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A NEW SUBCLASS OF HARMONIC MAPPINGS WITH POSITIVE REAL PART

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Abstract

Complex-valued harmonic functions that are univalent and sense preserving in the unit disk U can be written in the form $f = h + \bar{g}$, where h and g are analytic in U. In this paper, we introduce a class $HP(\beta, \alpha)$, $(\alpha \geq 0,\ 0 \leq \beta < 1)$ of all functions $f = h + \bar{g}$ for which $\Re e\{\alpha z(h'(z) + g'(z)) + h(z) + g(z)\} > \beta,\ f(0) = 1$. We give sufficient coefficient conditions for normalized harmonic functions to be in $HP(\beta, \alpha)$. These conditions are also shown to be necessary when the coefficients are negative. This leads to distortion bounds and extreme points.

Key Words: Harmonic mappings, extreme points, distortion bounds.

Mathematics Subject Classification: 30 C 45

1. Introduction

A continuous function $f = u + iv$ is a complex-valued harmonic function in a domain $D \subset \mathbb{C}$ if both u and v are real harmonic in D. In any simply connected domain we can write $f = h + \bar{g}$, where h and g are analytic in D. See Clunie and Sheil-Small [1].

There has been interest [2] in studying the class P_H of all the functions of the form $f = h + \bar{g}$ that are harmonic in $U = \{z : |z| < 1\}$ and such that for $z \in U$, $\Re e f(z) > 0$, where

$$h(z) = 1 + \sum_{n=1}^{\infty} a_n z^n \text{ and } g(z) = \sum_{n=1}^{\infty} b_n z^n \tag{1}$$

are analytic in U.

The class $P_H(\beta)$ of all functions of the form (1) with $\Re e f(z) > \beta,\ 0 \leq \beta < 1$ and $f(0) = 1$ is studied in [4]. Obviously, $P_H(0) = P_H$ and $P_H(\beta) \subset P_H$.

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We denote by $HP(\beta,\alpha)$ the class of all functions of the form (1) that satisfy the condition

$$\Re e\{\alpha z(h'(z)+g'(z))+h(z)+g(z)\}>\beta,\ \alpha\geq 0,\ 0\leq\beta<1. \tag{2}$$

Clearly, $HP(0,0)=P_H$ and $HP(\beta,0)=P_H(\beta)$. Moreover, if $0\leq\beta_1\leq\beta_2<1$, then $HP(\beta_2,\alpha)\subset HP(\beta_1,\alpha)$ and if $0\leq\alpha_1\leq\alpha_2$, then $HP(\beta,\alpha_2)\subset HP(\beta,\alpha_1)$. We further denote by $HR(\beta,\alpha)$ the subclass of $HP(\beta,\alpha)$ such that the functions h and g in $f=h+\bar{g}$ are of the form

$$h(z)=1-\sum_{n=1}^{\infty}a_nz^n \text{ and } g(z)=-\sum_{n=1}^{\infty}b_nz^n \tag{3}$$

with $a_n\geq 0$ and $b_n\geq 0$ for all $n\geq 1$.

2. Main Result

2.1. Theorem: *Let $f=h+\bar{g}$ be given by (1). Furthermore, let*

$$\sum_{n=1}^{\infty}\frac{\alpha n+1}{1-\beta}(|a_n|+|b_n|)\leq 1 \tag{4}$$

where $\alpha\geq 0$ and $0\leq\beta<1$. Then $f\in HP(\beta,\alpha)$.

Proof. We show that the inequality (4) is a sufficient condition for f to be in $HP(\beta,\alpha)$. According to the condition (2) we only need to show that if (4) holds then

$$|1-\beta+\alpha z(h'(z)+g'(z))+h(z)+g(z)| - |1+\beta-\alpha z(h'(z)+g'(z))-h(z)-g(z)|>0. \tag{5}$$

