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ON LOWEN'S COMPACTNESS IN FUZZY CLOSURE SPACES

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Abstract

In this note, we extend Lowen's notion of fuzzy compactness to fuzzy closure spaces and introduce some weak forms of compactness, namely almost compactness, near compactness, countable compactness and light compactness in fuzzy closure spaces. We obtain some equivalent characterizations of these notions and some implications.

Keywords: Fuzzy closure space, Lo-fuzzy compact, Lo-fuzzy almost compact, Lo-fuzzy nearly compact.

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1. Introduction

Fuzzy closure spaces (fcs for short) were first introduced and some of their fundamental properties studied by Mashour and Ghanim [5]. Among these, compactness was one of the important notions considered. They extended Chang's notion of compactness [2] to fuzzy closure spaces and introduced some weak forms of compactness, i.e. almost compactness and near compactness. In [3], Çoker and Eş introduced some other kinds of compactness, namely light compactness and countable compactness in fuzzy closure spaces, and they discussed some implications.

In this paper we adopt definition VII of Lowen [4], termed Lowen "fuzzy compact," and adapt this and the above-mentioned dilutions to fuzzy closure spaces. We obtain some equivalent characterizations and implications.

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2. Preliminaries

In this section we give some definitions, which we need for our further considerations.

2.1. Definition. (Mashhour and Ghanim [5]). A Ćech fuzzy closure operator (or ĆF-closure operator for short) on a set X is a function $c : I^X \to I^X$ satisfying the following three axioms.

C1 $c(\emptyset) = \emptyset$
C2 $\mu \leq c(\mu)$
C3 $c(\mu \vee \upsilon) = c(\mu) \vee c(\upsilon)$

The pair (X, c) is called a *fuzzy closure space* (or fcs for short).

2.2. Definition. (Bülbül [1]). Let $\mathcal{F} \subset I^X$ be a family of fuzzy sets. Then we say that $\mathcal{F}$ has the *finite intersection property in the sense of Lowen* (Lo-FIP for short) iff

$$\exists\, \alpha \in (0,1] \ \text{ and } \ \exists\, \epsilon \in (0,\alpha) \ \text{ such that } \ \bigwedge_{f \in \mathcal{F}^*} f > 1 - \alpha + \epsilon$$

holds for every finite subfamily $\mathcal{F}^* \subset \mathcal{F}$.

2.3. Definition. A fcs (X, c) is called *Lo-fuzzy compact* iff each family $\mathcal{F} \subset I^X$ and each $\alpha \in (0,1]$ such that $\bigvee_{\mu \in \mathcal{F}} \mu^* \geq \alpha$, where $\mu^* = 1 - c(1 - \mu)$, and for each $\epsilon \in (0, \alpha)$

$$\exists\, \mathcal{F}^* \subset \mathcal{F} \text{ finite, such that } \bigvee_{\mu \in \mathcal{F}^*} \mu^* \geq \alpha - \epsilon$$

2.4. Definition. A fcs (X, c) is called *Lo-fuzzy almost compact* iff each family $\mathcal{F} \subset I^X$ and each $\alpha \in (0,1]$ such that $\bigvee_{\mu \in \mathcal{F}} \mu^* \geq \alpha$, where $\mu^* = 1 - c(1 - \mu)$, and for each $\epsilon \in (0, \alpha)$

$$\exists\, \mathcal{F}^* \subset \mathcal{F} \text{ finite, such that } \bigvee_{\mu \in \mathcal{F}^*} c(\mu) \geq \alpha - \epsilon$$

2.5. Definition. A fcs (X, c) is called *Lo-fuzzy nearly compact* iff each family $\mathcal{F} \subset I^X$, with the property $\mu = (c(\mu))^*$ for each $\mu \in \mathcal{F}$, and each $\alpha \in (0,1]$ such that $\bigvee_{\mu \in \mathcal{F}} \mu \geq \alpha$, and for each $\epsilon \in (0, \alpha)$

$$\exists\, \mathcal{F}^* \subset \mathcal{F} \text{ finite, such that } \bigvee_{\mu \in \mathcal{F}^*} \mu \geq \alpha - \epsilon$$

3. Main Theorems

3.1. Theorem. *A fcs (X, c) is Lo-fuzzy compact iff each family $\mathcal{F} \subset I^X$ such that $\{c(f) \mid f \in \mathcal{F}\}$ has the Lo-FIP, also has the property*

$$\bigwedge_{f \in \mathcal{F}} c(f) \geq 1 - \alpha + \epsilon$$

for some $\alpha \in (0,1]$ and $\epsilon \in (0, \alpha)$.

