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Around Poisson–Mehler summation formula

Paweł J. Szabłowski *

Abstract

We study polynomials in x and y of degree $n+m$: $\{Q_{m,n}(x,y|t,q)\}_{n,m\geq 0}$ that are related to the generalization of Poisson–Mehler formula i.e. to the expansion $\sum_{i\geq 0}\frac{t^i}{[i]_q!}H_{i+n}(x|q)\,H_{m+i}(y|q)$ $=Q_{n,m}(x,y|t,q)\sum_{i\geq 0}\frac{t^i}{[i]_q!}H_i(x|q)\,H_m(y|q)$, where $\{H_n(x|q)\}_{n\geq -1}$ are the so-called $q-$Hermite polynomials (qH). In particular we show that the spaces $span\{Q_{i,n-i}(x,y|t,q):i=0,\ldots,n\}_{n\geq 0}$ are orthogonal with respect to a certain measure (two-dimensional $(t,q)-$Normal distribution) on the square $\left\{(x,y):|x|,|y|\leq 2/\sqrt{1-q}\right\}$ being a generalization of two-dimensional Gaussian measure. We study structure of these polynomials showing in particular that they are rational functions of parameters t and q. We use them in various infinite expansions that can be viewed as simple generalization of the Poisson-Mehler summation formula. Further we use them in the expansion of the reciprocal of the right hand side of the Poisson-Mehler formula.

Keywords: $q-$Hermite, big $q-$Hermite, Al-Salam–Chihara, orthogonal polynomials, Poisson-Mehler summation formula. Orthogonal polynomials on the plane.

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*Department of Mathematics and Information Sciences, Warsaw University of Technology ul. Koszykowa 75, 00-662 Warsaw, Poland
Email : pawel.szablowski@gmail.com

1. Introduction and auxiliary results

1.1. Preface. We consider various generalizations of the celebrated Poisson–Mehler formula (see e.g. [10], (13.1.24) or [1], (10.11.17)):

$$\sum_{n\geq 0}\frac{\rho^{n}}{[n]_{q}!}H_{n}\left(x|q\right)H_{n}\left(y|q\right)=\frac{\left(\rho^{2}\right)_{\infty}}{\prod_{j=0}^{\infty}\omega\left(x\sqrt{1-q}/2,y\sqrt{1-q}/2|\rho q^{j}\right)}, \tag{1.1}$$

where $\{H_{n}\}_{n\geq 0}$ denote $q-$Hermite polynomials and $\omega\left(x,y|t\right)$ are certain polynomials symmetric in x and y of degree two. These polynomials as well as symbols $[n]_{q}!$ and $\left(\rho^{2}\right)_{\infty}$are defined and explained in Sections 1.2 and 1.3. There exist many proofs of (1.1) (e.g. see [10], [1], [2], [21]). In [22] a certain generalization of (1.1) has been proved by the author. It was used in calculating moments of the so called Askey–Wilson distribution.

In the paper we consider functions

$$\gamma_{i,j}\left(x,y|\rho,q\right)=\sum_{n\geq 0}\frac{\rho^{n}}{[n]_{q}!}H_{n+i}\left(x|q\right)H_{n+j}\left(y|q\right). \tag{1.2}$$

for all $i,j\geq 0$. It was shown by the author in [21] (Lemma 3) that:

$$\gamma_{i,j}\left(x,y|\rho,q\right)=Q_{i,j}\left(x,y|\rho,q\right)\gamma_{0,0}\left(x,y|\rho,q\right), \tag{1.3}$$

where $Q_{i,j}\left(x,y|\rho,q\right)$ is a certain polynomial in x, y of degree $i+j$. Hence (1.3) can be viewed as a generalization of (1.1).

The main object of the paper is to study the properties and later the rôle of the polynomials $Q_{i,j}\left(x,y|\rho,q\right)$ in obtaining a family of two dimensional orthogonal polynomials as well as various expansions that can be viewed as either generalizations of (1.1) or expansions more or less directly related to this formula.

In particular we find generating function of these polynomials, we express them as linear combinations of polynomials belonging to families of polynomials of one variable.

We also analyze the measure (the so-called $(\rho,q)-2Normal$ measure) on the square $S\left(q\right)\times S\left(q\right)$ with the density defined by (2.5) below, that can be easily constructed from the densities of measures that make $q-$Hermite and the so-called Al-Salam–Chihara polynomials orthogonal and which can viewed as a generalization of bivariate Normal distribution. Interval $S\left(q\right)$ is defined by (1.5). The probabilistic aspects of this distribution were presented in [19]. We point out the rôle of the polynomials $Q_{n,m}$ in further analysis of this measure. In particular we introduce spaces of functions of two variables

$$\Lambda_{n}(x,y|\rho,q)=span\left\{Q_{i,n-i}(x,y,|\rho,q),i=0,\ldots,n\right\},n\geq 0 \tag{1.4}$$

and show that they are orthogonal with respect to $(\rho,q)-2Normal$ measure. Hence these spaces form the direct sum decomposition of the space of functions that are square integrable with respect to $(\rho,q)-2Normal$ measure.

Further we use these polynomials to obtain various infinite expansions. In particular we obtain an expansion of the reciprocal of the right hand side of (1.1) in an infinite series. In [21], (formula 5.3) one such expansion was presented. The expansions was non-symmetric in x and y (for each finite sum). This time the expansion is symmetric in x and y.

Among other possible views one can look at the results of paper as the generalization of the results of the two papers of Van der Jeugt et al. [12], [13]. The authors of these papers introduced convolutions of known families of classical orthogonal polynomials such as Hermite or Laguerre considered at two variables thus obtaining bivariate polynomials. They applied their results in Lie algebra and its generalizations.

Our "convolutions" concern generalizations of Hermite polynomials (q-Hermite, and Al-Salam–Chihara). As possible applications we mean the ones in analysis, two dimensional orthogonal polynomials theory or probability.

Since in our paper appear kernels built of mostly $q-$Hermite and Al-Salam–Chihara one should remark that some of the technics used in the proofs resemble those used in e.g. [8]. But by no means results are the same.

The paper is organized as follows. In the next two Subsections (i.e. 1.2 and 1.3) we provide simple introduction to $q-$series theory presenting typical notation used and presenting a few typical families of the so called basic orthogonal polynomials. The word basic comes from the base which is the parameter in most cases denoted by q. We do this since notation and terminology used in $q-$series theory is somewhat specific and not widely known to those not working within this field. We are also purposely not using notation based on hypergeometric series since it is mostly known to specialists of special functions theory. We believe that the results presented in the paper can be applied in various fields of traditional analysis like the theory of Fourier expansions, theory of reproducing kernels, orthogonal polynomials theory and last but not least probability theory. Then in Section 2 we present our main results, open questions and remarks are in Section 3 while laborious proofs are in Section 4.