Substituting $h(z)$ and $g(z)$ in (5) yields by (4),

$$\begin{aligned}&|1-\beta+\alpha z(h'(z)+g'(z))+h(z)+g(z)|\\ &\quad-|1+\beta-\alpha z(h'(z)+g'(z))-h(z)-g(z)|=\\ &\quad=\left|2-\beta+\sum_{n=1}^{\infty}(\alpha n+1)(a_n+b_n)z^n\right|-\left|\beta-\sum_{n=1}^{\infty}(\alpha n+1)(a_n+b_n)z^n\right|\\ &\quad\geq 2(1-\beta)-2\sum_{n=1}^{\infty}(\alpha n+1)(|a_n|+|b_n|)|z|^n\\ &\quad> 2(1-\beta)\left\{1-\sum_{n=1}^{\infty}\frac{\alpha n+1}{1-\beta}(|a_n|+|b_n|)\right\}\geq 0. \quad\square\end{aligned}$$

The harmonic mappings

$$f(z)=1+\sum_{n=1}^{\infty}\frac{1-\beta}{\alpha n+1}(x_nz^n+\overline{y_nz^n}), \tag{6}$$

where

$$\sum_{n=1}^{\infty}(|x_n|+|y_n|)=1,$$

show that the coefficient bound given by (4) is sharp.

The functions of the form (6) are in $HP(\beta,\alpha)$ because

$$\sum_{n=1}^{\infty}\frac{\alpha n+1}{1-\beta}(|a_n|+|b_n|)=\sum_{n=1}^{\infty}(|x_n|+|y_n|)=1.$$

The restriction imposed in Theorem 2.1 on the moduli of the coefficients of $f=h+\bar{g}$ enables us to conclude for arbitrary rotation of the coefficients of f that the resulting functions would still be harmonic and $f\in HP(\beta,\alpha)$. Our next theorem establishes that such coefficient bounds cannot be improved.

2.2. Theorem: *Let $f=h+\bar{g}$ be given by (3). Then $f\in HR(\beta,\alpha)$ if and only if*

$$\sum_{n=1}^{\infty}\frac{\alpha n+1}{1-\beta}(|a_n|+|b_n|)\leq 1, \tag{7}$$

where $\alpha\geq 0$ and $0\leq\beta<1$.

Proof. The if part follows from Theorem 2.1 upon nothing that if $f=h+\bar{g}\in HP(\beta,\alpha)$ are of the form (3) then $f\in HR(\beta,\alpha)$.

Suppose that $f\in HR(\beta,\alpha)$. Then we find from (2) that

$$\Re e\left\{1-\sum_{n=1}^{\infty}(\alpha n+1)(a_n+b_n)z^n\right\}>\beta,\ z\in U,\ \alpha\geq 0,\ 0\leq\beta<1.$$

If we choose z to be real and let $z\to 1^-$, we get

$$1-\sum_{n=1}^{\infty}(\alpha n+1)(a_n+b_n)\geq\beta$$

or equivalently,

$$\sum_{n=1}^{\infty}(\alpha n+1)(a_n+b_n)\leq 1-\beta,$$

which is precisely the assertion (7) of Theorem 2.2. □

2.3. Theorem: *If $f\in HR(\beta,\alpha)$, then*

$$|f(z)|\leq 1+\frac{1-\beta}{1+\alpha}r,\quad |z|<1$$

and

$$|f(z)|\geq 1-\frac{1-\beta}{1+\alpha}r,\quad |z|<1.$$

Proof. Let $f \in HR(\beta, \alpha)$. Taking the absolute value of f we obtain

$$\begin{aligned} |f(z)| &\le 1 + \sum_{n=1}^{\infty} (a_n + b_n)|z|^n \\ &\le 1 + \sum_{n=1}^{\infty} (a_n + b_n) r \\ &\le \ 1 + \frac{1-\beta}{1+\alpha} \sum_{n=1}^{\infty} \frac{\alpha n + 1}{1-\beta}(a_n + b_n) r \\ &\le 1 + \frac{1-\beta}{1+\alpha} r, \end{aligned}$$

and

$$\begin{aligned} |f(z)| &\ge 1 - \sum_{n=1}^{\infty} (a_n + b_n)|z|^n \\ &\ge 1 - \sum_{n=1}^{\infty} (a_n + b_n) r \\ &\ge 1 - \frac{1-\beta}{1+\alpha} \sum_{n=1}^{\infty} \frac{\alpha n + 1}{1-\beta}(a_n + b_n) r \\ &\ge 1 - \frac{1-\beta}{1+\alpha} r. \end{aligned}$$

The bounds given in Theorem 2.3 for the functions $f = h + \bar{g}$ of the form (3) also hold for functions of the form (1) if the coefficient condition (4) is satisfied. The functions

$$f(z) = 1 - \frac{1-\beta}{1+\alpha}\, z \quad \text{and} \quad f(z) = 1 - \frac{1-\beta}{1+\alpha}\, \bar{z}$$

for $0 \le \beta < 1$ and $\alpha \ge 0$ show that the bounds given in Theorem 2.3 are sharp.

