

Representation and characterization of rapidly varying functions

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Abstract

In this paper representations and characterizations of the class of rapidly varying functions in the sense of de Haan, for index $+\infty$, will be proved. The statements of these theorems will be given in a form that is used by Karamata. Also, some characterization of normalized rapidly varying functions are proved.

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1. Introduction and Results

Karamata's theory of regular variation (see e.g. [6]) was appeared during the thirties of last century as a result of the first serious study of Tauberian type theorems for integral transformations (see e.g. [7] and [8]). The main object in this theory is the class of slowly varying functions in the sense of Karamata which is denoted by SV .

A measurable function $f : [a, \infty) \mapsto (0, \infty)$ ($a > 0$) is called slowly varying in the sense of Karamata if it satisfies the following condition

$$(1.1) \quad \lim_{x \rightarrow \infty} \frac{f(\lambda x)}{f(x)} = 1,$$

for every $\lambda > 0$,

L. de Haan in [5] introduced the class of rapidly varying functions (denoted by R_∞), with the index of variability $+\infty$. In fact, this notion has already appeared in some Karamata's papers (see e.g. [11]), but in a less distinctly form. In recent years, the Theory of rapid variability and its generalizations have experienced great development in asymptotic analysis and in mathematics in general (see e.g. [1], [2], [3], [4] and [10]),

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simultaneously with Karamata's theory of regular variability (see [1]). Important properties of the class R_∞ can be seen in [4].

A measurable function $f : [a, \infty) \mapsto (0, \infty)$ ($a > 0$) is called rapidly varying in the sense of de Haan with the index of variability ∞ , if it satisfies the following condition

$$(1.2) \quad \lim_{x \rightarrow \infty} \frac{f(\lambda x)}{f(x)} = \infty,$$

for every $\lambda > 1$.

1.1. Remark. In this paper we will consider a function $f \in R_\infty$ defined on the interval $(0, \infty)$. Analogous results can be obtained if the domain of a function f is interval $[a, \infty)$, $a > 0$.

According to results from [1], rapidly varying function f satisfies condition

$$(1.3) \quad \lim_{x \rightarrow \infty} \inf_{\mu \geq \lambda} \frac{f(\mu x)}{f(x)} = \infty$$

for every $\lambda > 1$ and it follows that for some $x_0 > 0$ function f is bounded on every interval (x_0, x) . Also, $f(x) \rightarrow \infty$ for $x \rightarrow \infty$ holds.

Now, we give the representation of a functions from functional class R_∞ in Karamata's form.

1.2. Remark. In the following theorem, operator \underline{D} is lower Dini derivative (see [9])

$$\underline{D}g(x) = \lim_{y \rightarrow x} \frac{g(y) - g(x)}{y - x}, \quad \text{for } g : \mathbb{R} \rightarrow \mathbb{R}, x \in \mathbb{R},$$

and denotation \sim represents strong asymptotic equivalence relation.

1.3. Theorem. For a function $f : (0, \infty) \mapsto (0, \infty)$ the next assertions are mutually equivalent:

- (a) function f belongs to the class R_∞ ;
- (b) there is a non-decreasing, absolutely continuous function $g : \mathbb{R} \mapsto \mathbb{R}$ such that $\lim_{x \rightarrow \infty} \underline{D}g(x) = \infty$ and there is a measurable function $j : (0, \infty) \mapsto (0, \infty)$ such that $j(x) \sim x$ for $x \rightarrow \infty$, so that

$$f(x) = \exp(g(\log(j(x))))$$
 for all $x > 0$;
- (c) there are a measurable functions $j : (0, \infty) \mapsto (0, \infty)$ and $h : (0, \infty) \mapsto [0, \infty)$, such that $\lim_{x \rightarrow \infty} h(x) = \infty$ and $j(x) \sim x$ for $x \rightarrow \infty$, for which holds

$$f(x) = \exp \left\{ c + \int_0^{j(x)} h(u) \frac{du}{u} \right\},$$

for all $x > 0$ and for some $c \in \mathbb{R}$.