Proof. Necessity. Let (X, c) be Lo-fuzzy compact and $\mathcal{F} \subset I^X$ such that $\{c(f) \mid f \in \mathcal{F}\}$ has the Lo-FIP. Now suppose that

$$\bigwedge_{f \in \mathcal{F}} c(f) < 1 - \alpha + \epsilon$$

holds for every $\alpha \in (0,1]$ and $\epsilon \in (0, \alpha)$. Then

$$1 - \bigwedge_{f \in \mathcal{F}} c(f) > \alpha - \epsilon, \text{ i.e. } \bigvee_{f \in \mathcal{F}} (1 - c(f)) > \alpha - \epsilon$$

holds for every $\alpha \in (0,1]$ and every $\epsilon \in (0,\alpha)$. Therefore

$$\bigvee_{f\in\mathcal{F}} (1-c(f)) \geq \alpha, \text{ for each } \alpha \in (0,1].$$

Since $1-c(f) = (1-f)^*$, it follows that $\bigvee_{f\in\mathcal{F}} (1-f)^* \geq \alpha$, for each $\alpha \in (0,1]$. So by hypothesis, it follows that for each $\alpha \in (0,1]$ and $\epsilon \in (0,\alpha)$

$$\exists\, \mathcal{F}^* \subset \mathcal{F} \text{ finite, such that } \bigvee_{f\in\mathcal{F}^*} (1-f)^* \geq \alpha - \epsilon.$$

Hence $1-\bigvee_{f\in\mathcal{F}^*} (1-f)^* \leq 1-\alpha+\epsilon$, that is $\bigwedge_{f\in\mathcal{F}^*} c(f) \leq 1-\alpha+\epsilon$. Thus for every $\alpha \in (0,1]$ and every $\epsilon \in (0,\alpha)$ there is a finite $\mathcal{F}^* \subset \mathcal{F}$ such that $\bigwedge_{f\in\mathcal{F}^*} c(f) \leq 1-\alpha+\epsilon$ holds. But this contradicts the fact that $\{c(f) \mid f \in \mathcal{F}\}$ has the Lo-FIP.

Sufficiency. Suppose that every family $\mathcal{F} \subset I^X$ such that $\{c(f) \mid f \in \mathcal{F}\}$ has the Lo-FIP, also has the property

$$\bigwedge_{f\in\mathcal{F}} c(f) \geq 1-\alpha+\epsilon$$

for some $\alpha \in (0,1]$ and $\epsilon \in (0,\alpha)$. Let $\beta \subset I^X$ be a family of fuzzy sets and $\alpha \in (0,1]$ with

$$\bigvee_{\mu\in\beta} \mu^* \geq \alpha.$$

Now suppose that

$$\exists\, \epsilon \in (0,\alpha) \text{ such that } \bigvee_{\mu\in\beta^*} \mu^* < \alpha - \epsilon$$

holds for every finite subfamily $\beta^* \subset \beta$. Then

$$1-\bigvee_{\mu\in\beta^*} \mu^* > 1-\alpha+\epsilon \Longrightarrow \bigwedge_{\mu\in\beta^*} (1-\mu^*) = \bigwedge_{\mu\in\beta^*} c(1-\mu) > 1-\alpha+\epsilon.$$