The following covering result follows from the second inequality in Theorem 2.3.

2.4. Corollary. *If $f \in HR(\beta, \alpha)$, then*

$$\left\{ w : |w| < \frac{\alpha + \beta}{1+\alpha} \right\} \subset f(U).$$

As $HR(\beta, \alpha)$ is a convex family, $HR(\beta, \alpha)$ has a non-empty set of extreme points.

2.5. Theorem : *Set*

$$h_n(z) = 1 - \frac{1-\beta}{\alpha n + 1} z^n \text{ and } g_n(z) = 1 - \frac{1-\beta}{\alpha n + 1} \bar{z}^n, \text{ for } n = 1, 2, \ldots$$

Then $f \in HR(\beta, \alpha)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=1}^{\infty} (\lambda_n h_n + \gamma_n g_n), \tag{8}$$

where $\lambda_n \geq 0,\ \gamma_n \geq 0$ *and* $\sum_{n=1}^{\infty}(\lambda_n + \gamma_n) = 1$.

In particular, the extreme points of $HR(\beta, \alpha)$ *are* $\{h_n\}$ *and* $\{g_n\}$.

Proof. For functions f of the form (8) we have

$$f(z) = \sum_{n=1}^{\infty}(\lambda_n h_n + \gamma_n g_n) = 1 - \sum_{n=1}^{\infty} \frac{1-\beta}{\alpha n + 1}(\lambda_n z^n + \gamma_n \bar{z}^n).$$

Then

$$\sum_{n=1}^{\infty} \frac{\alpha n + 1}{1-\beta}\left[\frac{1-\beta}{\alpha n + 1}(\lambda_n + \gamma_n)\right] = \sum_{n=1}^{\infty}(\lambda_n + \gamma_n) = 1$$

and so $f \in HR(\beta, \alpha)$.

Conversely, suppose that $f \in HR(\beta, \alpha)$. Set

$$\lambda_n = \frac{\alpha n + 1}{1-\beta} a_n \text{ and } \gamma_n = \frac{\alpha n + 1}{1-\beta} b_n, \text{ for } n = 1, 2, \ldots.$$

Then by Theorem 2.2, $0 \leq \lambda_n \leq 1$ and $0 \leq \gamma_n \leq 1$, $(n = 1, 2, \ldots)$. Consequently, we obtain

$$f(z) = \sum_{n=1}^{\infty}(\lambda_n h_n + \gamma_n g_n),$$

as required. □

Following Ruscheweyh [3], we call the set

$$N_\delta(f) = \left\{F : F(z) = 1 - \sum_{n=1}^{\infty}(|A_n| z^n + |B_n| \bar{z}^n) \text{ and } \sum_{n=1}^{\infty} n(|a_n - A_n| + |b_n - B_n|) \leq \delta\right\}.$$

the δ–neighborhood of $f \in P_H$. In particular, for the constant function $I(z) = 1$, we immediately have

$$N_\delta(I) = \left\{f : f(z) = 1 - \sum_{n=1}^{\infty}(|a_n| z^n + |b_n| \bar{z}^n) \text{ and } \sum_{n=1}^{\infty} n(|a_n| + |b_n|) \leq \delta\right\}.$$

2.6. Theorem: *Let* $\delta = (1-\beta)/\alpha$. *Then* $HR(\beta, \alpha) \subset N_\delta(I)$.

Proof. Let f belong to $HR(\beta, \alpha)$. We have

$$\begin{aligned}\sum_{n=1}^{\infty} n(a_n + b_n) &= \frac{1}{\alpha}\sum_{n=1}^{\infty} \alpha n(a_n + b_n) \\ &\leq \frac{1}{\alpha}\sum_{n=1}^{\infty}(\alpha n + 1)(a_n + b_n) \\ &\leq \frac{1}{\alpha}(1-\beta) = \delta.\end{aligned}$$

Hence $f(z) \in N_\delta(I)$. □

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