, \tilde{\Psi}^*)$, $F^* = (\mu_0, \ldots, \mu_{n-1}, \tilde{F}^*)$, and $h(x) = a_0 + a_1 x + \ldots + a_{n-1} x^{n-1} + x^n \tilde{h}(x)$. Then using (27) we have

$$|\langle \mathcal{P}_{\Psi^*}(F^*), h \rangle| = |\langle \Psi^*, \phi_{F^*, h} \rangle|$$
$$= \left| a_0 \lambda_0 \mu_0 + \ldots + a_{n-1} \lambda_{n-1} \mu_{n-1} + \int_0^1 \int_0^1 \tilde{h}(\eta x)\, d\tilde{F}^*(x)\, d\tilde{\Psi}^*(\eta) \right|$$
$$\leq |\Psi^*|_{X^*(n+1)} |F^*|_{X^*(n)} |h|_{X(n)}.$$

This implies $|\mathcal{P}_{\Psi^*}| \leq |\Psi^*|_{X^*(n+1)}$. $\square$

Let $\Psi^* = (\lambda, \lambda_0, \ldots, \lambda_{n-1}, \tilde{\Psi}^*)$ and $F^* = (\mu_0, \ldots, \mu_{n-1}, \tilde{F}^*)$. Since

$$\phi_{F^*, x^s}(\eta) = \eta \langle F^*, (\eta y)^s \rangle = \eta^{s+1} \langle F^*, y^s \rangle = \eta^{s+1} \mu_s, \quad s = 0, 1, \ldots,$$

we have

$$\langle \mathcal{P}_{\Psi^*}(F^*), x^s \rangle = \langle \Psi^*, \phi_{F^*, x^s} \rangle = \langle \Psi^*, \eta^{s+1} \rangle \mu_s = \lambda_s \mu_s = \nu_{s+1} \mu_s, \quad s = 0, 1, \ldots,$$

where $\mu_s = \langle F^*, x^s \rangle$ and $\nu_s = \langle \Psi^*, x^s \rangle$, $s = 0, 1, \ldots$, that is, the operator $\mathcal{P}_{\Psi^*}$ satisfies (22).

4.3. Asymptotic behavior of the grinding process. Consider functions $f_m: [0,1] \to \mathbb{R}$, $m = \overline{1, M}$, describing the initial volume distributions of the particles with the shape m (see Section 2). Set $\bar{f}(V) = (f_1(V), \ldots, f_M(V))$. Then we have $f(V) = |\bar{f}(V)|$ and

$$\bar{\mu}_s = \int_0^1 V^s \bar{f}(V) dV.$$

Obviously

$$(28) \qquad |\bar{\mu}_s| = \int_0^1 V^s |\bar{f}(V)| dV \leq \int_0^1 f(V) dV = 1.$$

Below we obtain an asymptotic representation for the vectors $P_s^N \bar{\mu}_s$ when N goes to infinity. We obtain an especially interesting result if the components of the moment vectors $\bar{\mu}_s$, $s = 0, 1, \ldots$, satisfy condition (S). This take place if, for example, initially we have only particles of one shape, say shape 1, with the same volume V_0. In this case $\bar{f}(V) = (\delta(V - V_0), 0, \ldots, 0)$ and $\bar{\mu}_s = (V_0^s, 0, \ldots, 0)$.

4.4. Theorem. *Assume that conditions (C1) and (C2) are satisfied. Let p in condition (C1) be zero. Then there exists a bounded sequence $\alpha_s \geq 0$ such that*

$$(29) \qquad P_s^N \bar{\mu}_s = (\tilde{\mu}_s \eta^s)^N (\alpha_s \hat{\bar{\mu}}_s + r_s^{(N)}),$$

where $|r_s^{(N)}| \leq c e^{-\tau N}$, $\tau > 0$, and $\tilde{\mu}_s = \mu(\eta_1^s, \ldots, \eta_K^s)$. Moreover, if the components of the moment vectors $\bar{\mu}_s$, $s = 0, 1, \ldots$, satisfy condition (S), then there exist $n \geq 0$, $F^* \in X^*(n)$, and $\Psi^* \in X^*(n+1)$ such that $\langle F^*, x^s \rangle = \alpha_s$ and $\langle \Psi^*, x^{s+1} \rangle = \tilde{\mu}_s \eta^s = \lambda_s$, $s = 0, 1, \ldots$.