Thus the family $\{c(1-\mu) \mid \mu \in \beta\}$ has the Lo-FIP. So, by hypothesis $\bigwedge_{\mu\in\beta} c(1-\mu) \geq 1-\alpha+\epsilon$ must be hold for some $\alpha \in (0,1]$ and $\epsilon \in (0,\alpha)$. Therefore

$$1-\bigwedge_{\mu\in\beta} c(1-\mu) \leq \alpha-\epsilon \Longrightarrow \bigvee_{\mu\in\beta} [1-c(1-\mu)] = \bigvee_{\mu\in\beta} \mu^* \leq \alpha-\epsilon < \alpha$$

which is a contradiction. This completes the proof of the theorem. □

3.2. Theorem. *A fcs (X,c) is Lo-fuzzy almost compact iff it satisfies the following condition:*

If $\mathcal{F} \subset I^X$ is a family of fuzzy sets in X having the property

$$\exists\, \alpha \in (0,1] \text{ and } \exists\, \epsilon \in (0,\alpha) \text{ such that } \bigwedge_{f\in\mathcal{F}^*} f^* > 1-\alpha+\epsilon$$

holds for every finite subfamily $\mathcal{F}^* \subset \mathcal{F}$ (i.e. $\{f^* \mid f \in \mathcal{F}\}$ has the Lo-FIP), then

$$\bigwedge_{f\in\mathcal{F}} c(f) > 1-\alpha$$

Proof. *Necessity.* To prove that the condition is necessary, let (X, c) be a Lo-fuzzy almost compact fcs and $\mathcal{F} \subset I^X$ a family of fuzzy sets in X having the property

$$\exists\, \alpha \in (0,1] \text{ and } \exists\, \epsilon \in (0,\alpha) \text{ such that } \bigwedge_{f \in \mathcal{F}^*} f^* > 1 - \alpha + \epsilon \tag{1}$$

holds for every finite subfamily $\mathcal{F}^* \subset \mathcal{F}$. Now suppose that $\bigwedge_{f \in \mathcal{F}} c(f) \le 1 - \alpha$. Then

$$1 - \bigwedge_{f \in \mathcal{F}} c(f) \ge \alpha \implies \bigvee_{f \in \mathcal{F}} (1 - c(f)) = \bigvee_{f \in \mathcal{F}} (1 - f)^* \ge \alpha.$$

Since (X, c) is Lo-fuzzy almost compact and $\alpha \in (0,1]$ with $\bigvee_{f \in \mathcal{F}} (1-f)^* \ge \alpha$, then by the definition of Lo-fuzzy almost compactness it follows for the $\epsilon \in (0, \alpha)$ above that

$$\exists\, \mathcal{F}^* \subset \mathcal{F} \text{ finite, such that } \bigvee_{f \in \mathcal{F}^*} c(1 - f) \ge \alpha - \epsilon.$$

Therefore we obtain

$$1 - \bigvee_{f \in \mathcal{F}^*} c(1 - f) = \bigwedge_{f \in \mathcal{F}^*} (1 - c(1 - f)) = \bigwedge_{f \in \mathcal{F}^*} f^* \le 1 - \alpha + \epsilon,$$

contradicting the given property (1). Thus $\bigwedge_{f \in \mathcal{F}} c(f) > 1 - \alpha$ must hold, i.e. (X, c) must satisfy the given condition.

Sufficiency. To prove that the condition is sufficient, let (X, c) be a fcs satisfying the given condition and suppose that (X, c) is not fuzzy almost compact. Then

$$\exists \mathcal{F} \subset I^X \text{ and } \exists \alpha \in (0,1] \text{ and } \exists \epsilon \in (0,\alpha), \text{ such that } \bigvee_{f \in \mathcal{F}} f^* \ge \alpha, \text{ but } \bigvee_{f \in \mathcal{F}^*} c(f) < \alpha - \epsilon$$

holds for every finite $\mathcal{F}^* \subset \mathcal{F}$. Then

$$1 - \bigvee_{f \in \mathcal{F}^*} c(f) = \bigwedge_{f \in \mathcal{F}^*} 1 - c(f) = \bigwedge_{f \in \mathcal{F}^*} (1 - f)^* > 1 - \alpha + \epsilon$$

holds for the above $\alpha \in (0,1]$, $\epsilon \in (0, \alpha)$ and for every finite $\mathcal{F}^* \subset \mathcal{F}$. Then it follows by the given condition that

$$\bigwedge_{f \in \mathcal{F}} c(1 - f) > 1 - \alpha$$

must be hold. Therefore we obtain

$$1 - \bigwedge_{f \in \mathcal{F}} c(1 - f) = \bigvee_{f \in \mathcal{F}} (1 - c(1 - f)) = \bigvee_{f \in \mathcal{F}} f^* < \alpha,$$

contradicting the given property of the family $\mathcal{F}$. Thus (X, c) must be Lo-fuzzy almost compact. □