1.2. Notation. We use notation traditionally used in the so called $q-$series theory. Since not all readers are familiar with it we will recall now this notation.

Throughout the paper, q is a parameter. We will assume that $-1 < q \leq 1$ unless otherwise stated. Let us define $[0]_q = 0$; $[n]_q = 1 + q + \ldots + q^{n-1}$, $[n]_q! = \prod_{j=1}^{n} [j]_q$, with $[0]_q! = 1$ and

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \frac{[n]_q!}{[n-k]_q![k]_q!} & , \quad n \geq k \geq 0 \\ 0 & , \quad otherwise \end{cases}.$$

It will be useful to use the so called $q-$Pochhammer symbol for $n \geq 1$:

$$(a;q)_n = \prod_{j=0}^{n-1} \left(1 - aq^j\right),$$

$$(a_1, a_2, \ldots, a_k; q)_n = \prod_{j=1}^{k} (a_j; q)_n,$$

with $(a;q)_0 = 1$. Often $(a;q)_n$ as well as $(a_1, a_2, \ldots, a_k; q)_n$ will be abbreviated to $(a)_n$ and $(a_1, a_2, \ldots, a_k)_n$ respectively, if it will not cause misunderstanding.

It is easy to notice that for $|q| < 1$ we have $(q)_n = (1-q)^n [n]_q!$ and

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \frac{(q)_n}{(q)_{n-k}(q)_k} & , \quad n \geq k \geq 0 \\ 0 & , \quad otherwise \end{cases}.$$

Notice that $[n]_1 = n, [n]_1! = n!$, $\begin{bmatrix} n \\ k \end{bmatrix}_1 = \binom{n}{k}$, $(a;1)_n = (1-a)^n$ and $[n]_0 = \begin{cases} 1 & if \quad n \geq 1 \\ 0 & if \quad n = 0 \end{cases}$, $[n]_0! = 1$, $\begin{bmatrix} n \\ k \end{bmatrix}_0 = 1$, $(a;0)_n = \begin{cases} 1 & if \quad n = 0 \\ 1-a & if \quad n \geq 1 \end{cases}$.

In the sequel we shall also use the following useful notation:

$$S(q) = \begin{cases} [-\frac{2}{\sqrt{1-q}}, \frac{2}{\sqrt{1-q}}] & if \quad |q| < 1 \\ \mathbb{R} & if \quad q = 1 \end{cases}, \tag{1.5}$$

$$I_A(x) = \begin{cases} 1 & if \quad x \in A \\ 0 & if \quad x \notin A \end{cases}. \tag{1.6}$$

1.3. Polynomials.

1.3.1. *$q-$Hermite.* Let $\{H_n(x|q)\}_{n\geq 0}$ denote the family of the so called $q-$ Hermite (briefly qH) polynomials. That is the one parameter family of orthogonal polynomials satisfying the following three term recurrence:

$$H_{n+1}(x|q) = xH_n(x|q) - [n]_q H_{n-1}(x|q), \tag{1.7}$$

with $H_{-1}(x|q) = 0$ and $H_0(x|q) = 1$. In fact in the literature (see e.g. [1], [10], [15]) we encounter more often the re-scaled versions of these polynomials. Namely more often appear under the name of $q-$Hermite polynomials the following polynomials $\{h_n(x|q)\}_{n\geq 0}$ defined by their three term recurrence:

$$h_{n+1}(x|q) = 2xh_n(x|q) - (1-q^n)h_{n-1}(x|q), \tag{1.8}$$

with $h_{-1}(x|q) = 0$ and $h_0(x|q) = 1$. These polynomials are related to one another by the relationship $\forall n \geq -1$:

$$H_n(x|q) = \frac{h_n\left(x\sqrt{1-q}/2|q\right)}{(1-q)^{n/2}}, \tag{1.9}$$

for $|q| < 1$. For $q = 1$ we have $h_n(x|1) = 2^n x^n$ while $H_n(x|1) = H_n(x)$, where polynomials $H_n(x)$ are the so called 'probabilistic' Hermite polynomials i.e. classical, monic† polynomials orthogonal with respect to $\exp\left(-x^2/2\right)$. Observe further that $h_n(x|0) = U_n(x)$ and $H_n(x|0) = U_n(x/2)$, where U_n denotes the so called Chebyshev polynomial of the second kind (for details see e.g. [1]).

The polynomials H_n have nice probabilistic interpretation (see e.g. [22]) and besides they constitute the real generalization of the ordinary Hermite polynomials. That is why we will use them in this paper. The results presented here can be easily adopted and expressed in terms of polynomials h_n.

The generating function of these polynomials is given by the following formula that is in fact adapted to our setting formula (14.26.1) of [15]

$$\varphi_H(x|\rho,q) = \sum_{n\geq 0}\frac{\rho^n}{[n]_q!}H_n(x|q) = \frac{1}{\prod_{j=0}^{\infty} v\left(x\sqrt{1-q}/2|\rho q^j\sqrt{1-q}\right)}, \tag{1.10}$$

convergent for $|\rho(1-q)| < 1$, $x \in S(q)$, where we denoted

$$v(x|t) = 1 - 2xt + t^2. \tag{1.11}$$

Let us observe that $\forall x \in [-1,1]$, $t \in \mathbb{R} : v(x|t) \geq 0$.

Adapting formula (14.26.2) of [15] to our setting we have:

$$\int_{S(q)} H_n(x|q) H_m(x|q) f_N(x|q)\, dx = [n]_q!\delta_{mn}, \tag{1.12}$$

with

$$f_N(x|q) = \frac{\sqrt{(1-q)(4-(1-q)x^2)}\,(q)_\infty}{2\pi}\prod_{j=1}^{\infty} l\left(x\sqrt{1-q}/2|q^j\right), \tag{1.13}$$

for $x \in S(q)$, where

$$l(x|a) = (1+a)^2 - 4ax^2. \tag{1.14}$$

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†i.e. polynomials with leading coefficient equal to 1.

Ismail et al. showed that (see [11])

$$\lim_{q\to 1^-} f_N(x|q) = \frac{1}{\sqrt{2\pi}} \exp\left(-x^2/2\right), \tag{1.15}$$

$$\lim_{q\to 1^-} \varphi_H(x|\rho,q) = \exp\left(x\rho - x^2/2\right). \tag{1.16}$$

Apart from $q-$Hermite polynomials we will need the so called big $q-$Hermite (briefly bqH) polynomials $\{H_n(x|a,q)\}_{n\geq -1}$ with $a \in \mathbb{R}$. They are defined through their three term recurrence:

$$H_{n+1}(x|a,q) = (x - aq^n)H_n(x|a,q) - [n]_q H_{n-1}(x|a,q), \tag{1.17}$$

with $H_{-1}(x|a,q) = 0$, $H_0(x|a,q) = 1$. To support intuition let us remark that $H_n(x|a,1) = H_n(x-a)$ and $H_n(x|a,0) = U_n(x/2) - aU_{n-1}(x/2)$.