Remark This theorem implies that $\mathcal{P}^N(f) \approx \mathcal{P}_{\Psi^*}^N(F^*)$ when N is large enough. This justifies formula (3) for the partition operator, but this formula should be understood in the general form introduced above. A knowledge of the dominant eigenvalues of the matrices P_s allows us to compute the functional Ψ^*, and therefore to find the partition operator $\mathcal{P}_{\Psi^*}$.

Proof of Theorem 4.4. Setting $\bar{v} = \bar{\mu}_s$ in (24) we see that there exist α_s and $\bar{e}_s$ such that

$$(30) \qquad \bar{\mu}_s = \alpha_s \hat{\bar{\mu}}_s + \bar{e}_s,$$

$$(31) \qquad \langle \breve{\bar{\mu}}_s, \bar{e}_s \rangle = 0.$$

To show that the sequence α_s is bounded, suppose that there exists a subsequence s_i such that $\lim_{i \to \infty} \alpha_{s_i} = \infty$. Then multiplying (30) by $\breve{\bar{\mu}}_s$, dividing by α_s, and using (31), we obtain $\langle \bar{\mu}_s, \breve{\bar{\mu}}_s \rangle / \alpha_s = \langle \hat{\bar{\mu}}_s, \breve{\bar{\mu}}_s \rangle$. Since $\lim_{s \to \infty} \hat{\bar{\mu}}_s = \lim_{s \to \infty} \breve{\bar{\mu}}_s = \breve{\mu}$, using (28), setting $s = s_i$, and passing to the limit as i goes to infinity, we get $0 = \langle \breve{\mu}, \breve{\mu} \rangle$, a contradiction. Thus the sequence α_s is bounded. Combining (28) and (30), we see that the sequence $\bar{e}_s$ is also bounded.

Applying P_s^N to (30), we obtain (29) with

$$r_s^{(N)} = \tilde{\mu}_s^{-N} (D + \hat{P}_s)^N \bar{e}_s.$$

Since the eigenvalue $\tilde{\mu}_s$ is dominant, the modules of the eigenvalues of the linear operator $\tilde{\mu}_s^{-1}(D + \hat{P}_s)\big|_{E_s}$, where $E_s = E(\eta_1^s, \ldots, \eta_K^S)$, are less than 1. Hence there exist norms $\| \cdot \|_s$ in $\mathbb{R}^M$ such that $\|r_s^{(N)}\|_s \leq \gamma_s^N \|\bar{e}\|_s$, $\gamma_s \in [0, 1[$, $s, N = 0, 1, \ldots$. Since there are constants c_s satisfying $|\cdot| \leq c_s \| \cdot \|_s$, $s = 0, 1, \ldots$, we see that $|r_s^{(N)}|$, $s = 0, 1, \ldots$, tend to zero as N goes to infinity. Dividing (29) by $(\tilde{\mu}_s \eta^s)^N$ and passing to the limit as N goes to infinity, we obtain

$$\lim_{N \to \infty} \tilde{\mu}_s^{-N} (D + \hat{P}_s) \bar{\mu}_s = \alpha_s \hat{\bar{\mu}}.$$

Since the components of the vectors $\bar{\mu}_s$ are non-negative, this implies that all the numbers α_s, $s = 0, 1, \ldots$, are non-negative.

Set $E = \{\bar{e} \mid \langle \breve{\mu}, \bar{e} \rangle = 0\}$. Obviously if $\bar{e} \in E$, then we have

$$(32) \qquad |d_{\hat{m}}^{-1} D \bar{e}| \leq \gamma |\bar{e}|,$$

where $\gamma = \max_{m \neq \hat{m}} d_m / d_{\hat{m}} < 1$.

We show that there exists s_0 such that for all $\bar{e} \in E_s$, $s \geq s_0$ the inequality

$$|\tilde{\mu}_s^{-1} (D + \hat{P}_s) \bar{e}| \leq \frac{1 + \gamma}{2} |\bar{e}|$$

holds. Suppose that there exist a subsequence s_i, $i = 1, 2, \ldots$, and a sequence $\bar{e}_{s_i} \in E_{s_i}$, $|\bar{e}_{s_i}| = 1$, such that

$$(33) \qquad |\tilde{\mu}_{s_i}^{-1} (D + \hat{P}_{s_i}) \bar{e}_{s_i}| > \frac{1 + \gamma}{2}.$$

