Hacettepe Journal of Mathematics and Statistics
Volume 34 S (2005), 1 – 6

Doğan Çoker
Memorial Issue

GENERALIZED SET-THEORETIC OPERATIONS ON GENUINE SETS

Mustafa Demirci*

Dedicated to the memories of Prof. Dr. Doğan Çoker and Dr. Fikri Gökdal.

Received 20 : 05 : 2003 : Accepted 31 : 10 : 2005

Abstract

The determination of set-theoretic operations on genuine sets, other than the operations $\subseteq^g, =^g, {}^{gc}, \overset{g}{\cap}$ and $\overset{g}{\cup}$ presented in the author's paper *Genuine sets* (Fuzzy Sets and Systems **105** (1999), 377-384), is proposed as an open question in his paper *Some notices on genuine sets* (Fuzzy Sets and Systems **110** (2000), 275-278). The present paper gives a desirable answer to this problem, and introduces a general technique for the construction of set-theoretic operations on genuine sets. Furthermore, two examples are designed to demonstrate two significant special classes of set-theoretic operations on genuine sets.

Keywords: Fuzzy sets, Fuzzy Logic, Various kinds of fuzzy set, Type-m fuzzy sets, Genuine sets.

2000 AMS Classification: 03 E 72, 03 B 52.

1. Introduction

The notion of genuine set was introduced in [2] with the aim of establishing a general theory of fuzzily defined objects. It was demonstrated in [4] that genuine sets can be used to describe various notions of fuzzy set; for example, intuitionistic fuzzy sets, interval valued fuzzy sets, type-m fuzzy sets, rough sets and fuzzy rough sets, within the same framework.

Not only do genuine sets unify various kinds of fuzzy set within the same framework, but they also allow us to model various kinds of uncertainties comprehensively [4]. A special interest to the topological aspects of genuine sets is paid in [1].

The set-theoretic operations play a significant role in the development of the theory of genuine sets. The criteria behind the selection of set-theoretic operations on genuine sets is introduced in [3], and the construction problem of set-theoretic operations other

*Akdeniz University, Faculty of Sciences and Arts, Department of Mathematics, 07058-Antalya, Turkey. E-Mail: demirci@akdeniz.edu.tr

than the set-theoretic operations $\subseteq^g, =^g, ^{gc}, \overset{g}{\cap}$ and $\overset{g}{\cup}$ given in [2], is proposed in [3]. In this paper, utilizing the results in [3], we introduce a general technique for the building of set-theoretic operations on genuine sets, and establish two useful special classes of set-theoretic operations on genuine sets that include $\subseteq^g, =^g, ^{gc}, \overset{g}{\cap}$ and $\overset{g}{\cup}$ as special cases.

2. Preliminaries

In the present paper, the notations $\mathbb{N}$, $\mathbb{N}^+$, X, I and I^X always denote the set of all natural numbers, the set of all positive integers, a nonempty set, the real unit interval $[0, 1]$ and the family of all fuzzy subsets of X, respectively.

In this section, we first recall some definitions and results presented in [2, 3], which will be needed later on. For $n \in \mathbb{N}^+$, an n-th order genuine sets A in X is characterized by the reality function $g_A : X \times I^n \to I$. The family of all n-th order genuine sets in X is denoted by G_n^X.

A fuzzy subset of X is a 0-th order genuine set in X with the reality function as its membership function, i.e. $G_0^X = I^X$.