One knows its relationship with the $q-$Hermite polynomials:

$$H_n(x|a,q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-a)^k q^{\binom{k}{2}} H_{n-k}(x|q),$$

and that (see e.g. [15], (14.18.2) with an obvious modification for polynomials H_n):

$$\int_{S(q)} H_n(x|a,q) H_m(x|a,q) f_{bN}(x|a,q)\, dx = [n]_q! \delta_{mn},$$

$$\sum_{n\geq 0} \frac{t^n}{[n]_q!} H_n(x|a,q) = \varphi_H(x|t,q)\left((1-q)at\right)_\infty,$$

where

$$f_{bN}(x|a,q) = f_N(x|q)\, \varphi_H(x|a,q). \tag{1.18}$$

We will need the following Lemma concerning another relationship between polynomials $H_n(x|q)$ and $H_n(x|a,q)$.

1.1. Lemma. *Let us define for*
$\forall n \geq 0; x \in S(q); (1-q)t^2 < 1 : \eta_n(x|t,q) = \sum_{j\geq 0} \frac{t^j}{[j]_q!} H_{j+n}(x|q)$. *Then*

$$\eta_n(x|t,q) = H_n(x|t,q)\, \varphi_H(x|t,q),$$

where $H_n(x|t,q)$ is the bqH polynomial defined by (1.17).

Proof. In a version with continuous $q-$Hermite polynomials h defined by (1.8) and $h_n(x|t,q)$ are the big $q-$Hermite polynomials as defined in [15] (14.18.4) this formula has been proved as a particular case in [25] (2.1). We notice that $\eta_0(x|t,q) = \varphi_H(x|t,q)$. To switch to polynomials H_n using (1.9) is elementary. ☐

1.2. Remark. Let us remark that Carlitz in [7] considered similar shifted characteristic functions of the form $\sum_{j\geq 0} \frac{t^j}{(q)_j} w_{n+j}(x|q)$ with Rogers–Szegö polynomials w_n (see discussion below following formula (2.3)). From this result of Carlitz one can also deduce assertion of Lemma 1.1.

1.3.2. *Al-Salam–Chihara.* Next family of polynomials that we are going to consider depends on 2 (apart from q) parameters denoted by a and b, that satisfy the following three term recurrence (see e.g. [15],(14.8.4)):

$$A_{n+1}(x|a,b,q) = (2x-(a+b)q^n)A_n(x|a,b,q) - (1-abq^{n-1})(1-q^n)A_{n-1}(x|y,\rho,q), \tag{1.19}$$

with $A_{-1}(x|a,b,q) = 0$, $A_0(x|a,b,q) = 1$. These polynomials will be called Al-Salam–Chihara polynomials $\{A_n(x|a,b,q)\}_{n\geq -1}$ (briefly ASC). We will assume in the sequel that $|ab|<1$. This assumption together with $|q|\leq 1$ guarantees that the measure that makes these polynomials orthogonal is positive. It follows directly follows from Favard's theorem since then $(1-abq^{n-1})(1-q^{n})>0$.

In the sequel in fact we will consider these polynomials with complex parameters forming a conjugate pair and also re-scaled. Namely we will take $a=\frac{\sqrt{1-q}}{2}\rho(y-i\sqrt{\frac{4}{1-q}-y^{2}})$, $b=\frac{\sqrt{1-q}}{2}\rho(y+i\sqrt{\frac{4}{1-q}-y^{2}})$, with $y\in S\left(q\right)$ and $|\rho|<1$. More precisely we will consider polynomials $\left\{P_{n}\left(x|y,\rho,q\right)\right\}_{n\geq 0}$ defined by:

$$A_{n}\left(x\frac{\sqrt{1-q}}{2}|a,b,q\right)/(1-q)^{n/2}=P_{n}\left(x|y,\rho,q\right).$$

One can easily notice that $a+b=\rho y\sqrt{1-q}$, $ab=\rho^{2}$ and thus that the polynomials P_{n} satisfy the following three term recurrence:

$$P_{n+1}\left(x|y,\rho,q\right)=(x-\rho yq^{n})P_{n}\left(x|y,\rho,q\right)-[n]_{q}(1-\rho^{2}q^{n-1})P_{n-1}\left(x|y,\rho,q\right), \tag{1.20}$$

with $P_{-1}\left(x|y,\rho,q\right)=0$, $P_{0}\left(x|y,\rho,q\right)=1$.

1.3. Remark. To support intuition let us remark (following e.g. [22]) that $P_{n}\left(x|y,\rho,1\right)=H_{n}\left(\frac{x-\rho y}{\sqrt{1-\rho^{2}}}\right)\left(1-\rho^{2}\right)^{n/2}$. On the other hand $P_{n}\left(x|y,\rho,0\right)=U_{n}\left(x/2\right)-\rho yU_{n-1}\left(x/2\right)$ $+\rho^{2}U_{n-2}\left(x/2\right)$, where $U_{n}\left(x\right)$ denotes Chebyshev polynomial of the second kind.

It is known see e.g. [15], (formula (14.8.13) adapted to our setting), [6], [22] that the polynomials P_{n} have the following generating function:

$$\varphi_{P}\left(x|y,\rho,t,q\right)=\sum_{n\geq 0}\frac{t^{n}}{[n]_{q}!}P_{n}\left(x|y,\rho,q\right)=\prod_{j=0}^{\infty}\frac{v\left(y\sqrt{1-q}/2|\rho tq^{j}\sqrt{1-q}\right)}{v\left(x\sqrt{1-q}/2|tq^{j}\sqrt{1-q}\right)},$$

convergent for $\left|t\sqrt{1-q}\right|,|\rho|<1$, $x,y\in S\left(q\right)$.

We also have (see e.g. [22]) or :

$$\int_{S(q)}P_{n}(x|y,\rho,q)P_{m}\left(x|y,\rho,q\right)f_{CN}\left(x|y,\rho,q\right)dx=\delta_{nm}\left[n\right]_{q}!(\rho^{2})_{n}, \tag{1.21}$$

where

$$f_{CN}\left(x|y,\rho,q\right)=f_{N}\left(x|q\right)\frac{\left(\rho^{2}\right)_{\infty}}{\prod_{j=0}^{\infty}\omega\left(x\sqrt{1-q}/2,y\sqrt{1-q}/2|\rho q^{j}\right)},$$

with

$$\omega\left(x,y|\rho\right)=\left(1-\rho^{2}\right)^{2}-4\rho(1+\rho^{2})xy+4\rho^{2}\left(x^{2}+y^{2}\right). \tag{1.22}$$

1.4. Remark. It was shown in [26](Lemma 1, (v)) that for $|q|<1$ function $|f_{CN}(x|y,\rho,q)/f_{N}\left(x|q\right)|$ is bounded both from below and above hence square integrable on the square $S(q)\times S\left(q\right)$ with respect to the measure $f_{N}\left(x|q\right)f_{CN}\left(x|y,\rho,q\right)dxdy$. This will guarantee existence and convergence of some Fourier expansions considered in the next section.