Without loss of generality the sequence $\bar{e}_{s_i}$ converges to a vector $\bar{e} \in E$, $|\bar{e}| = 1$. Passing to the limit in (33) as i goes to infinity, we obtain

$$|d_{\hat{m}}^{-1} D\bar{e}| \geq \frac{1+\gamma}{2}.$$

This contradicts (32). Thus we have $|r_s^{(N)}| \leq ce^{-\tau N}$, where

$$\tau = \ln \max \left\{ \frac{1+\gamma}{2}, \gamma_0, \ldots, \gamma_{s_0} \right\}$$

and

$$c = \max \left\{ \sup_{s > s_0} |\bar{e}_s|, c_0 \|\bar{e}_0\|, \ldots, c_{s_0} \|\bar{e}_{s_0}\| \right\}.$$

Assume that the components of the moment vectors $\bar{\mu}_s$, $s = 0, 1, \ldots$, satisfy condition (S). Then by Proposition 3.3 the sequences $\{\alpha_s\}_{s \geq 0}$ and $\{\lambda_s\}_{s \geq 0}$ satisfy condition S. Hence Proposition 4.1 implies the existence of $n \geq 0$, $F^* \in X^*(n)$, and $\Psi^* \in X^*(n+1)$ such that $\langle F^*, x^s \rangle = \alpha_s$ and $\langle \Psi^*, x^{s+1} \rangle = \tilde{\mu}_s \eta^s = \lambda_s$, $s = 0, 1, \ldots$. $\hspace{1cm}\square$

4.4. The Classifier. Suppose that a function $F^* \in BV(0,1)$ is continuously differentiable and $(F^*)' = f^*$. Let $\alpha \in]0, 1[$. Then for any $h \in C(0,1)$ we have

$$\langle \mathcal{C}_\alpha(F^*), h \rangle = \int_\alpha^1 f^*(V)h(V)\, dV = \langle F^*, h\chi_{[\alpha,1]} \rangle,$$

where

$$\chi_{[\alpha,1]}(x) = \begin{cases} 1, & x \in [\alpha, 1], \\ 0, & x \notin [\alpha, 1]. \end{cases}$$

We use this identity to introduce a definition of the classifier operator $\mathcal{C}_\alpha : X^*(n) \to X^*(n)$.

Denote by $B(0,1)$ the space of bounded functions on the interval $[0,1]$ with the sup-norm. Consider the space

$$Y(n) = \{h(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} + x^n \tilde{h}(x) \mid \tilde{h} \in B(0,1)\}$$

with the norm

$$|h|_{Y(n)} = |a_0| + \cdots + |a_{n-1}| + |\tilde{h}|_{B(0,1)}.$$

Clearly $X(n) \subset Y(n)$. For example $Y(n)$ contains all functions of the form $h(x)\chi_{[\alpha,1]}(x)$. Let $F^* \in X^*(n)$. By the Hahn-Banach theorem, F^* can be considered as an element of the dual space $Y^*(n)$ and $|F^*|_{Y^*(n)} = |F^*|_{X^*(n)}$. Consider $\alpha \in]0, 1[$, $F^* \in X^*(n)$, and $h \in X(n)$. Introduce a linear operator $\mathcal{H}_\alpha : X(n) \to Y(n)$ defined by

$$\mathcal{H}_\alpha(h)(x) = x^n \tilde{h}(x)\chi_{[\alpha,1]}(x).$$

Define the classifier by

$$\langle \mathcal{C}_\alpha(F^*), h \rangle = \langle F^*, \mathcal{H}_\alpha(h) \rangle.$$

Obviously we have

$$\langle F^*, \mathcal{H}_\alpha(h) \rangle = \int_\alpha^1 \tilde{h}(x) d\tilde{F}^*(x).$$

4.5. Proposition. *The operator $\mathcal{C}_\alpha : X^*(n) \to X^*(n)$ is linear and continuous. Its norm is less than or equal to one.*