Now, we give the characterization of a elements from the class R_∞ in Karamata's form.

1.4. Theorem. Let $f : (0, \infty) \mapsto (0, \infty)$ be a measurable function. Then $f \in R_\infty$ if and only if for all $\alpha > 0$ there is a measurable function $j_\alpha : (0, \infty) \mapsto (0, \infty)$ such that $j_\alpha(x) \sim x$, for $x \rightarrow \infty$, and there is a non-decreasing function $k_\alpha : (0, \infty) \mapsto (0, \infty)$, so that

$$f(x) = x^\alpha \cdot k_\alpha(j_\alpha(x)), \quad \text{for } x > 0.$$

The following theorem gives a few characterization of elements of one proper subclass of class R_∞ , which could be called class of normalized rapidly varying functions (see, e.g., [1]).

1.5. Theorem. *For a measurable function $f : (0, \infty) \mapsto (0, \infty)$ the following assertions are mutually equivalent:*

- (a) $\lim_{\substack{x \rightarrow \infty \\ \lambda \rightarrow 1_+}} \log_\lambda \frac{f(\lambda x)}{f(x)} = \infty$;
- (b) $\frac{f(x)}{x} = o(\underline{D}f(x))$, for $x \rightarrow \infty$ (o is Landau symbol [1]);
- (c) *there exists a function $g : \mathbb{R} \mapsto \mathbb{R}$ such that $\lim_{x \rightarrow \infty} \underline{D}g(x) = \infty$ so that holds:*

$$f(x) = \exp(g(\log(x)))$$

for all $x > 0$;

- (d) *for all $\alpha \in \mathbb{R}$ function $\frac{f(x)}{x^\alpha}$ is increasing on some interval $[x_\alpha, \infty)$.*

- 1.6. Remark.**
- 1) Theorem 1.5 holds even without assumption that the function f is measurable, but this assumption should be included because in Theorem 1.5 one important subclass of class R_∞ is characterized.
 - 2) The fact that for a measurable function $f : (0, \infty) \mapsto (0, \infty)$ exists a measurable function $h : (0, \infty) \mapsto \mathbb{R}$ such that $\lim_{x \rightarrow \infty} h(x) = \infty$, and for which is $f(x) = \exp\{c + \int_0^x h(u) \frac{du}{u}\}$ for all $x > 0$ and some $c \in \mathbb{R}$, implicates (c) from Theorem 1.5 (and, also implicates (a), (b) and (d) from Theorem 1.5). The proof is analog to the proof (c) \Rightarrow (a) \Rightarrow (b) from Theorem 1.3. The opposite direction need not to be true without additional conditions.
 - 3) If f is absolutely continuous function, opposite direction in 2) is true. That can be proved analogously to the proof (b) \Rightarrow (c) from Theorem 1.3.

2. Proofs

Proof of Theorem 1.3. (a) \Rightarrow (b) Let $f \in R_\infty$. Let construct sequence (x_n) of positive real numbers with the following properties:

- 1° (x_n) is strictly increasing sequence and $\lim_{n \rightarrow \infty} x_n = \infty$,
- 2° $\lim_{n \rightarrow \infty} \frac{x_{n+1}}{x_n} = 1$, and
- 3° $\frac{f(x)}{f(y)} > 2$ for all $x > 0$ and all $y > 0$, for which $x \geq x_{n+1} > x_n \geq y \geq x_1$, where $n \in \mathbb{N}$.