We will call the densities f_{N} and f_{CN} respectively $q-$Normal and $(q,\rho)-$Conditional Normal. The names are justified by the nice probabilistic interpretations of these densities presented e.g. in [3], [4], [5], [6], [22] or [19]. Besides in [11] it was shown also that:

$$\lim_{q\rightarrow 1^{-}}f_{CN}\left(x|y,\rho,q\right)=\exp\left(-\frac{\left(x-\rho y\right)^{2}}{2(1-\rho^{2})}\right)/\sqrt{2\pi(1-\rho^{2})}. \tag{1.23}$$

1.5. Remark. Notice that convergence (1.15) and (1.23) in distribution of appropriate measures with these densities can be easily seen since we have $\lim_{q\longrightarrow 1^-} H_n\left(x|q\right) = H_n\left(x\right)$ and $\lim_{q\longrightarrow 1^-} P_n\left(x|y,\rho,q\right) = H_n\left(\frac{x-\rho y}{\sqrt{1-\rho^2}}\right)\left(1-\rho^2\right)^{n/2}$, hence we have convergence of appropriate moments. As stated above rigorous proofs of convergence of the densities can be found in [11].

We end up this section by recalling an auxiliary simple result that will be used in following sections many times. It has been formulated and proved in [25] Proposition 2.

1.6. Proposition. *Let $\sigma_n\left(\rho|q\right) = \sum_{j\geq 0}\frac{\rho^j}{[j]_q!}\xi_{n+j}$ for $|\rho|<1$, $-1<q\leq 1$ and certain sequence $\left\{\xi_m\right\}_{m\geq 0}$ such that σ_n exists for every n. Then*

$$\sigma_n\left(\rho q^m|q\right) = \sum_{k=0}^{m}\left(-1\right)^k \begin{bmatrix} m \\ k \end{bmatrix}_q q^{\binom{k}{2}}\left(1-q\right)^k\rho^k\sigma_{n+k}(\rho|q). \tag{1.24}$$

1.7. Remark. Notice that this Proposition is trivially true for both $q=0$ and $q=1$.

2. Main Results

One of our main interests in this paper are the generalizations of the Poisson-Mehler formula (1.1).

It is well known that convergence in (1.1) takes place for $x,y\in S\left(q\right)$, $|\rho|<1$ and for $|q|<1$ is uniform. For $q=1$ we have almost uniform convergence.

As a immediate corollary of Proposition 1.6 we have:

2.1. Corollary. *For $|q|<1$ we have:*

$$\gamma_{i,j}\left(x,y|\rho q^m,q\right) = \sum_{k=0}^{m}\left(-1\right)^k \begin{bmatrix} m \\ k \end{bmatrix}_q q^{\binom{k}{2}}\left(1-q\right)^k\rho^k\gamma_{i+k,j+k}\left(x,y|\rho,q\right), \tag{2.1}$$

$$H_i\left(x|q\right)H_j\left(y|q\right) = \sum_{k\geq 0}\left(-1\right)^k q^{\binom{k}{2}}\frac{\rho^k}{\left(q\right)_k}\gamma_{i+k,j+k}\left(x,y|\rho,q\right), \tag{2.2}$$

where $\gamma_{i,j}(x,y|\rho,q)$ is defined by (1.2). Formula (2.1) is also true trivially for $q=1$.

Proof. First assertion we get by applying directly (1.24) by setting $\sigma_{i,j}=\gamma_{i,j}$. Second assertion we get by passing in the first one with m to infinity and then noticing firstly that $\lim_{m\to\infty}\left[\begin{smallmatrix} m \\ k\end{smallmatrix}\right]_q = \frac{1}{[k]_q!}$ and finally that $\frac{(1-q)^k}{[k]_q!} = \frac{1}{(q)_k}$. ☐

Now let us turn to polynomials $Q_{i,j}\left(x,y|\rho,q\right)$ defined by (1.3). It was shown in [22] that for all $-1<q\leq 1$, $|\rho|<1$, $x,y\in\mathbb{R}$:

$$Q_{i,j}\left(x,y|\rho,q\right) = \sum_{s=0}^{j}(-1)^s q^{\binom{s}{2}}\begin{bmatrix} j \\ s \end{bmatrix}_q \rho^s H_{j-s}\left(y|q\right)P_{i+s}\left(x|y,\rho,q\right)/\left(\rho^2\right)_{i+s}, \tag{2.3}$$

and $Q_{i,j}\left(x,y|\rho,q\right) = Q_{j,i}\left(y,x|\rho,q\right)$.

2.2. Remark. It has to be remarked that Carlitz in [7] considered the sum $\xi_{k,j}(x,y|\rho,q)$ $=\sum_{n\geq 0}\frac{\rho^n}{(q)_n}w_{n+k}\left(x|q\right)w_{n+j}\left(y|q\right)$, where $w_n(x|q)$ are the so called Rogers–Szegö polynomials related to polynomials $h_n(x|q)$ by the formula: $h_n(x|q)=e^{ni\theta}w_n(e^{-2i\theta}|q)$ with x $=\cos\theta$, $i-$imaginary unit. Indeed it turned out that functions $\xi_{k,j}$ also have the property that

$$\xi_{k,j}(x,y|\rho,q) = \nu_{k,j}(x,y|\rho,q)\xi_{0,0}(x,y|\rho,q),$$

where $\nu_{k,j}$ are polynomials of degree $k+j$ in x and y. However to show that $\nu_{k,j}(e^{-i\theta},e^{-i\eta}|\rho,q)$ can be expressed as $Q_{k,j}(\cos\theta,\cos\eta|\rho,q)$ is not an easy task. Discussion on this subject is in [23]. In particular see the proof of Proposition 5.

In particular we have

$$Q_{k,0}\left(x,y|\rho,q\right)=P_{k}\left(x|y,\rho,q\right)/\left(\rho^{2}\right)_{k}. \tag{2.4}$$

To analyze further properties of polynomials $Q_{k,j}$ let us introduce the following 2 dimensional density defined for $S^{2}\left(q\right)\overset{df}{=}S\left(q\right)\times S\left(q\right).$

$$f_{2D}\left(x,y|\rho,q\right)=f_{CN}\left(x|y,\rho,q\right)f_{N}\left(y|q\right). \tag{2.5}$$

Measure that has density f_{2D} will be called $(\rho,q)-$bivariate Normal (briefly $(\rho,q)-2N$). Obviously $f_{2D}\left(x,y|\rho,q\right)=\gamma_{0,0}\left(x,y|\rho,q\right)f_{N}\left(x|q\right)f_{N}\left(y|q\right).$ Its applications in theories of probability and Markov stochastic processes have been presented in [19] and [20].

