

Hyperconnectedness and extremal disconnectedness in (a) topological spaces

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Abstract

The aim of this paper is to study hyperconnectedness and extremal disconnectedness in a space equipped with a countable number of topologies.

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1. Introduction

A bitopological space [7] is a nonempty set with two topologies. Kovár [8, 9] also studied the properties of a nonempty set equipped with three topologies. Datta and Roy Choudhuri [5], and Raut and Datta [13] introduced nontrivial infinitesimally small elements. With this, they defined a number system as an extension of real number system. Their study offers a natural framework for dealing with an infinite sequence of distinct topologies on a set. Also emergence of chaos in a deterministic system, in the theory of dynamical system, relates to an interplay of finite or infinite number of different topologies in the underlining set. All these matters motivated the authors to consider a countable number of topologies in (ω) topological spaces [2, 3, 4] and (\aleph_0) topological spaces [1]. In this paper, we introduce the notion of (a) topological spaces which is a set equipped with countable number of topologies. The notion of (a) topological spaces is more general than both the notions of (ω) topological spaces and (\aleph_0) topological spaces.

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Sarma [14] introduced the notion of pairwise extremal disconnectedness in bitopological spaces [7]. Among other results, she proved a result on pairwise extremal disconnectedness which is parallel to Urysohn's lemma on pairwise normal spaces. In [12], Mathew studied the hyperconnected topological spaces (Steen and Seebach [15]). In this paper, we study hyperconnectedness and extremal disconnectedness in the context of (a) topological spaces.

2. (a) topological spaces

2.1. Definition. If $\{\tau_n\}$ is a sequence of topologies on a set X , then the pair $(X, \{\tau_n\})$ is called an (a) topological space.

If there is no scope of confusion, we denote the (a) topological space $(X, \{\tau_n\})$, simply by X . Throughout the paper, N denotes the set of natural numbers. The elements of N are denoted by i, j, k, l, m, n etc. If a set G is open with respect to the topology τ_n i.e., if $G \in \tau_n$, then we say G is (τ_n) open. (τ_n) closed set, (τ_n) closure have the obvious meaning. The (τ_n) closure (resp. (τ_n) interior) of a set E is denoted by $(\tau_n)\text{cl}E$ (resp. $(\tau_n)\text{int}E$). Following Levine [11], we now introduce the following definitions.

2.2. Definition. Let X be an (a) topological space and let $m \neq n$. A set $A \subset X$ is said to be (m, n) semiopen if there exists a $U \in \tau_m$ such that $U \subset A \subset (\tau_n)\text{cl}U$.

Thus any (τ_m) open set is an (m, n) semiopen set for any $n \neq m$. The class of all (m, n) semiopen sets with $m \neq n$, is denoted by $SO_{(m,n)}(X)$. We write $SO(X) = \bigcup_{(m,n)} SO_{(m,n)}(X)$. A set belonging to $SO(X)$ is called an (a) semiopen set.

2.3. Definition. An (a) topology $\{\sigma_n\}$ on X is said to be stronger (resp. weaker) than an (a) topology $\{\tau_n\}$ if $\tau_n \subset \sigma_n$ (resp. $\sigma_n \subset \tau_n$) for each n . If, in addition, $\tau_n \neq \sigma_n$ for at least one n , then $\{\sigma_n\}$ is said to be strictly stronger (resp. weaker) than $\{\tau_n\}$.

2.4. Definition. An (a) topological space $(X, \{\tau_n\})$ with property P is said to be maximal (resp. minimal) with respect to P if for any other (a) topology $\{\sigma_n\}$ strictly stronger (resp. weaker) than $\{\tau_n\}$, the space $(X, \{\sigma_n\})$ can not have this property.

3. Hyperconnected spaces

Recall that a topological space (X, \mathcal{T}) is hyperconnected if the intersection of any two nonempty open sets is nonempty.

3.1. Definition. An (a) topological space $(X, \{\tau_n\})$ is said to be hyperconnected if for any two nonempty sets U and V with $U \in \tau_m, V \in \tau_n$ and $m \neq n$, we have $U \cap V \neq \emptyset$.

It follows that if X is hyperconnected, then for any nonempty set $U \in \tau_m, (\tau_n)\text{cl}U = X$ for any $n \neq m$.