The set-theoretic operations of equality, inclusion, complement, intersection and union for fuzzy subsets of X and n-th order genuine sets in X will be denoted by the symbols $=^f$, $\subseteq^f$, ^{fc}, $\cap^f$ and $\cup^f$; and by the symbols $=^n$, $\subseteq^n$, $^{c(n)}$, $\cap^n$ and $\cup^n$, respectively. The relation $=$ of absolute equality on G_n^X is defined in the usual manner, i.e.

$$A = B \iff g_A(x, \varphi_1, \varphi_2, \ldots, \varphi_n) = g_B(x, \varphi_1, \varphi_2, \ldots, \varphi_n),$$
$$\forall\, x \in X,\ \forall\, \varphi_1, \varphi_2, \ldots, \varphi_n \in I,\ \forall\, A, B \in G_n^X.$$

For an n-th order genuine set A in X ($n \in \mathbb{N}$), the $(n+1)$-th order identification of A, denoted by $I_n^{n+1}(A)$, is the $(n+1)$-th order genuine set in X whose reality function is given by

$$g_{I_n^{n+1}(A)}(x, \varphi_1, \varphi_2, \ldots, \varphi_{n+1}) = \begin{cases} 0 : & \varphi_{n+1} \neq g_A(x, \varphi_1, \varphi_2, \ldots, \varphi_n), \\ 1 : & \varphi_{n+1} = g_A(x, \varphi_1, \varphi_2, \ldots, \varphi_n), \end{cases}$$
$$\forall x \in X,\ \forall\, \varphi_1, \varphi_2, \ldots, \varphi_{n+1} \in I.$$

An n-th order genuine set A in X ($n \in \mathbb{N}$), and its $(n+1)$-th order identification $I_n^{n+1}(A)$ can regarded as being the same thing. Under this simple assumption, G_n^X becomes a subfamily of G_{n+1}^X.

2.1. Definition. Let Q be a class of some objects, and S a subclass of Q. Suppose that the set-theoretic operations of equality, inclusion, complement, intersection and union for the objects of Q and for the objects of S are defined and represented by the symbols $=^Q$, $\subseteq^Q$, ^{Qc}, $\bigcap^Q$, $\bigcup^Q$ and $=^S$, $\subseteq^S$, ^{Sc}, $\bigcap^S$ and $\bigcup^S$, respectively. Then, $(Q, =^Q, \subseteq^Q, ^{Qc}, \cap^Q, \cup^Q)$ is said to be an extension of $(S, =^S, \subseteq^S, ^{Sc}, \cap^S, \cup^S)$ iff

(i) $(\forall\, A, \forall\, B \in S)\,(A =^Q B \iff A =^S B)$,
(ii) $(\forall\, A, \forall\, B \in S)\,(A \subseteq^Q B \iff A \subseteq^S B)$,
(iii) $(\forall\, A, \forall\, B \in S)\,(A =^Q B^{Qc} \iff A =^S B^{Sc})$,
(iv) $(\forall\, A \in S)(\forall\, \{A_i : i \in J\} \subseteq S)(A =^Q \bigcap\limits_{i \in J}{}^Q A_i \iff A =^S \bigcap\limits_{i \in J}{}^S A_i)$,
(v) $(\forall\, A \in S)(\forall\, \{A_i : i \in J\} \subseteq S)(A =^Q \bigcup\limits_{i \in J}{}^Q A_i \iff A =^S \bigcup\limits_{i \in J}{}^S A_i)$.

If $(Q, =^Q, \subseteq^Q, ^{Qc}, \cap^Q, \cup^Q)$ is an extension of $(S, =^S, \subseteq^S, ^{Sc}, \cap^S, \bigcup^S)$, then $=^S$, $\subseteq^S$, ^{Sc}, $\cap^S$, $\cup^S$ can be replaced by $=^Q$, $\subseteq^Q$, ^{Qc}, $\cap^Q$, $\cup^Q$, respectively.

Definition 2.1 provides a reasonable criteria for the construction of set-theoretic operations $=^n$, $\subseteq^n$, $^{c(n)}$, $\cap^n$ and $\cup^n$ on G_n^X, in other words, it is reasonable to expect that for each $m, n \in \mathbb{N}$ with $m < n$, $(G_n^X, =^n, \subseteq^n, {}^{c(n)}, \cap^n, \cup^n)$ should be an extension of $(G_m^X, =^m, \subseteq^m, {}^{c(m)}, \cap^m, \cup^m)$. The question as to whether or not it may be of interest to consider additional conditions in this context is left open in the present paper.