Here below we give another interpretation of the polynomials $Q_{n,m}$ in particular its connection with the big $q-$Hermite polynomials.

2.3. Proposition. *For $|q|<1$, $|\rho|<1$, $x,y\in\mathbb{R}$ we have:*

i) $\forall i,j,m,k,i+j\neq m+k,$

$$\int_{S^{2}(q)}Q_{i,j}\left(x,y|\rho,q\right)Q_{m,k}\left(x,y|\rho,q\right)f_{2D}\left(x,y|\rho,q\right)dxdy=0,$$

ii) $\forall i,j,m,k,\ i+j=m+k,k>j:$

$$\int_{S^{2}(q)}Q_{n-j,j}\left(x,y|q\right)Q_{n-k,k}\left(x,y|\rho,q\right)f_{2D}\left(x,y|\rho,q\right)dxdy=$$
$$(-1)^{k-j}\frac{\rho^{k-j}q^{\binom{k-j}{2}}\left[j\right]_{q}!\left[n-j\right]_{q}!}{(\rho^{2})_{n}}\sum_{s=0}^{j}q^{s(s-1)+ns}\begin{bmatrix}k\\k-j+s\end{bmatrix}_{q}$$
$$\times\begin{bmatrix}n-j+s\\s\end{bmatrix}_{q}\rho^{2s}\left(\rho^{2}q^{n-j+s}\right)_{j-s}.$$

iii)

$$\sum_{n,m\geq0}\frac{t^{n}s^{m}}{\left[n\right]_{q}!\left[m\right]_{q}!}Q_{n,m}\left(x,y|\rho,q\right)=\frac{f_{bN}\left(x|t,q\right)f_{bN}\left(y|s,q\right)}{f_{2D}(x,y|\rho,q)}\sum_{k\geq0}\frac{\rho^{k}}{\left[k\right]_{q}!}H_{k}\left(x|t,q\right)H_{k}\left(y|s,q\right),$$

where function f_{bN} is defined by (1.18). The above mentioned formulae are also true for $q=1$.

iv) $\forall m\geq0:$

$$Q_{i,j}\left(x,y|\rho q^{m},q\right)\prod_{i=0}^{m-1}\omega\left(x\sqrt{1-q}/2,y\sqrt{1-q}/2|\rho q^{i}\right)= \tag{2.6}$$
$$\left(\rho^{2}\right)_{2m}\sum_{k=0}^{m}(-1)^{k}\begin{bmatrix}m\\k\end{bmatrix}_{q}q^{\binom{k}{2}}\left(1-q\right)^{k}\rho^{k}Q_{i+k,j+k}\left(x,y|\rho,q\right),$$

where polynomial ω is defined by (1.22). In particular we have:

$$\prod_{j=0}^{n-1}\omega\left(x\sqrt{1-q}/2,y\sqrt{1-q}/2|\rho q^{j}\right)= \left(\rho^{2}\right)_{2n}\sum_{k=0}^{n}(-1)^{k}\begin{bmatrix}n\\k\end{bmatrix}_{q}q^{\binom{k}{2}}(1-q)^{k}\rho^{k}Q_{k,k}(x,y|\rho,q),\tag{2.7}$$

and

$$q^{\binom{n}{2}}\rho^{n}(1-q)^{n}Q_{n,n}(x,y|\rho,q)= \sum_{k=0}^{n}(-1)^{k}q^{\binom{n-k}{2}}\begin{bmatrix}n\\k\end{bmatrix}_{q}\frac{\prod_{j=0}^{k-1}\omega\left(x\sqrt{1-q}/2,y\sqrt{1-q}/2|\rho q^{j}\right)}{\left(\rho^{2}\right)_{2k}},\tag{2.8}$$

with understanding that $\prod_{j=0}^{k-1}$ for $k=0$ is equal to 1.

Proof. Is shifted to section 4. □

Our main results follow in fact directly the results presented above.

2.4. Theorem. *Either for $|q|<1; x,y\in S(q); |\rho|<1$ we have:*
i)

$$H_{i}(x|q)H_{j}(y|q)\frac{\prod_{k=0}^{\infty}\omega\left(x\sqrt{1-q}/2,y\sqrt{1-q}/2|\rho q^{k}\right)}{\left(\rho^{2}\right)_{\infty}} =\sum_{k=0}^{\infty}(-1)^{k}q^{\binom{k}{2}}\frac{\rho^{k}}{[k]_{q}!}Q_{i+k,j+k}(x,y|\rho,q).$$

In particular we get:
ii)

$$1/\sum_{n\geq0}\frac{\rho^{n}}{[n]_{q}!}H_{n}(x|q)H_{n}(y|q)=\frac{\prod_{k=0}^{\infty}\omega\left(x\sqrt{1-q}/2,y\sqrt{1-q}/2|\rho q^{k}\right)}{\left(\rho^{2}\right)_{\infty}} =\sum_{k=0}^{\infty}(-1)^{k}q^{\binom{k}{2}}\frac{\rho^{k}}{[k]_{q}!}Q_{k,k}(x,y|\rho,q).\tag{2.9}$$

The last formula is valid also for $x,y\in\mathbb{R}$, $q=1$ and $|\rho|<1/2$.

Proof. To get i) we pass in (2.6) with m to infinity noting by (2.3) and (1.20) that $Q_{n,m}(x,y|0,q)=H_{n}(x|q)H_{m}(y|q)$. On the way we observe that $\lim_{m\to\infty}\begin{bmatrix}m\\k\end{bmatrix}_{q}=\frac{1}{(q)_{k}}=(1-q)^{-k}\frac{1}{[k]_{q}!}$. As far as the case $q=1$ is concerned denote by $g_{N}(x,y,\rho)$ density of the bivariate Normal density with parameters $\sigma_{1}=\sigma_{2}=1$, correlation coefficient ρ. Then notice that function $\exp(-\frac{1}{2}(x^{2}+y^{2}))/g_{N}(x,y,\rho)$ is square integrable on the plane with respect to $g_{N}(x,y,\rho)$ if $|\rho|<1/2$. □

3. Open problems and comments

3.1. Remark. The non-symmetric kernels constructed of bqH polynomials were given in [18]. Formula ii) of Proposition 2.3 gives its new interpretation. Besides, recall that these kernels were expressed using basic hypergeometric function ${}_{3}\phi_{2}$. Expansion on the left hand side of Proposition 2.3ii) gives new outlook on the properties of this function.