3.3. Theorem. *A fcs (X, c) is Lo-fuzzy nearly compact iff each family $\mathcal{F} \subset I^X$ with $f = c(f^*)$ for each $f \in \mathcal{F}$, and having the Lo-FIP (i.e. $\bigwedge_{f \in \mathcal{F}^*} f > 1 - \alpha + \epsilon$ holds for each finite $\mathcal{F}^* \subset \mathcal{F}$, for some $\alpha \in (0,1]$ and for some $\epsilon \in (0,\alpha)$), also has the property*

$$\bigwedge_{f \in \mathcal{F}} f > 1 - \alpha.$$

Proof. Necessity. Let (X, c) be Lo-fuzzy nearly compact and let $\mathcal{F} \subset I^X$ be a family of fuzzy sets in X having the Lo-FIP, with $f = c(f^*)$ for each $f \in \mathcal{F}$. Now suppose $\bigwedge_{f\in\mathcal{F}} f \leq 1 - \alpha$ for each $\alpha \in (0, 1]$. Then

$$1 - \bigwedge_{f\in\mathcal{F}} f \geq \alpha, \text{ i.e. } \bigvee_{f\in\mathcal{F}} (1 - f) \geq \alpha.$$

Further $1 - f = (c(1 - f))^*$, since $f = c(f^*)$ for each $f \in \mathcal{F}$ and $f^* = 1 - c(1 - f)$. So, by hypothesis, since (X, c) is Lo-fuzzy nearly compact, we get

$$\forall \epsilon \in (0, \alpha)\ \exists \mathcal{F}^* \subset \mathcal{F} \text{ finite, such that } \bigvee_{f\in\mathcal{F}^*} (1 - f) \geq \alpha - \epsilon.$$

Therefore

$$1 - \bigvee_{f\in\mathcal{F}^*} (1 - f) \leq 1 - \alpha + \epsilon, \text{ i.e. } \bigwedge_{f\in\mathcal{F}^*} (1 - (1 - f)) = \bigwedge_{f\in\mathcal{F}^*} f \leq 1 - \alpha + \epsilon.$$

That is, $\bigwedge_{f\in\mathcal{F}^*} f \leq 1-\alpha+\epsilon$. But this contradicts the fact that $\mathcal{F}$ has the Lo-FIP. Therefore $\bigwedge_{f\in\mathcal{F}} f > 1 - \alpha$ must be true for some $\alpha \in (0, 1]$.

Sufficiency. Suppose that (X, c) is not Lo-fuzzy nearly compact. Then there is a family $\mathcal{F} \subset I^X$ with $f = (c(f))^*$ for each $f \in \mathcal{F}$ and there is an $\alpha \in (0, 1]$ with $\bigwedge_{f\in\mathcal{F}} f \geq \alpha$ and

$$\exists\ \epsilon \in (0, \alpha) \text{ such that } \bigvee_{f\in\mathcal{F}^*} f < \alpha - \epsilon$$

holds for every finite $\mathcal{F}^* \subset \mathcal{F}$. Therefore

$$1 - \bigvee_{f\in\mathcal{F}^*} f > 1 - \alpha + \epsilon, \text{ i.e. } \bigwedge_{f\in\mathcal{F}^*} (1 - f) > 1 - \alpha + \epsilon.$$

So, the family $\mathcal{F}' = \{1 - f \mid f \in \mathcal{F}\}$ has the Lo-FIP. Moreover, for each $f \in \mathcal{F}$, since $f = (c(f))^*$ and $(1 - f)^* = 1 - c(f)$, we have

$$1 - f = 1 - (c(f))^* = c(1 - c(f)) = c((1 - f)^*).$$

Therefore the family $\mathcal{F}'$ satisfies the properties of the hypothesis. So, by hypothesis, it also has the property

$$\bigwedge_{f\in\mathcal{F}} (1 - f) > 1 - \alpha.$$

Hence we obtain

$$1 - \bigwedge_{f\in\mathcal{F}} (1 - f) = \bigvee_{f\in\mathcal{F}} f < \alpha,$$

contradicting the given property of the family $\mathcal{F}$. Thus (X, c) must be Lo-fuzzy nearly compact and this completes the proof of the theorem. □

3.4. Theorem. *In a fcs the following implications hold:*

Lo-fuzzy almost compact $\Longleftarrow$ *Lo-fuzzy compact* $\Longrightarrow$ *Lo-fuzzy nearly compact.*

Proof. The first implication follows directly from the definitions, since $f^* \leq f \leq c(f)$ for each fuzzy set f in the space. In fact, $1 - f \leq c(1 - f)$ implies $1 - c(1 - f) \leq f$, that is $f^* \leq f$. The second inequality is clear.

