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Non-selfadjoint matrix Sturm-Liouville operators with eigenvalue-dependent boundary conditions

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Abstract

In this paper we investigate discrete spectrum of the non-selfadjoint matrix Sturm-Liouville operator L generated in $L^2(\mathbb{R}_+, S)$ by the differential expression

$$\ell(y) = -y'' + Q(x)y \quad , \qquad x \in \mathbb{R}_+ : [0, \infty),$$

and the boundary condition $y\prime(0) - (\beta_0 + \beta_1\lambda + \beta_2\lambda^2)\, y(0) = 0$ where Q is a non-selfadjoint matrix valued function. Also using the uniqueness theorem of analytic functions we prove that L has a finite number of eigenvalues and spectral singularities with finite multiplicities.

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1. Introduction

The study of the spectral analysis of non self-adjoint Sturm-Liouville operators was begun by Naimark [23] in 1954. He studied the spectral analysis of non-selfadjoint differential operators with continuous and discrete spectrum. Also he investigated the existence of spectral singularities in the continuous spectrum of the non-selfadjoint differential operator. Spectral singularities are poles of the resolvent's kernel which are in the continuous spectrum and are not eigen-values [26]. General notion of the sets of spectral singularities for closed linear operators on a Banach space was given by Nagy in [22]. Let L_0 denote the operator generated in $L^2(\mathbb{R}_+)$ by the differential expression

$$\ell_0(y) = -y'' + v(x)y \quad , \qquad x \in \mathbb{R}_+ \tag{1.1}$$

and the boundary condition

$$y\prime(0) - hy(0) = 0$$

where v is a complex valued function and $h \in \mathbb{C}$.

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In [23] it is shown that if

$$\int_0^\infty \exp(\varepsilon x)\,|v(x)|\,dx < \infty,$$

for some $\varepsilon > 0$, then L_0 has a finite number of eigenvalues and spectral singularities with a finite multiplicities. Pavlov [25] established the dependence of the structure of the spectral singularities of L_0 on the behavior of the potential function at infinity. The spectral analysis of the non-selfadjoint operator, generated in $L^2(\mathbb{R}_+)$ by (1.1) and the integral boundary condition

$$\int_0^\infty B(x)\,y(x)\,dx + \alpha y'(0) - \beta y(0) = 0$$

where $B \in L^2(\mathbb{R}_+)$ is a complex-valued function, and $\alpha, \beta \in \mathbb{C}$, was investigated in detail by Krall [15],[16].

Some problems of spectral theory of differential and some other types of operators with spectral singularities were also studied in [1],[3]-[7],[17],[18]. The spectral analysis of the non self-adjoint operator, generated in $L^2(\mathbb{R}_+)$ by (1.1) and the boundary condition

$$\frac{y'(0)}{y(0)} = \alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2$$

where $\alpha_i \in \mathbb{C}$, $i = 0,\ 1,\ 2$ with $\alpha_2 \neq 0$ was investigated by Bairamov et al. [8].

The all above mentioned papers related with differential and difference operators are of scalar coefficients.Spectral analysis of the selfadjoint differential and difference operators with matrix coefficients are studied in [2],[9]-[11],[14].

Let S be a n-dimensional $(n < \infty)$ Euclidian space. We denote by $L^2(\mathbb{R}_+, S)$ the Hilbert space of vector-valued functions with values in S and the norm

$$\|f\|^2_{L_2(\mathbb{R}_+,S)} = \int_0^\infty \|f(x)\|^2_S\,dx.$$

Let L denote the operator generated in $L^2(\mathbb{R}_+, S)$ by the matrix differential expression

$$\ell(y) = -y'' + Q(x)\,y, \qquad x \in \mathbb{R}_+$$

and the boundary condition $y(0) = 0$, where Q is a non-selfadjoint matrix-valued function (i.e. $Q \neq Q^*$). In [24], [12] discrete spectrum of the non-selfadjoint matrix Sturm-Liouville operator was investigated. Let us consider the BVP in $L_2(\mathbb{R}_+, S)$

$$-y'' + Q(x)y = \lambda^2 y,\ x\epsilon\mathbb{R}_+, \tag{1.2}$$

$$y'(0) - \left(\beta_0 + \beta_1\lambda + \beta_2\lambda^2\right) y(0) = 0 \tag{1.3}$$

where Q is a non self-adjoint matrix-valued function and β_0, β_1, β_2 are non self-adjoint matrices with $\det\beta_2 \neq 0$.