Notice also that for $q=1$ we have $\eta(x|t,1)=\exp(xt-\frac{t^2}{2})$, $H_n(x|t,1)=H_n(x-t)$ and

$$\sum_{n\geq 0}\frac{\rho^n}{n!}H_n(x)H_n(y)=\exp(\frac{x^2}{2}-\frac{(x-\rho y)^2}{2(1-\rho^2)}),$$

hence generating function of polynomials $Q_{i,j}$ can be calculated explicitly.

Similarly for $q=0$ we have $\eta(x|t,0)=\frac{1}{1-xt+t^2}$ (characteristic function of the Chebyshev polynomials) and $H_n(x|t,0)=U_n(x/2)-tU_{n-1}(x/2)$ (see e.g. [24]) hence also in this case we can get explicit form of the characteristic function of polynomials $Q_{i,j}$.

3.2. Remark. First of all notice that the left hand side of (2.9) is equal to $1/\gamma_{0,0}(x,y|\rho,q)$ $=f_N(x|q)/f_{CN}(x|y,\rho,q)$ and that it is a symmetric (with respect to x and y) function. In [21] there was presented (formula 5.3) an expansion of this function involving polynomials P_n and certain polynomials related to $q-$Hermite ones. The expansion was non-symmetric for every partial sum. Thus we get another expansion of known important special function.

3.3. Remark. Assertion i) of Proposition 2.3 states that polynomials $Q_{n,m}$ and $Q_{i,j}$ are orthogonal with respect to two dimensional measure μ_{2D} with the density given by (2.5) if only the $n+m\neq i+j$. Let us define space $\mathcal{L}=L_2\left(S^2(q),\mathcal{B},\mu_{2D}\right)$ of functions $f:S^2(q)\longrightarrow\mathbb{R}$ square integrable with respect to the measure μ_{2D}. Do polynomials $Q_{m,n}$ constitute a base of this space? It seems that yes, but not orthogonal. We can define subspaces of $\Lambda_m=span\{Q_{m,0},\ldots,Q_{0,m}\}$ of polynomials that are linear combinations of polynomials $Q_{i,j}$ such that $i+j=m$. Subspaces Λ_m are mutually orthogonal. Besides following argument that polynomials are dense in $\mathcal{L}$ we deduce that $\mathcal{L}=\bigoplus_{n=0}^{\infty}\Lambda_n$. What is the orthogonal base of $\mathcal{L}$? We have calculated covariances between polynomials $Q_{i,j}$ from Λ_m following (2.3) and (1.21). Thus we can follow Gram-Schmidt orthogonalization procedure within the spaces Λ_m. Is the union of orthogonal bases of Λ_m an orthogonal base of $\mathcal{L}$? Again it seems that yes. It would be interesting to find this base. Note that orthogonal polynomials on the plane are not an easy extension of the one-dimesional case. There are problems in defining them. For details see e.g. [14], [17], [16]. Recently in [9] there was defined a family of two dimensional polynomials that are two dimensional analogies of $q-$Hermite polynomials. Analogy is in the sense that many properties of the one-dimesional $q-$Hermite polynomials are retained in its two dimensional version.

3.4. Remark. In 2001 Wünsche in [27] considered Hermite and Laguerre polynomials on the plane. He has not however related his Hermite polynomials to any particular measure on the plane. In particular he defined Hermite polynomials depending on parameters forming a $2x2$ matrix. This matrix is however not connected in any way to the covariance matrix of the measure with respect to which these polynomials are supposed to be orthogonal.

On the other hand definition of polynomials $Q_{i,j}$ depends heavily on the measure with the density f_{2D}. For $q=1$ following (2.3), we have

$$Q_{i,j}(x,y|\rho,1)=\sum_{k=0}^{j}(-1)^k\binom{j}{k}H_{j-k}(y)H_{k+i}\left(\frac{x-\rho y}{\sqrt{1-\rho^2}}\right)/\left(\sqrt{1-\rho^2}\right)^{k+i}.$$

Hence polynomials $Q_{i,j}(x,y,\rho,1)$ are in fact another (different from that of Wünsche's) family of two dimensional generalization of Hermite polynomials.

4. Proofs

Proof of Proposition 2.3. i) We use (2.3), assume that $i > m$. We have:

$$
\begin{aligned}
&\int_{S^2(q)} Q_{i,j}(x,y|q)\, Q_{m,k}(x,y|\rho,q)\, f_{2D}(x,y|\rho,q)\, dxdy = \\
&\sum_{s=0}^{j}\sum_{t=0}^{k}(-1)^{s+t} q^{\binom{s}{2}} q^{\binom{t}{2}} \begin{bmatrix} j \\ s \end{bmatrix}_q \begin{bmatrix} k \\ t \end{bmatrix}_q \rho^{s+t} \frac{1}{(\rho^2)_{i+s}(\rho^2)_{m+t}} \\
&\times \int_{S(q)} H_{j-s}(y|q)\, H_{k-t}(y|q)\, f_N(y|q) \\
&\times \int_{S(q)} P_{i+s}(x|y,\rho,q)\, P_{m+t}(x|y,\rho,q)\, f_{CN}(x|y,\rho,q)\, dxdy = \\
&(-1)^{i-m}\rho^{i-m} \sum_{s=0\vee m-i}^{j\wedge k+m-i} q^{\binom{s}{2}+\binom{i-m+s}{2}} \begin{bmatrix} j \\ s \end{bmatrix}_q \begin{bmatrix} k \\ i+s-m \end{bmatrix}_q \times \\
&\rho^{2s}\frac{[i+s]_q!}{(\rho^2)_{i+s}} \int_{S(q)} H_{j-s}(y|q)\, H_{k+m-i-s}(y|q)\, f_N(y|q)\, dy = 0,
\end{aligned}
$$

if $j-s \neq k+m-i-s$ i.e. if $j+i \neq k+m$.

Now for $j+i = k+m$ and assuming that $k \geq j$ we get:

$$
\begin{aligned}
&\int_{S^2(q)} Q_{n-j,j}(x,y|q)\, Q_{n-k,k}(x,y|\rho,q)\, f_{2D}(x,y|\rho,q)\, dxdy = \\
&\sum_{s=0}^{j}\sum_{t=0}^{k}(-1)^{s+t} q^{\binom{s}{2}} q^{\binom{t}{2}} \begin{bmatrix} j \\ s \end{bmatrix}_q \begin{bmatrix} k \\ t \end{bmatrix}_q \rho^{s+t} \frac{1}{(\rho^2)_{n-j+s}(\rho^2)_{n-k+t}} \\
&\times \int_{S(q)} H_{j-s}(y|q)\, H_{k-t}(y|q)\, f_N(y|q) \\
&\times \int_{S(q)} P_{n-j+s}(x|y,\rho,q)\, P_{n-k+t}(x|y,\rho,q)\, f_{CN}(x|y,\rho,q)\, dxdy = \\
&(-1)^{k-j}\rho^{k-j}\sum_{s=0}^{j} q^{\binom{s}{2}+\binom{k-j+s}{2}} \begin{bmatrix} j \\ s \end{bmatrix}_q \begin{bmatrix} k \\ k-j+s \end{bmatrix}_q \rho^{2s}\frac{[n-j+s]_q!}{(\rho^2)_{n-j+s}} \\
&\times \int_{S(q)} H_{j-s}(y|q)\, H_{j-s}(y|q)\, f_N(y|q)\, dy
\end{aligned}
$$