Let $x_1 > 0$ so that f is locally bounded on the interval $[x_1, \infty)$ and let $x_{n+1} = (\lambda_n + \frac{1}{n})x_n$ for $n \in \mathbb{N}$, where $\lambda_n = \sup\{\lambda \geq 1 \mid f(\lambda x_n) \leq 2 \sup_{x_1 \leq t \leq x_n} f(t)\}$. Clearly, for all $n \in \mathbb{N}$ there exists λ_n in \mathbb{R} and $x_n \leq \lambda_n x_n < x_{n+1}$. If $x \geq x_{n+1} > x_n \geq y > 0$, then $x > \lambda_n x_n$ for $n \in \mathbb{N}$, and according to the definition of the sequence (λ_n) it is $f(x) > 2f(y)$ for $x_1 \leq y \leq x_n$. Especially, $f(x_{n+1}) > 2f(x_n)$ for $n \in \mathbb{N}$, which yields $\lim_{n \rightarrow \infty} f(x_n) = \infty$. As f is locally bounded function on the interval $[x_1, \infty)$, it follows $\lim_{n \rightarrow \infty} x_n = \infty$.

According to the definition of sequence (λ_n) it can be concluded that sequences (μ_n) and (y_n) are such that, for every $n \in \mathbb{N}$, it follows that $\mu_n \in (\lambda_n - \frac{1}{n}, \lambda_n)$ and $y_n \in [x_1, x_n]$, for which it is $\frac{f(\mu_n x_n)}{f(y_n)} \leq 2$. Then, according to the theorem of uniform convergence for

rapidly varying functions (see (1.3) or [1]), it follows $\limsup_{n \rightarrow \infty} \frac{\mu_n x_n}{y_n} \leq 1$, i.e., $\limsup_{n \rightarrow \infty} \mu_n \leq 1$, so it follows that $\lim_{n \rightarrow \infty} \lambda_n = 1$. Thus, $\lim_{n \rightarrow \infty} \frac{x_{n+1}}{x_n} = 1$.

Let $g : \mathbb{R} \mapsto \mathbb{R}$ be a linear function on $[t_n, t_{n+1}]$ such that $g(t_n) = \ln f(x_n)$, where $t_n = \ln x_n$, for every $n \in \mathbb{N}$. Also, $g(t) = e^t - x_1 + g(t_1)$, for $t < t_1$. Now, we have that g is a continuous, piecewise smooth and strictly increasing (hence, absolutely continuous and non-decreasing) function, and (from 1° and 3°) it satisfies

$$g'(t) = \frac{g(t_{n+1}) - g(t_n)}{t_{n+1} - t_n} = \frac{\ln f(x_{n+1}) - \ln f(x_n)}{\ln x_{n+1} - \ln x_n} > 0,$$

for any $t \in (t_n, t_{n+1})$, $n \in \mathbb{N}$. Furthermore, (from 2° and 3°) it satisfies

$$\lim_{t \rightarrow \infty} g'(t) = \lim_{n \rightarrow \infty} \frac{\ln \frac{f(x_{n+1})}{f(x_n)}}{\ln \frac{x_{n+1}}{x_n}} \geq \lim_{n \rightarrow \infty} \frac{\ln 2}{\ln \frac{x_{n+1}}{x_n}} = \infty,$$

for $\mathbb{R} \ni t \neq t_n$, for every $n \in \mathbb{N}$. Thus, $\lim_{x \rightarrow \infty} \underline{D}g(x) = \infty$.

Now, let $j(x) = e^{g^{-1}(\ln f(x))}$, for $x > 0$. A function $j(x)$ is measurable, because $f(x)$ is a measurable function, and $\exp(g^{-1}(\log(t)))$ is a piecewise smooth function (and hence it is absolutely continuous function), for $t > 0$.