For each $n \in \mathbb{N}$, if $=^n$, $\subseteq^n$, ${}^{c(n)}$, $\cap^n$ and $\cup^n$ are chosen in particular to be the set-theoretic operations $=^g$, $\subseteq^g$, ^{gc}, $\overset{g}{\cap}$ and $\overset{g}{\cup}$, respectively, of [2], it is shown in [3] that this expectation is satisfied. The determination of set-theoretic operations $=^n$, $\subseteq^n$, ${}^{c(n)}$, $\cap^n$ and $\cup^n$ other than $=^g$, $\subseteq^g$, ^{gc}, $\overset{g}{\cap}$ and $\overset{g}{\cup}$ is proposed in [3] as an open question, and is still somewhat problematic. In the forthcoming section, we are interested in the solution of this problem, and establish some satisfactory results in this direction.

3. Results

3.1. Theorem. *For each $m, n \in \mathbb{N}$ with $m < n$, $(G_n^X, =^n, \subseteq^n, {}^{c(n)}, \cap^n, \bigcup^n)$ is an extension of $(G_m^X, =^m, \subseteq^m, {}^{c(m)}, \cap^m, \cup^m)$ iff the following conditions are satisfied:*

(GG1) $\left(\forall A, B \in G_k^X\right)$ $(I_k^{k+1}(A) =^{k+1} I_k^{k+1}(B) \iff A =^k B)$,

(GG2) $\left(\forall A, B \in G_k^X\right)$ $(I_k^{k+1}(A) \subseteq^{k+1} I_k^{k+1}(B) \iff A \subseteq^k B)$,

(GG3) $\left(\forall A, B \in G_k^X\right)$ $(I_k^{k+1}(A) =^{k+1} (I_k^{k+1}(B))^{c(k+1)} \iff A =^k B^{c(k)})$,

(GG4) $(\forall A \in G_k^X)$ $(\forall \{A_i : i \in J\} \subseteq G_k^X)$,

$$\left(I_k^{k+1}(A) =^{k+1} \bigcap_{i\in J}^{k+1} I_k^{k+1}(A_i) \iff A =^k \bigcap_{i\in J}^{k} A_i\right),$$

(GG5) $(\forall A \in G_k^X)$ $(\forall \{A_i : i \in J\} \subseteq G_k^X)$,

$$\left(I_k^{k+1}(A) =^{k+1} \bigcup_{i\in J}^{k+1} I_k^{k+1}(A_i) \iff A =^k \bigcup_{i\in J}^{k} A_i\right),$$

for each $k \in \mathbb{N}$.

Proof. This is an immediate consequence of Theorem 2.1 and [3, Corollary 1]. □

3.2. Theorem. *For $n \in \mathbb{N}^+$ and $k = 1, \dots, n$, let $R^k : G_k^X \to G_{k-1}^X$ be a function satisfying $R^k(I_{k-1}^k(A)) = A$ for each $A \in G_{k-1}^X$. Let us denote by G^k the composition function $R^1 \circ R^2 \circ \cdots \circ R^k : G_k^X \to I^X$ for $k \in \mathbb{N}^+$ and the identity map $id_{I^X} : I^X \to I^X$ for $k = 0$. For each $k \in \mathbb{N}$, if the set-theoretic operations $\subseteq^k, =^k, {}^{c(k)}, \cap^k$ and $\cup^k$ on G_k^X are defined by the following equivalences:*

(i) $(\forall A, B \in G_k^X)$, $(A =^k B \iff G^k(A) =^f G^k(B))$,

(ii) $(\forall A, B \in G_k^X)$, $(A \subseteq^k B \iff G^k(A) \subseteq^f G^k(B))$,

(iii) $(\forall A, B \in G_k^X)$, $(A =^k B^{c(k)} \iff G^k(A) =^f (G^k(B))^{fc})$,