$$
\begin{aligned}
&= (-1)^{k-j}\rho^{k-j}\sum_{s=0}^{j} q^{\binom{s}{2}+\binom{k-j+s}{2}} \begin{bmatrix} j \\ s \end{bmatrix}_q \begin{bmatrix} k \\ k-j+s \end{bmatrix}_q \rho^{2s}\frac{[n-j+s]_q!}{(\rho^2)_{n-j+s}} [j-s]_q! \\
&= (-1)^{k-j}\frac{\rho^{k-j} q^{\binom{k-j}{2}} [j]_q!\, [n-j]_q!}{(\rho^2)_n}\sum_{s=0}^{j} q^{s(s-1)+ns} \begin{bmatrix} k \\ k-j+s \end{bmatrix}_q \\
&\times \begin{bmatrix} n-j+s \\ s \end{bmatrix}_q \rho^{2s}\left(\rho^2 q^{n-j+s}\right)_{j-s}
\end{aligned}
$$

we use here $s(s-1)/2 + (s+n)(s-1+n)/2 - s(s-1) - n(n-1)/2 = ns$

ii) We have

$$\sum_{i\geq 0,j\geq 0}\frac{s^{i}t^{j}}{[i]_q!\,[j]_q!}Q_{i,j}\left(x,y|\rho,q\right)=$$

$$\frac{1}{\gamma_{0,0}\left(x,y|\rho,q\right)}\sum_{i\geq 0,j\geq 0}\frac{t^{i}s^{j}}{[i]_q!\,[j]_q!}\sum_{n\geq 0}\frac{\rho^{n}}{[n]_q!}H_{i+n}\left(x|q\right)H_{n+j}\left(y|q\right)$$

$$=\sum_{n\geq 0}\frac{\rho^{n}}{[n]_q!}\sum_{j=0}^{\infty}\frac{s^{j}}{[j]_q!}H_{n+j}\left(y|q\right)\sum_{i\geq 0}\frac{t^{i}}{[i]_q!}H_{n+i}\left(x|q\right).$$

Now we use Lemma 1.1 twice and get

$$\sum_{i\geq 0,j\geq 0}\frac{s^{i}t^{j}}{[i]_q!\,[j]_q!}Q_{i,j}\left(x,y|\rho,q\right)=\frac{\varphi_H(x|t,q)\varphi_H\left(y|s,q\right)}{\gamma_{0,0}(x,y|\rho,q)}\sum_{n\geq 0}\frac{\rho^{n}}{[n]_q!}H_n\left(x|t,q\right)H_n(y|s,q)$$

$$=\frac{1}{(\rho^2)_\infty}\prod_{j=0}^{\infty}\frac{\omega\left(x\sqrt{1-q}/2,y\sqrt{1-q}/2|\rho q^{j}\right)}{v\left(x\sqrt{1-q}/2|t\sqrt{1-q}q^{j}\right)v\left(y\sqrt{1-q}/2|s\sqrt{1-q}q^{j}\right)}$$

$$\times\sum_{n\geq 0}\frac{\rho^{n}}{[n]_q!}H_n\left(x|t,q\right)H_n(y|s,q).$$

iii) First we notice that from (1.3) it follows that for $x,y\in S\left(q\right);\rho^{2}<1,-1<q\leq 1:$

$$\gamma_{i,j}\left(x,y|\rho q^{m},q\right)=Q_{i,j}\left(x,y|\rho q^{m},q\right)\frac{\left(\rho^{2}q^{2m}\right)_\infty}{\prod_{i=0}^{\infty}\omega\left(x\sqrt{1-q}/2,y\sqrt{1-q}/2|\rho q^{m+i}\right)}$$

$$=Q_{i,j}\left(x,y|\rho q^{m},q\right)\frac{\prod_{i=0}^{m-1}\omega\left(x\sqrt{1-q}/2,y\sqrt{1-q}/2|\rho q^{i}\right)}{(\rho^2)_{2m}}\gamma_{0,0}\left(x,y,\rho,q\right),$$

and also that $\gamma_{0,0}\left(x,y|\rho,q\right)=\frac{\left(\rho^{2}\right)_\infty}{\prod_{i=0}^{\infty}\omega\left(x\sqrt{1-q}/2,y\sqrt{1-q}/2|\rho q^{i}\right)}$. Then we apply (2.1) to $\gamma_{i,j}$ above and then use (1.24) and cancel out $\gamma_{0,0}$ on both sides of (2.1). Finally we observe that on both sides we have polynomials hence one can extend the identity for all values of the variables. To get other formula of this assertion we argue by induction checking that the equality is true for $n=0$. Then we put (2.7) into (2.8) and get:

$$\sum_{k=0}^{n}\left(-1\right)^{k}q^{\binom{n-k}{2}}\begin{bmatrix}n\\k\end{bmatrix}_q\frac{\prod_{i=0}^{k-1}\omega\left(x\sqrt{1-q}/2,y\sqrt{1-q}/2|\rho q^{i}\right)}{(\rho^2)_{2k}}=$$

$$\sum_{k=0}^{n}\left(-1\right)^{k}q^{\binom{n-k}{2}}\begin{bmatrix}n\\k\end{bmatrix}_q\sum_{j=0}^{k}(-1)^{j}\begin{bmatrix}k\\j\end{bmatrix}_q q^{\binom{j}{2}}\rho^{j}Q_{j,j}\left(x,y|\rho,q\right)=$$

$$\sum_{j=0}^{n}(-1)^{j}q^{\binom{j}{2}}\begin{bmatrix}n\\j\end{bmatrix}_q\rho^{j}Q_{j,j}\left(x,y|\rho,q\right)\sum_{k=j}^{n}\left(-1\right)^{k}q^{\binom{n-k}{2}}\begin{bmatrix}n-j\\k-j\end{bmatrix}_q=$$

$$\sum_{j=0}^{n}(-1)^{j}q^{\binom{j}{2}}\begin{bmatrix}n\\j\end{bmatrix}_q\rho^{j}Q_{j,j}\left(x,y|\rho,q\right)\sum_{m=0}^{n-j}\left(-1\right)^{m+j}q^{\binom{m}{2}}\begin{bmatrix}n-j\\m\end{bmatrix}_q=$$

$$q^{\binom{n}{2}}\rho^{n}\left(1-q\right)^{n}Q_{n,n}\left(x,y|\rho,q\right)$$

since $\forall n\geq 1:\sum_{i=0}^{n}(-1)^{i}q^{\binom{i}{2}}\begin{bmatrix}n\\i\end{bmatrix}_q=0.$ $\square$