In this paper using the uniqueness theorem of analytic functions we investigate the eigenvalues and the spectral singularities of L. In particular we prove that L has a finite number of eigenvalues and spectral singularities with finite multiplicities, if the condition

$$\lim_{x\to\infty} Q(x) = 0\ ,\ \int_0^\infty e^{\epsilon x}\left\|Q'(x)\right\|dx < \infty,\ \epsilon > 0,$$

holds, where $\|.\|$ denote norm in S. We also show that the analogue of the Pavlov condition for L is the form

$$\lim_{x\to\infty} Q(x) = 0\ ,\ \int_0^\infty e^{\epsilon\sqrt{x}}\left\|Q'(x)\right\| dx < \infty,\ \epsilon > 0.$$

2. Jost Solution

Let us consider the matrix Sturm-Liouville equation

$$-y'' + Q(x)\, y = \lambda^2 y,\ x \in \mathbb{R}_+ \tag{2.1}$$

where Q is a non-selfadjoint matrix-valued function and

$$\int_0^\infty x\left\|Q(x)\right\| dx < \infty \tag{2.2}$$

holds. The bounded matrix solution of (2.1) satisfying the condition

$$\lim_{x\to\infty} y(x,\lambda)\, e^{-i\lambda x} = I,\ \lambda \in \overline{\mathbb{C}}_+ := \{\lambda : \lambda \in \mathbb{C},\ \ \mathrm{Im}\,\lambda \geq 0\}$$

will be denoted by $F(x,\lambda)$. The solution $F(x,\lambda)$ is called Jost solution of (2.1). It has been shown that, under the condition (2.2), the Jost solution has the representation

$$F(x,\lambda) = e^{i\lambda x} I + \int_x^\infty K(x,t)\, e^{i\lambda t} dt \tag{2.3}$$

where I denotes the identity matrix in S and the matrix function $K(x,t)$ satisfies

(2.4)

$$K(x,t) = \frac{1}{2}\int_{\frac{x+t}{2}}^\infty Q(s)ds + \frac{1}{2}\int_x^{\frac{x+t}{2}}\int_{t+x-s}^{t+s-x} Q(s)K(s,v)dvds \quad + \frac{1}{2}\int_{\frac{x+t}{2}}^\infty \int_s^{t+s-x} Q(s)K(s,v)dvds.$$

$K(x,t)$ is continuously differentiable with respect to their arguments and

$$\left\|K(x,t)\right\| \leq c\alpha\left(\frac{x+t}{2}\right) \tag{2.5}$$

$$\left\|K_x(x,t)\right\| \leq \frac{1}{4}\left\|Q\left(\frac{x+t}{2}\right)\right\| + c\alpha\left(\frac{x+t}{2}\right) \tag{2.6}$$

$$\left\|K_t(x,t)\right\| \leq \frac{1}{4}\left\|Q\left(\frac{x+t}{2}\right)\right\| + c\alpha\left(\frac{x+t}{2}\right) \tag{2.7}$$

where $\alpha(x) = \int_x^\infty \left\|Q(s)\right\| ds$ and $c > 0$ is a constant. Therefore, $F(x,\lambda)$ is analytic with respect to λ in $\mathbb{C}_+ := \{\lambda : \lambda \in \mathbb{C}_+, \mathrm{Im}\,\lambda > 0\}$ and continuous on the real axis ([2], [17], [19]).

We will denote the matrix solution of (2.1) satisfying the initial conditions

$$G(0,\lambda) = I\ \ ,\quad G'(0,\lambda) = \beta_0 + \beta_1\lambda + \beta_2\lambda^2$$

by $G(x,\lambda)$. Let us define the following functions:

$$A_\pm(\lambda) = F_x(0,\pm\lambda) - (\beta_0 + \beta_1\lambda + \beta_2\lambda^2)F(0,\pm\lambda) \qquad \lambda\epsilon\bar{\mathbb{C}}_\pm, \tag{2.8}$$

where $\bar{\mathbb{C}}_{\pm} = \{\lambda : \lambda \in \mathbb{C}, \ \pm \operatorname{Im} \lambda \geq 0\}$. It is obvious that the functions $A_{+}(\lambda)$ and $A_{-}(\lambda)$ are analytic in $\mathbb{C}_{+}$ and $\mathbb{C}_{-}$, respectively and continuous on the real axis.It is clear that the resolvent of L defined by the following

$$\mathbf{R}_{\lambda}(L)\varphi = \int_0^{\infty} R(x,\xi;\lambda)\varphi(\xi)\,d\xi, \qquad \varphi \in L^2(\mathbb{R}_+, S) \tag{2.9}$$