3.2. Theorem. *Suppose for any (τ_m) open set G and (τ_n) open set $H, G \cap H \in \tau_l$ for some l . Then the (a) topological space $(X, \{\tau_n\})$ and the topological spaces $(X, \tau_n), n \in N$ are hyperconnected iff the set D of all nonempty (a) semiopen sets is a filter.*

Proof. Let the (a) topological space $(X, \{\tau_n\})$ and the topological spaces $(X, \tau_n), n \in N$ be hyperconnected and $A, B \in D$. Then there exist two pairs (m, n) and (k, l) with $m \neq n$ and $k \neq l$ such that for some $U \in \tau_m$ and $V \in \tau_k, U \subset A \subset (\tau_n)\text{cl}U, V \subset B \subset (\tau_l)\text{cl}V$. Since $(X, \{\tau_n\})$ and $(X, \tau_n), n \in N$ are hyperconnected, we have $U \cap V \neq \emptyset$. Also $U \cap V \in \tau_i$ for some i . Therefore $(\tau_j)\text{cl}(U \cap V) = X$ for any $j \neq i$. Hence $U \cap V \subset A \cap B \subset (\tau_j)\text{cl}(U \cap V)$. Thus $A \cap B \in D$. Now let A be a nonempty (a) semiopen set and $B \supset A$. Then for some (m, n) with $m \neq n$, there exists $U \in \tau_m$ such that

$U \subset A \subset (\tau_n)\text{cl}U$. But $(\tau_n)\text{cl}U = X$. Therefore $U \subset B \subset (\tau_n)\text{cl}U$. Hence B is a nonempty (a) semiopen set. Thus D is a filter.

Since for any n , a (τ_n) open set is an (a) semiopen set, the converse follows. \square

For a pair (m, n) , the union of an arbitrary number of (m, n) semiopen sets is an (m, n) semiopen set. If X is hyperconnected, then the intersection of a finite number of (m, n) semiopen sets is (m, n) semiopen. Hence in this case, the class $SO_{(m,n)}(X)$ forms a topology on X . Since the class $\{SO_{(m,n)}(X) \mid m, n \in N\}$ is countable, and any countable class can be represented as a sequence, we rewrite the class $\{SO_{(m,n)}(X) \mid m, n \in N\}$ as a sequence $\{S_k\}$. So $(X, \{S_k\})$ is an (a) topological space. From Theorem 3.2, it follows that if $(X, \{\tau_n\})$ is hyperconnected and if for $G \in \tau_m$ and $H \in \tau_n$, $G \cap H \in \tau_l$ for some l , then $(X, \{S_k\})$ is hyperconnected.

3.3. Corollary. *If $(X, \{\tau_n\})$ is maximal hyperconnected, then $\{\tau_n\} = \{S_k\}$.*

Levine [10] introduced the concept of a simple extension of a topological space. Let (X, \mathcal{P}) be a topological space. A family \mathcal{Q} of subsets of X is a simple extension of \mathcal{P} if \mathcal{Q} contains \mathcal{P} and there exists a $P \notin \mathcal{P}$ such that $\mathcal{Q} = \{G \cup (H \cap P) \mid G, H \in \mathcal{P}\}$.

3.4. Theorem. *If for any (τ_m) open set G and (τ_n) open set H , $G \cap H \in \tau_l$ for some l and if the space $(X, \{\tau_n\})$ is maximal hyperconnected and the space (X, τ_n) , $n \in N$ are hyperconnected, then the set D of all nonempty (a) semiopen sets is an ultrafilter.*

Proof. Since by Theorem 3.2, D is a filter, it is sufficient to show that for a nonempty set E , $X - E \in D$ if $E \notin D$. Let us suppose that $E \notin D$. Then $E \notin \bigcup_n \tau_n$. Let $\tau_n(E)$ denote a simple extension of τ_n . Then the space $(X, \{\tau_n(E)\})$ which is stronger than $(X, \{\tau_n\})$ is not hyperconnected. So for some nonempty set $U \in \tau_m(E)$ and nonempty set $V \in \tau_n(E)$ with $m \neq n$ we have $U \cap V = \emptyset$. By the definition of simple extension, $U = U_1 \cup (U_2 \cap E)$ and $V = V_1 \cup (V_2 \cap E)$ for some $U_i \in \tau_m$ and $V_i \in \tau_n$, $i = 1, 2$. Since $U \cap V = \emptyset$, $U_1 \cap V_1 = \emptyset$. But $(X, \{\tau_n\})$ is hyperconnected and so either $U_1 = \emptyset$ or $V_1 = \emptyset$. Suppose without loss of generality, $U_1 = \emptyset$. Now we consider the cases (i) $V_1 = \emptyset$, and (ii) $V_1 \neq \emptyset$.