Proof. The linearity of this operator is obvious. Let $F^* = (\mu_0, \ldots, \mu_{n-1}, \tilde{F}^*)$ and $h(x) = a_0 + a_1 x + \ldots + a_{n-1} x^{n-1} + x^n \tilde{h}(x)$. Then we have

$$\left| \langle \mathcal{C}_\alpha(F*), h \rangle \right| = \left| \int_\alpha^1 \tilde{h}(x)\, d\tilde{F}^*(x) \right|$$

$$\leq \sup_{x \in [\alpha, 1]} |\tilde{h}(x)| \bigvee_\alpha^1 (\tilde{F}^*)$$

$$\leq |h|_{X(n)} |F^*|_{X^*(n)}.$$

This implies $|\mathcal{C}_\alpha| \leq 1$. □

5. The Grinding Equation

Let $\Psi^* \in X(n+1)$. We can write (5) and (7) as equalities in the space $X^*(n)$:

$$F^*_{\text{out}} = (\mathcal{I} - \mathcal{C}_\alpha)(G^*)$$

and

(34) $\qquad G^* = \mathcal{P}_{\Psi^*} \circ \mathcal{C}_\alpha(G^*) + F^*_{\text{in}}.$

Applying $\mathcal{C}_\alpha$ to (34) and setting $H^* = \mathcal{C}_\alpha(G^*)$, we obtain

(35) $\qquad H^* = \mathcal{C}_\alpha \circ \mathcal{P}_{\Psi^*}(H^*) + \mathcal{C}_\alpha(F^*_{\text{in}}).$

Applying $\mathcal{I} - \mathcal{C}_\alpha$ to (34) we have

$$F^*_{\text{out}} = \mathcal{P}_{\Psi^*}(H^*) - \mathcal{C}_\alpha \circ \mathcal{P}_{\Psi^*}(H^*) + (\mathcal{I} - \mathcal{C}_\alpha)(F^*_{\text{in}}),$$

that is, to find the output F^*_{out}, it suffices to solve equation (35).

5.1. Theorem. *Let $\Psi^* = (\lambda, \lambda_0, \ldots, \lambda_{n-1}, \tilde{\Psi}^*) \in X^*(n+1)$. Assume that*

(36) $\qquad \lim_{\sigma \uparrow 1} \int_\sigma^1 |d\tilde{\Psi}^*| = 0.$

Then there exists $m \geq 0$ such that equation (35) has one and only one solution $H^* \in X^*(n+m)$.

Proof. We show that the norm of the operator $\mathcal{C}_\alpha \circ \mathcal{P}_{\Psi^*} : X^*(n+m) \to X^*(n+m)$ is less than one, whenever m is big enough. The functional Ψ^* can be considered as an element of the space $X^*(n+m+1)$:

$$\Psi^* = \left(\lambda, \lambda_0, \ldots, \lambda_{n+m-1}, \int_0^x y^m d\tilde{\Psi}^*(y) \right),$$

where $\lambda_s = \langle \Psi^*, x^{s+1} \rangle$, $s = \overline{n, n+m-1}$. Let $F^* = (\mu_0, \ldots, \mu_{n+m-1}, \tilde{F}^*) \in X^*(n+m)$ and $h(x) = a_0 + \cdots + a_{n+m-1} x^{n+m-1} + x^{n+m} \tilde{h}(x) \in X(n+m)$. The functional $\mathcal{P}_{\Psi^*}(F^*) \in X^*(n+m)$ is given by

$$\mathcal{P}_{\Psi^*}(F^*) = (\lambda_0 \mu_0, \ldots, \lambda_{n+m-1} \mu_{n+m-1}, \tilde{\Phi}^*).$$

Therefore we have

(37) $\qquad \langle \mathcal{P}_{\Psi^*}(F^*), h \rangle = \sum_{j=0}^{n+m-1} \lambda_j \mu_j a_j + \int_0^1 \tilde{h}(y) d\tilde{\Phi}^*(y).$

The functional $\mathcal{C}_\alpha \circ \mathcal{P}_{\Psi^*}(F^*) \in X^*(n+m)$ has the form

$$\mathcal{C}_\alpha \circ \mathcal{P}_{\Psi^*}(F^*) = (0, \ldots, 0, \tilde{\Phi}^*(x) \chi_{[\alpha, 1]}(x)),$$

and the equality

$$(38) \qquad \langle \mathcal{C}_\alpha \circ \mathcal{P}_{\Psi^*}(F^*), h \rangle = \int_\alpha^1 x^m \tilde{h}(x) \, d\tilde{\Phi}^*(x)$$

holds. Since

$$\langle \mathcal{P}_{\Psi^*}(F^*), h \rangle = \sum_{j=0}^{n+m-1} \lambda_j \mu_j a_j + \int_0^1 \int_0^1 \eta^m \tilde{h}(\eta x) \, d\tilde{F}^*(x) d\tilde{\Psi}^*(\eta)$$