(iv) $(\forall A \in G_k^X)$ $(\forall \{A_i : i \in K\} \subseteq G_k^X)$,

$$\left(A =^k \bigcap_{i\in K}^{k} A_i \iff G^k(A) =^f \bigcap_{i\in K}^{f} G^k(A_i)\right),$$

(v) $(\forall A \in G_k^X)$ $(\forall \{A_i : i \in K\} \subseteq G_k^X)$,

$$\left(A =^k \bigcup_{i\in K}^{k} A_i \iff G^k(A) =^f \bigcup_{i\in K}^{f} G^k(A_i)\right),$$

then for each $m, n \in \mathbb{N}$ with $m < n$, $(G_n^X, =^n, \subseteq^n, {}^{c(n)}, \cap^n, \cup^n)$ is an extension of $(G_m^X, =^m, \subseteq^m, {}^{c(m)}, \cap^m, \cup^m)$.

Proof. To prove the required result, we invoke Theorem 3.1. Let us assume that the set-theoretic operations $\subseteq^k, =^k, {}^{c(k)}, \cap^k$ and $\cup^k$ on G_k^X are given by the equivalences (i–v), respectively. It is sufficient to see that the conditions (GG1–GG5) in Theorem 3.1 are satisfied. For each $k \in \mathbb{N}^+$ and for each $A, B \in G_k^X$, by considering the definition of the map G^k, the hypothesis on R^k, and using (i), we observe that

$$I_k^{k+1}(A) =^{k+1} I_k^{k+1}(B) \iff G^{k+1}(I_k^{k+1}(A)) =^f G^{k+1}(I_k^{k+1}(B).$$

However,

$$\begin{aligned} G^{k+1}(I_k^{k+1}(A)) &= [R^1 \circ R^2 \circ \cdots \circ R^k \circ R^{k+1}](I_k^{k+1}(A)) \\ &= [R^1 \circ R^2 \circ \cdots \circ R^k](R^{k+1}(I_k^{k+1}(A)) \\ &= [R^1 \circ R^2 \circ \cdots \circ R^k](A) \\ &= G^k(A), \end{aligned}$$

and likewise,

$$\begin{aligned} G^{k+1}(I_k^{k+1}(B)) &= [R^1 \circ R^2 \circ \cdots \circ R^k \circ R^{k+1}](I_k^{k+1}(B)) \\ &= [R^1 \circ R^2 \circ \cdots \circ R^k](R^{k+1}(I_k^{k+1}(B)) \\ &= [R^1 \circ R^2 \circ \cdots \circ R^k](B) \\ &= G^k(B). \end{aligned}$$

Hence

$$I_k^{k+1}(A) =^{k+1} I_k^{k+1}(B) \iff G^k(A) =^f G^k(B) \iff A =^k B.$$

Thus (GG1) is verified. In a similar fashion, one can easily deduce from (ii), (iii), (iv) and (v) the conditions (GG2), (GG3), (GG4) and (GG5), respectively. For this reason, the proof of the properties (GG2–GG5) is skipped here. □

As a consequence of Theorem 3, the construction problem for set-theoretic operations on genuine sets turns into the determination of mappings $R^k : G_k^X \to G_{k-1}^X$ fulfilling the condition $R^k(I_{k-1}^k(A)) = A$ for each $A \in G_{k-1}^X$ and for each $k \in \mathbb{N}^+$. The following examples are designed to demonstrate some special classes of mappings $R^k : G_k^X \to G_{k-1}^X$ satisfying the above condition.