Case (i): $V_1 = \emptyset$. Since $U, V \neq \emptyset$, we have $U_2 \neq \emptyset$, $V_2 \neq \emptyset$. Therefore $U_2 \cap V_2 \neq \emptyset$, since $(X, \{\tau_n\})$ is hyperconnected. Now

$$\begin{aligned} U \cap V &= \emptyset \\ \Rightarrow U_2 \cap V_2 \cap E &= \emptyset \\ \Rightarrow U_2 \cap V_2 &\subset X - E \\ \Rightarrow X - E &\in D, \text{ since } D \text{ is a filter.} \end{aligned}$$

Case (ii): $V_1 \neq \emptyset$. Since $U_2 \neq \emptyset$, we have $U_2 \cap V_1 \neq \emptyset$. Therefore $U_2 \cap V_1 \in D$. Again since $U \cap V = \emptyset$, we have $(U_2 \cap E) \cap V_1 = \emptyset \Rightarrow U_2 \cap V_1 \subset X - E$. Therefore $X - E \in D$. Thus D is an ultrafilter. \square

Recall that a topological space (X, \mathcal{F}) is a door space if for each $A \subset X$, either $A \in \mathcal{F}$ or $X - A \in \mathcal{F}$.

3.5. Definition. The space $(X, \{\tau_n\})$ is said to be a door space if for every subset E of X , there exists an n_0 such that either $E \in \tau_{n_0}$ or $X - E \in \bigcup_{n \neq n_0} \tau_n$.

3.6. Theorem. *If the (a) topological space $(X, \{\tau_n\})$ is door and hyperconnected, then $\bigcap \tau_n - \{\emptyset\}$ is a filter.*

Proof. Let $A, B \in \bigcap_n \tau_n - \{\emptyset\}$. Then A, B are nonempty (τ_n) open sets for all n and so $A \cap B \in \bigcap_n \tau_n$. Since $(X, \{\tau_n\})$ is hyperconnected, $A \cap B \neq \emptyset$. Thus $A \cap B \in \bigcap_n \tau_n - \{\emptyset\}$. Now suppose $B \supset A \in \bigcap_n \tau_n - \{\emptyset\}$. Suppose that $B \notin \bigcap_n \tau_n - \{\emptyset\}$. Then $B \notin \tau_{n_0} - \{\emptyset\}$

for some n_0 and so $X - B \in \bigcup_{n \neq n_0} \tau_n$, since $(X, \{\tau_n\})$ is door. So we have $A \in \tau_{n_0}$ and $A \cap (X - B) = \emptyset$, which contradicts the hyperconnectivity of $(X, \{\tau_n\})$. Hence $B \in \bigcap_n \tau_n - \{\emptyset\}$. Therefore, $\bigcap_n \tau_n - \{\emptyset\}$ is a filter. \square

3.7. Theorem. *If the (a)topological space $(X, \{\tau_n\})$ is door and hyperconnected and the topological spaces (X, τ_n) , $n \in N$ are door [6], then $\bigcap_n \tau_n - \{\emptyset\}$ is an ultrafilter.*

Proof. By Theorem 3.6, $\bigcap_n \tau_n - \{\emptyset\}$ is a filter. If E is a nonempty set with $E \notin \bigcap_n \tau_n - \{\emptyset\}$, then $E \notin \tau_{n_0} - \{\emptyset\}$ for some n_0 . Then $X - E \in \tau_{n_0}$, since (X, τ_{n_0}) is door. Therefore, $E \notin \bigcup_{n \neq n_0} \tau_n$, since $(X, \{\tau_n\})$ is hyperconnected. And so $E \notin \tau_n$ for all $n \neq n_0 \Rightarrow X - E \in \tau_n$ for all $n \neq n_0$, since (X, τ_n) , $n \in N$ are door. Therefore $X - E \in \bigcap_n \tau_n - \{\emptyset\}$. Hence $\bigcap_n \tau_n - \{\emptyset\}$ is an ultrafilter. \square

3.8. Theorem. *If $(X, \{\tau_n\})$ is door and hyperconnected, then $(X, \{\tau_n\})$ is minimal door and maximal hyperconnected.*

Proof. Let $(X, \{\sigma_n\})$ be a door space which is weaker than $(X, \{\tau_n\})$. Suppose that $G \in \tau_m - \sigma_m$. Then $X - G \in \bigcup_{n \neq m} \sigma_n \subset \bigcup_{n \neq m} \tau_n$. But this is not possible, since $(X, \{\tau_n\})$ is hyperconnected. Hence $\sigma_n = \tau_n$ for all n .