Now, we will show that $j(x) \sim x$, for $x \rightarrow \infty$. From condition 3° it follows that $\frac{f(x)}{f(x_{n-1})} > 2$ and $\frac{f(x_{n+2})}{f(x)} > 2$, for some $x \in [x_n, x_{n+1})$ and $n \in \mathbb{N}$, $n \geq 2$. For those x and n we obtain

$$f(x_{n-1}) < f(x) < f(x_{n+2}),$$

so, we have

$$g^{-1}(\ln f(x_{n-1})) < g^{-1}(\ln f(x)) < g^{-1}(\ln f(x_{n+2})).$$

Furthermore, for those x and n , we have

$$t_{n-1} = \ln x_{n-1} < \ln j(x) < \ln x_{n+2} = t_{n+2},$$

and finally we obtain that

$$\frac{x_{n-1}}{x_{n+1}} < \frac{x_{n-1}}{x} < \frac{j(x)}{x} < \frac{x_{n+2}}{x} < \frac{x_{n+2}}{x_n}.$$

Hence, from 2° it is satisfied that $j(x) \sim x$, for $x \rightarrow \infty$, and $f(x) = \exp(g(\log(j(x))))$ for $x > 0$.

(b) \Rightarrow (c) Let functions g and j have properties given in (b). Let

$$g_0 = \begin{cases} g(x), & \text{for } x > 0, \\ g(0), & \text{for } x \leq 0. \end{cases}$$

Let $h(x) = \underline{D}g_0(\ln x)$ for $x > 0$. Then h is measurable, locally integrable, and $\lim_{x \rightarrow \infty} h(x) = \infty$ holds. Also, it is

$$\begin{aligned} \int_0^{j(x)} h(u) \frac{du}{u} &= \int_0^{j(x)} \underline{D}g_0(\ln u) \frac{du}{u} = \int_{-\infty}^{\ln j(x)} \underline{D}g_0(t) dt = \\ &= \int_0^{\ln j(x)} \underline{D}g(t) dt = g(\ln j(x)) - g(0) = \ln f(x) - c, \end{aligned}$$

for a constant $c = g(0) \in \mathbb{R}$, and for all $x > 0$.

Thus, $f(x) = \exp \left\{ c + \int_0^{j(x)} h(u) \frac{du}{u} \right\}$, for all $x > 0$ and for mentioned $c \in \mathbb{R}$.

(c) \Rightarrow (a) Let $\lambda > 1$ and $M \in \mathbb{R}$. Then, there is $x_0 > 0$ such that $h(x) > \frac{2M}{\ln \lambda}$ and $\lambda^{-\frac{1}{4}} < \frac{j(x)}{x} < \lambda^{\frac{1}{4}}$, for $x > \frac{x_0}{\lambda}$. Hence, it follows

$$\begin{aligned} \ln \frac{f(\lambda x)}{f(x)} &= \int_0^{j(\lambda x)} h(u) \frac{du}{u} - \int_0^{j(x)} h(u) \frac{du}{u} = \int_{j(x)}^{j(\lambda x)} h(u) \frac{du}{u} > \\ &> \frac{2M}{\ln \lambda} \cdot (\ln j(\lambda x) - \ln j(x)) = \frac{2M}{\ln \lambda} \cdot \ln \left(\lambda \cdot \frac{j(\lambda x)}{\lambda x} \cdot \frac{x}{j(x)} \right) > \\ &> \frac{2M}{\ln \lambda} \cdot \ln \left(\lambda \cdot \lambda^{-\frac{1}{4}} \cdot \lambda^{-\frac{1}{4}} \right) = M, \end{aligned}$$

for $x > x_0$. Therefore, it holds that $\lim_{x \rightarrow \infty} \frac{f(\lambda x)}{f(x)} = \infty$, for every $\lambda > 1$. Also, f is a measurable function, as a composition of three function: a measurable function j , an absolutely continuous function $\int_0^x h(u) \frac{du}{u}$ and an exponential function. Hence, $f \in R_\infty$. \square