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References

[1] Andrews, George E.; Askey, Richard; Roy, Ranjan. Special functions. Encyclopedia of Mathematics and its Applications, 71. *Cambridge University Press,* Cambridge, 1999. xvi+664 pp. ISBN: 0-521-62321-9; 0-521-78988-5 MR1688958 (2000g:33001)

[2] Bressoud, D. M. A simple proof of Mehler's formula for q-Hermite polynomials. *Indiana Univ. Math. J.* **29** (1980), no. 4, 577–580. MR0578207 (81f:33009)

[3] Bożejko, Marek; Kümmerer, Burkhard; Speicher, Roland. q-Gaussian processes: non-commutative and classical aspects. *Comm. Math. Phys.* **185** (1997), no. 1, 129–154. MR1463036 (98h:81053)

[4] Bryc, Wlodzimierz. Stationary Markov chains with linear regressions. *Stochastic Process. Appl.* **93** (2001), no. 2, 339–348. MR1828779 (2002d:60058)

[5] Bryc, Włodzimierz. Stationary random fields with linear regressions. *Ann. Probab.* **29** (2001), no. 1, 504–519. MR1825162 (2002d:60014)

[6] Bryc, Włodzimierz; Matysiak, Wojciech; Szabłowski, Paweł J. Probabilistic aspects of Al-Salam-Chihara polynomials. *Proc. Amer. Math. Soc.* **133** (2005), no. 4, 1127–1134 (electronic). MR2117214 (2005m:33033)

[7] Carlitz, L. Generating functions for certain q-orthogonal polynomials. *Collect. Math.* **23** (1972), 91–104. MR0316773 (47 #5321)

[8] Ismail, Mourad E. H.; Stanton, Dennis. Classical orthogonal polynomials as moments. *Canad. J. Math.* **49** (1997), no. 3, 520–542. MR1451259 (98f:33033)

[9] M. E. H. Ismail, Ruiming Zhang, On some 2D orthogonal q-polynomials. arXiv:1411.5223.math.CA.

[10] Ismail, Mourad E. H. Classical and quantum orthogonal polynomials in one variable. With two chapters by Walter Van Assche. With a foreword by Richard A. Askey. Encyclopedia of Mathematics and its Applications, 98. *Cambridge University Press,* Cambridge, 2005. xviii+706 pp. ISBN: 978-0-521-78201-2; 0-521-78201-5 MR2191786 (2007f:33001)

[11] Ismail, Mourad E. H.; Stanton, Dennis; Viennot, Gérard. The combinatorics of q-Hermite polynomials and the Askey-Wilson integral. *European J. Combin.* **8** (1987), no. 4, 379–392. MR0930175 (89h:33015)

[12] Van der Jeugt, J. Coupling coefficients for Lie algebra representations and addition formulas for special functions. *J. Math. Phys.* **38** (1997), no. 5, 2728–2740. MR1447890 (98e:33004)

[13] Koelink, H. T.; Van Der Jeugt, J. Convolutions for orthogonal polynomials from Lie and quantum algebra representations. *SIAM J. Math. Anal.* **29** (1998), no. 3, 794–822. MR1617724 (2000i:33022)

[14] Jackson, Dunham. Formal properties of orthogonal polynomials in two variables. *Duke Math. J.* 2 (1936), no. 3, 423–434. MR1545933

[15] Koekoek, Roelof; Lesky, Peter A.; Swarttouw, René F. Hypergeometric orthogonal polynomials and their q-analogues. With a foreword by Tom H. Koornwinder. *Springer Monographs in Mathematics. Springer-Verlag*, Berlin, 2010. xx+578 pp. ISBN: 978-3-642-05013-8 MR2656096 (2011e:33029)

[16] Koornwinder, Tom H. Orthogonal polynomials in two variables which are eigenfunctions of two algebraically independent partial differential operators. *I. Nederl. Akad. Wetensch. Proc. Ser. A 77=Indag. Math.* 36 (1974), 48–58. MR0340673 (49 #5425a)

[17] Krall, H. L.; Sheffer, I. M. Orthogonal polynomials in two variables. *Ann. Mat. Pura Appl.* (4) 76 1967 325–376. MR0228920 (37 #4499)

[18] Askey, Richard A.; Rahman, Mizan; Suslov, Sergeĭ K. On a general q-Fourier transformation with nonsymmetric kernels. *J. Comput. Appl. Math.* **68** (1996), no. 1-2, 25–55. MR1418749 (98m:42033).

[19] Szabłowski, Paweł J. Multidimensional q-normal and related distributions—Markov case. *Electron. J. Probab.* 15 (2010), no. **40**, 1296–1318. MR2678392

[20] Szabłowski, Paweł J. $q-$Wiener and $(\alpha, q)-$ Ornstein–Uhlenbeck processes. A generalization of known processes, *Theory of Probability and Its Applications,* **56** (4), 2012, 742–772, http://arxiv.org/abs/math/0507303

[21] Szabłowski, Paweł J., Expansions of one density via polynomials orthogonal with respect to the other., *J. Math. Anal. Appl.* **383** (2011) 35–54,

[22] Szabłowski, Paweł J., On the structure and probabilistic interpretation of Askey–Wilson densities and polynomials with complex parameters. *J. Funct. Anal.* **262**(2011), 635-659,

[23] Szabłowski, Paweł J. Befriending Askey–Wilson polynomials, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* , Vol . **17**, No. 3 (2014) 1450015 (25 pages), http://arxiv.org/abs/1111.0601

[24] Szabłowski, Paweł J. On the $q-$Hermite polynomials and their relationship with some other families of orthogonal polynomials, *Demonstr. Math.* **66(**2013) no.4, 679-708, http://arxiv.org/abs/1101.2875

[25] Szabłowski, Paweł J. On peculiar properties of generating functions of some orthogonal polynomials, *J. Phys. A: Math. Theor.* **45** (2012) 365207 (12pp) http://arxiv.org/abs/1204.0972

[26] Szabłowski, Paweł J., On summable form of Poisson-Mehler kernel for big q-Hermite and Al-Salam-Chihara polynomials, *Infinite Dimensional Analysis, Quantum Probability and Related Topics*, Vol. **15**, No. 3 (2012), http://arxiv.org/abs/1011.1848

[27] Wünsche, A. Hermite and Laguerre 2D polynomials. Proceedings of the Fifth International Symposium on Orthogonal Polynomials, Special Functions and their Applications (Patras, 1999). *J. Comput. Appl. Math.* **133** (2001), no. 1-2, 665–678. MR1858322 (2002k:33011)