3.3. Example. Let $h : I \to I$ be a bijective function with $h(0) = 0$ and $h(1) = 1$, and let T be a t-norm [5], i.e. $T : I \times I \to I$ is a mapping for which $([0,1], T)$ is a commutative monoid with identity element 1 and T satisfies the isotonicity:

$$((\alpha_1 \le \beta_1) \text{ and } (\alpha_2 \le \beta_2)) \implies T(\alpha_1, \alpha_2) \le T(\beta_1, \beta_2),\ \forall\, \alpha_1, \alpha_2, \beta_1, \beta_2 \in I.$$

It is easy to see that the map $R_h^k : G_k^X \to G_{k-1}^X$, $A \in G_k^X \mapsto R_h^k(A) \in G_{k-1}^X$, defined by

$$g_{R_h^k(A)}(x, \lambda_1, \lambda_2, \ldots, \lambda_{k-1}) = h^{-1}\left(\sup_{\lambda_k \in I} \left\{ T(h(g_A(x, \lambda_1,\ \lambda_2, \ldots, \lambda_k)), h(\lambda_k)) \right\} \right),$$
$$\forall\, k \in \mathbb{N}^+,\ \forall\, A \in G_k^X, \forall x \in X,\ \forall\, \lambda_1, \lambda_2, \ldots, \lambda_{k-1} \in I,$$

satisfies the equality $g_{R_h^k(I_{k-1}^k(A))} = g_A$, i.e. $R_h^k(I_{k-1}^k(A)) = A$ for each $k \in \mathbb{N}^+$ and for each $A \in G_{k-1}^X$. Let

$$G_h^k = \begin{cases} R_h^1 \circ R_h^2 \circ \cdots \circ R_h^k & \text{if } k \in \mathbb{N}^+ \\ id_{I^X} & \text{if } k = 0 \end{cases} \quad \forall k \in \mathbb{N}.$$

Theorem 3.2 gives us that for the operations $\subseteq^k$, $=^k$, ${}^{c(k)}$, $\cap^k$ and $\cup^k$ defined by the equivalences (i–v) in Theorem 3.2, and for each $m, n \in \mathbb{N}$ with $m < n$, we have that

$(G_n^X,\ =^n,\ \subseteq^n,\ ^{c(n)},\ \cap^n,\ \cup^n)$ forms an extension of $(G_m^X,\ =^m,\ \subseteq^m,\ ^{c(m)},\ \cap^m,\ \cup^m)$. For the particular t-norm $T = \min = \wedge$, if $h : I \to I$ is taken as an order isomorphism, we observe that

$$\begin{aligned} g_{R_h^k(A)}(x, \lambda_1, \lambda_2, \ldots, \lambda_{k-1}) &= h^{-1}\Bigg(\sup_{\lambda_k \in I} \Big\{ h(g_A(x, \lambda_1, \lambda_2, \ldots, \lambda_k)) \wedge h(\lambda_k) \Big\} \Bigg) \\ &= h^{-1}\Bigg(\sup_{\lambda_k \in I} \Big\{ h(g_A(x, \lambda_1, \lambda_2, \ldots, \lambda_k) \wedge \lambda_k) \Big\} \Bigg) \\ &= h^{-1}\Bigg(h\Bigg(\sup_{\lambda_k \in I} \Big\{ g_A(x, \lambda_1, \lambda_2, \ldots, \lambda_k) \wedge \lambda_k \Big\} \Bigg) \Bigg) \\ &= \sup_{\lambda_k \in I} \Big\{ g_A(x, \lambda_1, \lambda_2, \ldots, \lambda_k) \wedge \lambda_k \Big\}. \end{aligned}$$

Therefore we get

$$\begin{aligned} &g_{G_h^k(A)}(x) = sup_{\lambda_1, \lambda_2, \ldots, \lambda_k \in I} \{ g_A(x, \lambda_1, \lambda_2, \ldots, \lambda_k) \wedge \lambda_1 \wedge \lambda_2 \wedge \ldots \wedge \lambda_k \}, \\ &\qquad \forall\, k \in \mathbb{N}^+,\ \forall\, x \in X. \end{aligned}$$

Thus, for $T = \wedge$ and for an order isomorphism $h : I \to I$ fulfilling $h(0) = 0$ and $h(1) = 1$, the operations $\subseteq^k, =^k, ^{c(k)}, \cap^k$ and $\cup^k$ given by (i–v) in Theorem 3.2 are nothing but the operations $\subseteq^g, =^g, ^{gc}, \overset{g}{\cap}$ and $\overset{g}{\cup}$, respectively, presented in [2].