Now let $(X, \{\rho_n\})$ be a hyperconnected space stronger than $(X, \{\tau_n\})$. Suppose that $G \in \rho_m - \tau_m$. Then $X - G \in \bigcup_{n \neq m} \tau_n \subset \bigcup_{n \neq m} \rho_n$ which is not possible, since $(X, \{\rho_n\})$ is hyperconnected. Thus $\rho_n = \tau_n$ for all n . \square

The following example shows that for an (a)topological door space $(X, \{\tau_n\})$, the topological space (X, τ_n) may not be door even for a single n .

3.9. Example. Suppose R is the set of real numbers. Let τ_n be the topology on R , generated by the subbase $\{\emptyset\} \cup \{E \subset (-\infty, n) \mid 0 \in E\} \cup \{E \subset R \mid 0 \in E \text{ and } E \text{ is not bounded above}\}$. Then the (a)topological space $(X, \{\tau_n\})$ is door but for no n , the topological space (X, τ_n) is door.

3.10. Definition. A set E in an (a)topological space $(X, \{\tau_n\})$ is said to be $(n \neq n_0)$ dense if $\bigcap_{n \neq n_0} (\tau_n) \text{cl} E = X$.

3.11. Definition. The space $(X, \{\tau_n\})$ is said to be submaximal if for any n_0 , every $(n \neq n_0)$ dense subset is (τ_{n_0}) open.

3.12. Theorem. *If the space $(X, \{\tau_n\})$ is hyperconnected and submaximal, then it is maximal hyperconnected.*

Proof. Let $(X, \{\sigma_n\})$ be a hyperconnected space stronger than $(X, \{\tau_n\})$. Let G be a nonempty set belonging to σ_{n_0} for some n_0 . Then $(\sigma_n) \text{cl} G = X$ for all $n \neq n_0$. Hence $\bigcap_{n \neq n_0} (\sigma_n) \text{cl} G = X \Rightarrow \bigcap_{n \neq n_0} (\tau_n) \text{cl} G = X$. Therefore G is $(n \neq n_0)$ dense in $(X, \{\tau_n\})$ and so $G \in \tau_{n_0}$. Hence $\tau_{n_0} = \sigma_{n_0}$. Thus $\tau_n = \sigma_n$ for all n . \square

4. Extremely disconnected spaces

4.1. Definition. An (a)topological space $(X, \{\tau_n\})$ is said to be extremely disconnected if for any $G \in \tau_m$ and $H \in \tau_n$ with $G \cap H = \emptyset$ and $m \neq n$, there exist $k, l \in N$ with $m \neq k$, $n \neq l$ and $k \neq l$ such that for some (τ_k) closed set F and (τ_l) closed set K , we have $G \subset F$, $H \subset K$ and $F \cap K = \emptyset$.

If the sequence $\{\tau_n\}$ consists of two topologies \mathcal{P} and \mathcal{Q} only, and if the bitopological space $(X, \mathcal{P}, \mathcal{Q})$ is pairwise extremely disconnected [14], then the space $(X, \{\tau_n\})$ is extremely disconnected.

4.2. Theorem. *The (a)topological space $(X, \{\tau_n\})$ is extremally disconnected iff for any (τ_m) open set G and (τ_n) closed set C with $G \subset C$ and $m \neq n$, there exist $k, l \in N$ with $m \neq k, n \neq l$ and $k \neq l$ such that for some (τ_k) closed set F and (τ_l) open set U , we have $G \subset F \subset U \subset C$.*

Proof. Suppose the space X is extremally disconnected. Let us consider a (τ_m) open set G and a (τ_n) closed set C with $G \subset C$ and $m \neq n$. Then $X - C$ is a (τ_n) open set and $G \cap (X - C) = \emptyset$. Therefore there exist k, l with $m \neq k, n \neq l$ and $k \neq l$ such that for some (τ_k) closed set F and (τ_l) closed set K , we have $G \subset F, X - C \subset K, F \cap K = \emptyset \Rightarrow G \subset F \subset U \subset C$ where $U = X - K \in \tau_l$.