where

$$R(x,\xi;\lambda) = \begin{cases} R_{+}(x,\xi;\lambda) & , \quad \lambda \in \mathbb{C}_{+} \\ R_{-}(x,\xi;\lambda) & , \quad \lambda \in \mathbb{C}_{-} \end{cases}$$

$$R_{\pm}(x,\xi;\lambda) = \begin{cases} -F(x,\pm\lambda)\,A_{\pm}^{-1}(\lambda)\,G^{t}(\xi,\lambda), & 0 \leq \xi \leq x \\ -G(x,\lambda)\left[A_{\pm}^{t}(\lambda)\right]^{-1} F(\xi,\pm\lambda), & x \leq \xi < \infty, \end{cases} \tag{2.10}$$

and $G^{t}(\xi,\lambda)$ and $A_{\pm}^{t}(\lambda)$ denotes the transpose of the matrix function $G(\xi,\lambda)$ and $A_{\pm}(\lambda)$ respectively.

In the following we will denote the class of non self-adjoint matrix-valued valued absolutely continuous functions in $\mathbb{R}_{+}$ by $AC(\mathbb{R}_{+})$.

2.1. Lemma. *If*

$$Q\epsilon AC(\mathbb{R}_{+}) \ , \quad \lim_{x\to\infty} Q(x) = 0 \ , \quad \int_0^{\infty} x^3 \left\|Q'(x)\right\| < \infty \tag{2.11}$$

then $K_{tt}(x,t)$ *exist and*

$$\begin{aligned} K_{tt}(x,t) = &-\frac{1}{8}Q'(\frac{t}{2}) + \frac{1}{2}\int_0^{\infty} Q(s)K_t(s,t+s)ds \\ &- \frac{1}{4}Q(\frac{t}{2})K(\frac{t}{2},\frac{t}{2}) \\ &- \frac{1}{2}\int_0^{\frac{t}{2}} Q(s)\left[K_t(s,t-s) + K_t\left(t-x+s\right)\right]ds. \end{aligned} \tag{2.12}$$

Proof. The proof of lemma direct consequently of (2.4). ∎

From (2.5)-(2.7) and (2.12) we obtain that

$$\|K_{tt}(0,t)\| \leq c\left\{\left\|Q'(\frac{t}{2})\right\| + t\left\|Q(\frac{t}{2})\right\| + t\alpha(\frac{t}{2}) + \alpha_1(\frac{t}{2})\right\} \tag{2.13}$$

holds, where $\alpha_1(t) = \int_t^{\infty} \alpha(s)\,ds$ and $c > 0$ is a constant.

2.2. Lemma. *Under the condition* (2.11)*,* A_{+} *and* A_{-} *have the representations*

$$A_{+}(\lambda) = -\beta_2\lambda^2 + A\lambda + B + \int_0^{\infty} F^{+}(t)e^{i\lambda t}dt, \quad \lambda \in \bar{\mathbb{C}}_{+}, \tag{2.14}$$

$$A_{-}(\lambda) = -\beta_2\lambda^2 + C\lambda + D + \int_0^{\infty} F^{-}(t)e^{-i\lambda t}dt, \quad \lambda \in \bar{\mathbb{C}}_{-}, \tag{2.15}$$

where $A, \ B, \ C, \ D$ *are non self-adjoint matrices in* S*, and* $F^{\pm} \in L_1(\mathbb{R}_{+})$.

Proof. Using (2.3), (2.4) and (2.8) we get (2.14), where

$$\begin{aligned} A &= i-\beta_1-i\beta_2 K(0,0), \\ B &= -K(0,0)-\beta_0-i\beta_1 K(0,0)+\beta_2 K_t(0,0), \\ F^{+}(t) &= K_x(0,t)-\beta_0 K(0,t)-i\beta_1 K_t(0,t)+\beta_2 K_{tt}(0,0). \end{aligned} \tag{2.16}$$

From (2.5) – (2.7) and (2.13), $F^{+}\in L_1(\mathbb{R}_+)$. By similar way we obtain (2.15) and $F^{-}\in L_1(\mathbb{R}_+)$. ∎

2.3. Theorem. *$A_+(\lambda)$ and $A_-(\lambda)$ have the asymptotic behavior:*

(2.17) $A_{\pm}(\lambda)=-\beta_2\lambda^2+A\lambda+B+o(1) \qquad \lambda\in\bar{\mathbb{C}}_{\pm},\ |\lambda|\to\infty.$