Proof of Theorem 1.4. (\Rightarrow) If $f \in R_\infty$, then $\frac{f(x)}{x^\alpha} \in R_\infty$, for $x > 0$, and every fixed $\alpha > 0$. From Theorem 1.3, it follows that $\frac{f(x)}{x^\alpha} = \exp \left\{ g_\alpha(\log(j_\alpha(x))) \right\}$, for that α and every $x > 0$, where $g_\alpha : \mathbb{R} \mapsto \mathbb{R}$ is a non-decreasing function and $j_\alpha : (0, \infty) \mapsto (0, \infty)$ is a measurable function such that $j_\alpha(x) \sim x$, for $x \rightarrow \infty$. If we take that $k_\alpha(t) = \exp \{ g_\alpha(\log(t)) \}$, for $t > 0$, we obtain that Theorem holds for this direction. (\Leftarrow) For arbitrary $\alpha > 0$, if there is a measurable function $j_\alpha : (0, \infty) \mapsto (0, \infty)$ such that $\lim_{x \rightarrow \infty} \frac{j_\alpha(x)}{x} = 1$ and a non-decreasing function $k_\alpha : (0, \infty) \mapsto (0, \infty)$, that is satisfied $f(x) = x^\alpha \cdot k_\alpha(j_\alpha(x))$, for $x > 0$ we obtain that

$$\frac{f(\lambda x)}{f(x)} = \lambda^\alpha \cdot \frac{k_\alpha(j_\alpha(\lambda x))}{k_\alpha(j_\alpha(x))} \geq \lambda^\alpha,$$

for $\lambda > 1$ and sufficiently large x . The previous inequality holds because $j_\alpha(\lambda x) \geq \sqrt{\lambda}x \geq j_\alpha(x)$ for mentioned α , λ and sufficiently large x . Therefore, it follows that $\lim_{x \rightarrow \infty} \frac{f(\lambda x)}{f(x)} = \infty$, for $\lambda > 1$. Also, f is a measurable function. Finally, $f \in R_\infty$. \square

Proof of Theorem 1.5. (a) \Leftrightarrow (c) Let introduce function f in the following way: $f(x) = \exp(g(\log(x)))$, for $x > 0$. Then, equivalence (a) \Leftrightarrow (c) follows from:

$$\lim_{\substack{x \rightarrow \infty \\ \lambda \rightarrow 1_+}} \log_\lambda \frac{f(\lambda x)}{f(x)} = \lim_{\substack{t \rightarrow \infty \\ \delta \rightarrow 0_+}} \frac{g(t + \delta) - g(t)}{\delta} = \lim_{t \rightarrow \infty} \underline{D}g(t).$$

(b) \Leftrightarrow (c) Again, let introduce function f as $f(x) = \exp(g(\log(x)))$, for $x > 0$. Then, from the fact that $\underline{D}g(x) = \underline{D}(\ln f(e^x)) = \frac{\underline{D}f(e^x)e^x}{f(e^x)}$, for all $x \in \mathbb{R}$, it follows $\lim_{x \rightarrow \infty} \underline{D}g(x) = \infty$ is equal to the fact that $\frac{f(t)}{t} = o(\underline{D}g(t))$, for $t \rightarrow \infty$. This proves the equivalence (b) \Leftrightarrow (c).

(c) \Leftrightarrow (d) Once more, let introduce function f by $f(x) = \exp(g(\log(x)))$, for $x > 0$. Then the function $\frac{f(x)}{x^\alpha}$, $x > 0$ and $\alpha \in \mathbb{R}$, is increasing on an interval $[x_\alpha, \infty)$ if and only if the function $\ln \frac{f(e^t)}{e^{\alpha t}} = g(t) - \alpha t$, for $t \in \mathbb{R}$ and the same $\alpha \in \mathbb{R}$, is increasing on an

interval $[t_\alpha, \infty)$, and this last condition is equivalent to the fact that $\liminf_{t \rightarrow \infty} \underline{Dg}(t) \geq \alpha$ for all $\alpha \in \mathbb{R}$. The last fact is equivalent to the fact that $\lim_{t \rightarrow \infty} \underline{Dg}(t) = \infty$. \square

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