$$= \sum_{j=0}^{n+m-1} \lambda_j \mu_j a_j + \int_0^1 \eta^m \int_0^\eta \tilde{h}(y) \, d\tilde{F}^*\left(\frac{y}{\eta}\right) d\tilde{\Psi}^*(\eta),$$

from (37) and (38) we obtain

$$\int_\alpha^1 x^m \tilde{h}(x) \, d\tilde{\Phi}^*(x) = \int_\alpha^1 \eta^m \int_\alpha^\eta \tilde{h}(y) \, d\tilde{F}^*\left(\frac{y}{\eta}\right) d\tilde{\Psi}^*(\eta)$$

$$= \int_\alpha^1 \eta^m \int_{\alpha/\eta}^1 \tilde{h}(\eta x) \, d\tilde{F}^*(x) \, d\tilde{\Psi}^*(\eta).$$

Thus we have

$$\langle \mathcal{C}_\alpha \circ \mathcal{P}_{\Psi^*}(F^*), h \rangle = \int_\alpha^1 \eta^m \int_{\alpha/\eta}^1 \tilde{h}(\eta x) \, d\tilde{F}^*(x) \, d\tilde{\Psi}^*(\eta)$$

$$\leq |\tilde{h}|_{C(0,1)} \bigvee_0^1 (\tilde{F}^*) \int_\alpha^1 \eta^m |\, d\tilde{\Psi}^*(\eta)|$$

$$(39) \qquad\qquad \leq |h|_{X(n+m)} |F^*|_{X^*(n+m)} \int_\alpha^1 \eta^m |\, d\tilde{\Psi}^*(\eta)|.$$

We have that $\lim_{m\to\infty} \int_\alpha^1 \eta^m |\, d\tilde{\Psi}^*(\eta)| = 0$. Indeed, if $\sigma \in]\alpha, 1[$, then we have

$$\int_\alpha^1 \eta^m |\, d\tilde{\Psi}^*(\eta)| \leq \sigma^m \int_\alpha^\sigma |\, d\tilde{\Psi}^*(\eta)| + \int_\sigma^1 |d\tilde{\Psi}^*(\eta)|$$

$$(40) \qquad\qquad \leq \sigma^m \bigvee_\alpha^1 (\tilde{\Psi}^*) + \int_\sigma^1 |d\tilde{\Psi}^*(\eta)|.$$

Consider a number $\epsilon > 0$. From (36) we see that there exists $\sigma \in]\alpha, 1[$ such that the second integral in (40) is less than $\epsilon/2$. Hence sum (40) is less than ϵ, whenever m is big enough. From (39) we see that the norm of the operator $\mathcal{C}_\alpha \circ \mathcal{P}_{\Psi^*} : X^*(n+m) \to X^*(n+m)$ is less than one, if m is big enough. Therefore there exists $m \geq 0$ such that equation (35) has one and only one solution $H^* \in X^*(n+m)$. $\qquad\square$

Consider a geometric partition model satisfying conditions (C1) and (C2). Assume that p in condition (C1) is equal to zero. Then from Proposition 4.1 we see that there exist a functional $\Psi^* \in X^*(n+1)$ such that $\lambda_s = \langle \Psi^*, x^{s+1} \rangle$, where λ_s, $s = 0, 1, \ldots$, is the sequence of the dominant eigenvalues λ_s of the matrices P_s. The condition $p = 0$ implies that $q^{(0)} = 0$ in (26). Therefore condition (36) is satisfied, and we get the following result.

5.2. Theorem. *Assume that conditions (C1) and (C2) are satisfied. Let p in condition (C1) be zero. Then there exist a functional $\Psi^* \in X^*(n+1)$ such that $\lambda_s = \langle \Psi^*, x^{s+1} \rangle$, where λ_s, $s = 0, 1, \ldots$, is the sequence of the dominant eigenvalues of the matrices P_s, and a number m such that equation (35) has one and only one solution $H^* \in X^*(n+m)$.*

The condition $p = 0$ essential in Theorems 4.4 and 5.2 is not restrictive. Indeed, if $p > 0$, then we have $\Psi^* = \Psi_0^* + p\Psi_1^*$, where $\Psi_0^* = (\lambda, \lambda_0, \ldots, \lambda_{n-1}, \tilde{\Psi}_0^*)$ and $\Psi_1^* = (1, \ldots, 1, \tilde{\Psi}_1^*)$. The function $\tilde{\Psi}_0^*$ satisfies condition (36) and

$$\tilde{\Psi}_1^* = \begin{cases} 0, & x \in [0, 1[, \\ 1, & x = 1. \end{cases}$$