Proof. The proof is obvious from (2.5) – (2.7) and (2.13)). ∎

We will denote the continuous spectrum of L by σ_c. From Theorem 2 ([22], page 303) we get that

(2.18) $\sigma_c=\mathbb{R}.$

3. Eigenvalues and Spectral Singularities of L

Let us suppose that

(3.1) $f_{\pm}(\lambda):=\det A_{\pm}(\lambda).$

We denote the set of eigenvalues and spectral singularities of L by $\sigma_d(L)$ and $\sigma_{ss}(L)$, respectively. By the definition of eigenvalues and spectral singularities of differential operators we can write

(3.2) $\sigma_d(L)=\{\lambda{:}\lambda\in\mathbb{C}_+,\ \ f_+(\lambda)=0\}\cup\{\lambda{:}\lambda\in\mathbb{C},\ \ f_-(\lambda)=0\}$

(3.3) $\sigma_{ss}(L)=\{\lambda:\lambda\in\mathbb{R}\backslash\{0\},\ f_+(\lambda)=0\}\cup\{\lambda:\lambda\in\mathbb{R}\backslash\{0\},\ f_-(\lambda)=0\}$

[22], [23], [26]. It is clear that $\sigma_{ss}(L)\subset\mathbb{R}$.

3.1. Definition. The multiplicity of a zero of f_+ in $\overline{\mathbb{C}}_+$ (or f_- in $\overline{\mathbb{C}}_-$) is defined as the multiplicity of the corresponding eigenvalue and spectral singularity of L.

In order to investigate the quantitative properties of the eigenvalues and the spectral singularities of L, we need to discuss the quantitative properties of the zeros of f_+ and f_- in $\bar{\mathbb{C}}_+$ and $\bar{\mathbb{C}}_-$, respectively. Assume that

$$M_1^{\pm}=\{\lambda:\lambda\in\mathbb{C}_{\pm},\ f_{\pm}(\lambda)=0\}$$

and

$$M_2^{\pm}=\{\lambda:\lambda\in\mathbb{R},\ f_{\pm}(\lambda)=0\}.$$

From (3.3) and (3.4), we get

(3.4) $\sigma_d(L)=M_1^{+}\cup M_1^{-},$

and

(3.5) $\sigma_{ss}(L)=M_2^{+}\cup M_2^{-}-\{0\}.$

3.2. Theorem. *Under the condition* (2.11)

i) *The set $\sigma_d(L)$ is bounded and has at most countable number of elements and its limit points can lie only in a bounded subinterval of the real axis.*
ii) *The set $\sigma_{ss}(L)$ is bounded and $\mu(\sigma_{ss}(L))=0$, where $\mu(\sigma_{ss}(L))$ denotes the linear Lebesque measure of $\sigma_{ss}(L)$.*

Proof. Using (2.5) and (3.1) we get that the function $f_{\pm}$ is analytic in $\mathbb{C}_+$ continuous on the real axis and

$$f_{\pm}(\lambda) = -\lambda^2 \det \beta_2 + O(\lambda), \ \lambda \in \overline{\mathbb{C}}_{\pm}, |\lambda| \to \infty, \tag{3.6}$$

Equation (3.6) shows the boundedness of the sets $\sigma_d(L)$ and $\sigma_{ss}(L)$. From the analyticity of the function $f_{\pm}$ in $\mathbb{C}_{\pm}$ we obtain that $\sigma_d(L)$ has at most countable number of elements and its limit points can lie only in a bounded subinterval of the real axis . By the boundary value uniqueness theorem of analytic functions, we find that $\mu(\sigma_{ss}(L)) = 0$, [13]. ∎

We will denote the sets of limit points of M_1^+ and M_2^+ by M_3^+ and M_4^+ respectively and the set of all zeros of A_+ with infinite multiplicity in $\bar{\mathbb{C}}_+$ by M_5^+. Analogously define the sets M_3^-, M_4^- and M_5^-.

It is explicit from the boundary uniqueness theorem of analytic functions that [13]

$$\begin{aligned} &M_1^{\pm} \cap M_5^{\pm} = \varnothing, \quad M_3^{\pm} \subset M_2^{\pm}, \quad M_4^{\pm} \subset M_2^{\pm}, \\ &M_5^{\pm} \subset M_2^{\pm}, \quad M_3^{\pm} \subset M_5^{\pm}, \quad M_4^{\pm} \subset M_5^{\pm} \end{aligned} \tag{3.7}$$

and $\mu(M_3^{\pm}) = \mu(M_4^{\pm}) = \mu(M_5^{\pm}) = 0$.