For the second implication, let (X, c) be Lo-fuzzy compact, let $\mathcal{F} \subset I^X$ satisfy $f = (c(f))^*$ for each $f \in \mathcal{F}$, and let $\alpha \in (0, 1]$ with $\bigvee_{f\in\mathcal{F}} f \geq \alpha$ and $\epsilon \in (0, \alpha)$ be given. Clearly $\bigvee_{f\in\mathcal{F}} (c(f))^* \geq \alpha$. So by Lo-fuzzy compactness of (X, c),

$$\exists\, \mathcal{F}^* \subset \mathcal{F} \text{ finite, such that } \bigvee_{f\in\mathcal{F}^*} (c(f))^* = \bigvee_{\in\mathcal{F}^*} f \geq \alpha - \epsilon.$$

Thus (X, c) is Lo-fuzzy nearly compact. □

In the following, we suggest two more definitions, namely:

3.5. Definition. A fcs (X, c) is called *Lo-fuzzy countably compact* iff for each countable family $\{f_n \mid n \in \mathbb{N}\}$ of fuzzy sets in X and each $\alpha \in (0, 1]$ such that $\bigvee_{n\in\mathbb{N}} f_n^* \geq \alpha$, where $f_n^* = 1 - c(1 - f_n)$, and for each $\epsilon \in (0, \alpha)$,

$$\exists\, \{f_{i_1}, \ldots, f_{i_n}\} \text{ such that } \bigvee_{k=1}^{n} f_{i_k}^* \geq \alpha - \epsilon.$$

3.6. Definition. A fcs (X, c) is called *Lo-fuzzy lightly compact* iff each countable family $\{f_n \mid n \in \mathbb{N}\}$ of fuzzy sets in X and each $\alpha \in (0, 1]$ such that $\bigvee_{n\in\mathbb{N}} f_n^* \geq \alpha$, where $f_n^* = 1 - c(1 - f_n)$, and for each $\epsilon \in (0, \alpha)$,

$$\exists\, \{f_{i_1}, \cdots, f_{i_n}\} \text{ such that } \bigvee_{k=1}^{n} c(f_{i_k}) \geq \alpha - \epsilon.$$

Now we obtain the following theorem by changing the term "family of fuzzy sets" in Theorem 3.2 to "countable family of fuzzy sets". That is, we have

3.7. Theorem. *A fcs (X, c) is Lo-fuzzy lightly compact iff it satisfies the following condition:*

If $\mathcal{F} \subset I^X$ is a countable family of fuzzy sets in X having the property

$$\exists\, \alpha \in (0, 1] \text{ and } \exists\, \epsilon \in (0, \alpha) \text{ such that } \bigwedge_{f\in\mathcal{F}^*} f^* > 1 - \alpha + \epsilon$$

holds for every finite subfamily $\mathcal{F}^* \subset \mathcal{F}$ (i.e., $\{f^* \mid f \in \mathcal{F}\}$ has the Lo-FIP), then

$$\bigwedge_{f\in\mathcal{F}} c(f) > 1 - \alpha.$$

Proof. As for the proof of Theorem 3.2. □

Finally, in the following diagram, we summarize the implications between all the notions discussed in this paper. The implications below, which are not included in Theorem 3.4, follow immediately from the definitions.

$$\begin{array}{ccc}
\text{Lo-fuzzy nearly compact} & & \\
\Uparrow & & \\
\text{Lo-fuzzy compact} & \Longrightarrow & \text{Lo-fuzzy almost compact} \\
\Downarrow & & \Downarrow \\
\text{Lo-fuzzy countably compact} & \Longrightarrow & \text{Lo-fuzzy lightly compact.}
\end{array}$$

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