To prove the converse, suppose $G \in \tau_m$ and $H \in \tau_n$ with $G \cap H = \emptyset$ and $m \neq n$. Then $G \subset X - H$ and $X - H$ is (τ_n) closed. Therefore there exist k, l with $m \neq k, n \neq l$ and $k \neq l$ such that for some (τ_k) closed set F and (τ_l) open set U , we have $G \subset F \subset U \subset X - H$. If $K = X - U$, then K is (τ_l) closed, $H \subset K$ and $F \cap K = \emptyset$. Thus the space is extremally disconnected. \square

4.3. Theorem. *The space $(X, \{\tau_n\})$ is extremally disconnected if for every m and for every $G \in \tau_m$, we have $(\tau_k)clG \in \tau_m$ for all $k \neq m$.*

Proof. Let $G \in \tau_m$ and $H \in \tau_n$ with $G \cap H = \emptyset$ and $m \neq n$. Then $((\tau_n)clG) \cap H = \emptyset$. Since $m \neq n$, by the given condition we have $(\tau_n)clG \in \tau_m$. Therefore $((\tau_n)clG) \cap ((\tau_m)clH) = \emptyset$. \square

Let \mathcal{T} denote the smallest topology on X containing τ_n for all n .

4.4. Theorem. *Let $(X, \{\tau_n\})$ be extremally disconnected. Then for any $G \in \tau_m$ and $H \in \tau_n$ with $m \neq n$ and $G \cap H = \emptyset$, there exists a function $f : X \rightarrow [0, 1]$ such that*

- (i) $f(G) = \{0\}, f(H) = \{1\}$
- (ii) f is (\mathcal{T}) continuous.

Proof. Since $G \subset X - H$ and $C = X - H$ is (τ_n) closed, by Theorem 4.2, there exist $k(\frac{1}{2}), l(\frac{1}{2}) \in N$ with $k(\frac{1}{2}) \neq m$ and $l(\frac{1}{2}) \neq n$ such that for some $(\tau_{k(\frac{1}{2})})$ closed set $F(\frac{1}{2})$ and $(\tau_{l(\frac{1}{2})})$ open set $U(\frac{1}{2})$ we have $G \subset F(\frac{1}{2}) \subset U(\frac{1}{2}) \subset C$. Again applying Theorem 4.2 to the pair $(G, F(\frac{1}{2}))$ and $(U(\frac{1}{2}), C)$ there exist $k(\frac{1}{4}), l(\frac{1}{4}), k(\frac{3}{4}), l(\frac{3}{4}) \in N$ such that for some $(\tau_{k(\frac{1}{4})})$ closed set $F(\frac{1}{4}), (\tau_{k(\frac{3}{4})})$ closed set $F(\frac{3}{4}), (\tau_{l(\frac{1}{4})})$ open set $U(\frac{1}{4})$ and $(\tau_{l(\frac{3}{4})})$ open set $U(\frac{3}{4})$ we have $G \subset F(\frac{1}{4}) \subset U(\frac{1}{4}) \subset F(\frac{1}{2}) \subset U(\frac{1}{2}) \subset F(\frac{3}{4}) \subset U(\frac{3}{4}) \subset C$ and $k(\frac{1}{4}) \neq m, l(\frac{1}{4}) \neq k(\frac{1}{2}), k(\frac{3}{4}) \neq l(\frac{1}{2}), l(\frac{3}{4}) \neq n$. By repeating the process, we obtain $t \in D = \{\frac{i}{2^j} \mid 0 < i < 2^j, i, j \in N\}$, a $(\tau_{k(t)})$ closed set $F(t)$ and a $(\tau_{l(t)})$ open set $U(t)$ with $k(t), l(t) \in N$ such that if $s, t \in D$ with $s < t$, then $F(s) \subset U(s) \subset F(t) \subset U(t)$ and $k(t) \neq l(s)$. If we take $F(0) = \emptyset$ and $U(1) = X$, then the above relation is true when s, t coincide with 0 or 1. For $t \neq 0, 1$, we have, $G \subset F(t) \subset U(t) \subset C$. Now we define the function $f : X \rightarrow [0, 1]$ by $f(x) = \sup\{t \mid x \notin U(t)\}$. Then $f(x) = 0$ for $x \in G$ and $f(x) = 1$ for $x \in H$. It is easy to verify that for $a \in (0, 1), \{x \in X \mid f(x) < a\} = \bigcup_{t < a} U(t)$ and $\{x \in X \mid f(x) > a\} = \bigcup_{t > a} (X - F(t))$. Since $U(t) \in \tau_{l(t)} \subset \mathcal{T}$ and $X - F(t) \in \tau_{k(t)} \subset \mathcal{T}$, it follows that f is (\mathcal{T}) continuous. \square

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