3.4. Example. As in the previous example, let $h : I \to I$ stand for a bijection satisfying the properties $h(0) = 0$ and $h(1) = 1$, and let S be a s-conorm [5], i.e. $S : I \times I \to I$ is a mapping for which $([0, 1], S)$ is a commutative monoid with identity element 0, and S satisfies the isotonicity property. Let us define the map $R_h^k : G_k^X \to G_{k-1}^X$, $A \in G_k^X \mapsto R_h^k(A) \in G_{k-1}^X$, by

$$\begin{aligned} g_{R_h^k(A)}(x, \lambda_1, \lambda_2, \ldots, \lambda_{k-1}) = 1 - h^{-1}\Bigg(\inf_{\lambda_k \in I} \Big\{ &S(h(1 - g_A(x, \lambda_1, \lambda_2, \ldots, \lambda_k)), \\ &h(1 - \lambda_k)) \Big\} \Bigg), \end{aligned}$$

$$\forall\, k \in \mathbb{N}^+,\ \forall\, A \in G_k^X,\ \forall\, x \in X,\ \forall\, \lambda_1, \lambda_2, \ldots, \lambda_{k-1} \in I.$$

In a similar fashion to Example 3.3, it is easy to observe that $g_{R_h^k(I_{k-1}^k(A))} = g_A$, i.e. $R_h^k(I_{k-1}^k(A)) = A$ for each $k \in \mathbb{N}^+$ and for each $A \in G_{k-1}^X$. Similarly, for the particular s-conorm $S = \max = \vee$ and for an order isomorphism h, we easily obtain

$$g_{R_h^k(A)}(x, \lambda_1, \lambda_2, \ldots, \lambda_{k-1}) = \sup_{\lambda_k \in I} \Big\{ g_A(x, \lambda_1, \lambda_2, \ldots, \lambda_k) \wedge \lambda_k \Big\},$$

and so we have

$$\begin{aligned} &g_{G_h^k(A)}(x) = \sup_{\lambda_1, \lambda_2, \ldots, \lambda_k \in I} \Big\{ g_A(x, \lambda_1, \lambda_2, \ldots, \lambda_k) \wedge \lambda_1 \wedge \lambda_2 \wedge \ldots \wedge \lambda_k \Big\}, \\ &\qquad \forall\, k \in \mathbb{N}^+,\ \forall\, x \in X. \end{aligned}$$

Hence, for $S = \vee$ and for an order isomorphism $h : I \to I$ fulfilling $h(0) = 0$ and $h(1) = 1$, the operations $\subseteq^k, =^k, ^{c(k)}, \cap^k$ and $\cup^k$ defined by (i–v) in Theorem 3.2 will also be the operations $\subseteq^g, =^g, ^{gc}, \overset{g}{\cap}$ and $\overset{g}{\cup}$, respectively.

Acknowledgment. The present work is supported by the Turkish Academy of Sciences under the framework of the Young Scientist Award Program (MD/TÜBA-GEBİP/2002-1-8).

References

[1] Çoker, D. and Demirci, M. *An invitation to topological structures on first order genuine sets*, Fuzzy Sets and Systems **119**, 521–527, 2001.

[2] Demirci, M. *Genuine sets*, Fuzzy Sets and Systems **105**, 377–384, 1999.

[3] Demirci, M. *Some notices on genuine sets*, Fuzzy Sets and Systems **110**, 275–278. 2000.

[4] Demirci, M. *Genuine sets, various kinds of fuzzy sets and fuzzy rough sets*, Int. J. Uncertainty, *Fuzziness and Knowledge-Based Systems* **11** (4), 467–494, 2003.

[5] Klement, E. P., Mesiar, R. and Pap, E. *Triangular norms* (Trends in Logic, Vol. 8, Kluwer Academic Publishers, Dordrecht, 2000).