3.3. Theorem. *If*

$$Q\epsilon AC(\mathbb{R}_+) \ \ , \ \lim_{x\to\infty} Q(x) = 0 \ \ , \ \int_0^\infty e^{\epsilon x} \left\|Q'(x)\right\| dx < \infty, \quad \epsilon > 0 \tag{3.8}$$

the operator L has a finite number of eigenvalues and spectral singularities and each of them is of finite multiplicity.

Proof. By (2.5) , (2.13) , (2.14) and (3.8) we observe that, the function A_+ has an analytic continuation to the half plane $\operatorname{Im}\lambda > -\frac{\varepsilon}{4}$. So, the limit points of zeros of A_+ in $\overline{\mathbb{C}}_+$ can not lie in $\mathbb{R}$. From analyticity of A_+ for $\operatorname{Im}\lambda > -\frac{\varepsilon}{4}$, we obtain that all zeros of A_+ in $\overline{\mathbb{C}}_+$ have a finite multiplicity. We obtain similar results for A_-. Consequently by (3.4) and (3.5) the sets $\sigma_d(L)$ and $\sigma_{ss}(L)$ have a finite number of elements with a finite multiplicity. ∎

Now let us suppose that hold, the conditions which is weaker than (3.8).

3.4. Theorem. *If*

$$Q\epsilon AC(\mathbb{R}_+) \ \ , \ \lim_{x\to\infty} Q(x) = 0 \ \ , \sup_{x\in\mathbb{R}_+} \left[\exp\left(\varepsilon\sqrt{x}\right) \left\|Q'(x)\right\|\right] < \infty, \ \ \varepsilon > 0 \tag{3.9}$$

holds , then $M_5^+ = M_5^- = \phi$.

Proof. From (3.1) and (3.9) we have f_+ is analytic in $\mathbb{C}_+$ and all of its derivatives are continuous on the $\overline{\mathbb{C}}_+$.For sufficiently large $P > 0$ we have

$$\left|\frac{d^m}{d\lambda^m} f_+(\lambda)\right| \le T_m, \quad m = 0, 1, 2, ..., \lambda \in \bar{\mathbb{C}}_+, |\lambda| < P \tag{3.10}$$

where

$$T_m := 2^m c \int_0^\infty t^m e^{-(\varepsilon/2)\sqrt{t}} dt, \ m = 0, 1, 2, ..., \tag{3.11}$$

where $c > 0$ is a constant. Since the function f_+ is not equal to zero identically, using Pavlov's Theorem [25] we get that M_5^+ satisfies

$$\int_0^a \ln G(s)\, d\mu\left(M_5^+, s\right) > -\infty \tag{3.12}$$

where $G(s) = \inf_m \frac{T_m s^m}{m!}$, $\mu\left(M_5^+, s\right)$ is the linear Lebesgue measure of s-neighborhood of M_5^+ and $a > 0$ is a constant .

We obtain the following estimates for T_m

(3.13) $\quad T_m \leq B b^m m! m^m$

where B and b are constants depending on c and ε. Substituting (3.13) in the definition of $G(s)$, we arrive at

$$G(s) = \inf_m \frac{T_m s^m}{m!} \leq B \exp\left(-e^{-1} b^{-1} s^{-1}\right).$$

Now by (3.12), we get

(3.14) $$\int_0^a s^{-1} d\mu\left(M_5^+, s\right) < \infty.$$

Consequently (3.14) holds for an arbitrary s if and only if $\mu\left(M_5^+, s\right) = 0$ or $M_5^+ = \phi$. In a similar way we can show $M_5^- = \phi$ ∎

3.5. Theorem. *Under the condition* (3.9) *the operator L has a finite number of eigenvalues and spectral singularities and each of them is of a finite multiplicity.*

Proof. We have to show that the functions f_+ and f_- have a finite number of zeros with a finite multiplicities in $\overline{\mathbb{C}}_+$ and $\overline{\mathbb{C}}_-$, respectively. We prove only for f_+.

It follows from (3.7) and Theorem 3.4 that $M_3^+ = M_4^+ = \phi$. So the bounded set M_1^+ and M_1^+ have no limit points, i.e. the function f_+ has only finite number of zeros in $\overline{\mathbb{C}}_+$. Since $M_5^+ = \phi$, these zeros are of finite multiplicity. ∎

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