Equation (35) takes the form

$$H^* = \mathcal{C}_\alpha \circ \mathcal{P}_{\Psi_0^*}(H^*) + p\mathcal{C}_\alpha \circ \mathcal{P}_{\Psi_1^*}(H^*) + \mathcal{C}_\alpha(F_{\mathrm{in}}^*).$$

From this, after simple calculations we have

$$(1 - p)H^* = \mathcal{C}_\alpha \circ \mathcal{P}_{\Psi_0^*}(H^*) + \mathcal{C}_\alpha(F_{\mathrm{in}}^*).$$

Dividing by $(1 - p)$, we obtain an equation with a partition operator satisfying the conditions of Theorems 4.4 and 5.2:

$$H^* = \mathcal{C}_\alpha \circ \mathcal{P}_{(1-p)^{-1}\Psi_0^*}(H^*) + (1 - p)^{-1}\mathcal{C}_\alpha(F_{\mathrm{in}}^*).$$

6. Conclusion

We have shown that the grinding process can be described in term of partition operators $\mathcal{P}_{\Psi^*}$ and classifier operators $\mathcal{C}_\alpha$ defined in the spaces $X^*(n)$. The dynamics of the grinding process is approximately described as $\mathcal{P}^N(f) \approx \mathcal{P}_{\Psi^*}^N(F^*)$ whenever N is big enough. Here $F^* \in X^*(n)$ is a generalized distribution function. This implies that the input and the output of a grinding system are related (approximately) by the grinding equation. The equation has a unique solution in an appropriate space $X^*(n)$. The study of geometric partition models, especially the study of the dominant eigenvalues of the matrices P_s, allows one to compute the functional Ψ^*, and therefore to determine the partition operator $\mathcal{P}_{\Psi^*}$.

Let us mention possible applications of the theory developed above. A grinding system can be composed of various grinders and classifiers. The grinding equation is an adequate mathematical model, which allows us to predict the results of grinding and construct grinding systems with desired properties.

An experimental verification of mathematical models describing grinding processes is rather difficult because different instruments used in particle size measurement give different results [1]. Usually sizers interpret all particles as spheres independently of the particle shapes. For example, if a sizer uses laser light scattering, then the results of the scattered radiation measurements are interpreted as the far-field diffraction pattern of an assembly of spheres. The moment analysis can help to recover the 'true' particle size distributions from the results of the measurements.

Consider a particle of volume V and the shape m having an orientation $\omega \in \Omega$, where Ω is a set of possible orientations of particles in the sizer. Assume that the sizer interprets the particle as a ball of volume $T(m, \omega)V$. Then the density function measured by the device is given by

$$f^{\mathrm{b}}(V) = \sum_{m=1}^{M} \int_\Omega f_m\left(\frac{V}{T(m, \omega)}\right) h_m(\omega)\, d\omega,$$

where h_m are the density functions of the random value ω, the orientation of a particle of the shape m. It is easy to calculate

$$\mu_s^{\mathrm{b}} = \int_0^\infty V^s f^{\mathrm{b}}(V) dV$$

$$= \sum_{m=1}^M \int_\Omega \int_0^\infty V^s f_m \left(\frac{V}{T(m,\omega)} \right) dV \, h_m(\omega) \, d\omega$$

$$= \sum_{m=1}^M \theta_{s+1}^{(m)} \mu_s^{(m)},$$

where

$$\theta_s^{(m)} = \int_\Omega T^s(m,\omega) h_m(\omega) \, d\omega.$$

As we know from Theorem 4.4, the vector $\bar{\mu}_s$ (see (21)) approximately has the form $\bar{\mu}_s \approx \mu_s \hat{\bar{\mu}}_s$, where $\hat{\bar{\mu}}_s$ is an eigenvector of the matrix P_s with the coordinates $\hat{\mu}_s^{(m)}$, $m = \overline{1,M}$, satisfying $\sum_m \hat{\mu}_s^{(m)} = 1$. Therefore

$$\mu_s^{\mathrm{b}} = \mu_s \sum_{m=1}^M \theta_{s+1}^{(m)} \hat{\mu}_s^{(m)}.$$

From this we can find the moments μ_s and estimate the density function f.

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