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M-Cofaithful modules and correspondences of closed submodules with coclosed submodules

T. Amouzegar*† and Y. Talebi‡

Abstract

In this paper we introduce and investigate M-cofaithful modules. A module $N \in \sigma[M]$ is called M-cofaithful if for every $o \neq f \in Hom_R(N, X)$ with $X \in \sigma[M]$, $Hom_R(X, M)f \neq 0$. We show that if N is an M-cofaithful weak supplemented module and $Hom_R(N, M)$ a noetherian S-module, then there exists an order-preserving correspondence between the cocolsed R-submodules of N and the closed S-submodules of $Hom_R(N, M)$, where $S = End_R(M)$. Some applications are: (1) the connection between M's being a lifting module and $End_R(M)$'s being an extending ring; (2) the equality between the hollow dimension of a quasi-injective coretractable module M and the uniform dimension of $End_R(M)$.

Keywords: M-Cofaithful modules, Coretractable modules, Closed and coclosed submodules.

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1. Introduction

Throughout this paper, R will denote an arbitrary associative ring with identity, M and N unitary right R-modules with $U = Hom_R(N, M)$ the set of R-homomorphisms of N in M and $S = End_R(M)$ the ring of all R-endomorphisms of M; U is then a left S-module. By $\sigma[M]$ we mean the full subcategory of Mod-R whose objects are submodules of M-generated modules.

Following [5], a module $N \in \sigma[M]$ is said to be *M-faithful* if for every $0 \neq f \in Hom_R(X, N)$ with $X \in \sigma[M]$, $fHom_R(M, X) \neq 0$. When M is itself M-faithful, M is called a *self-faithful* module. Self-faithful modules have been studied by some authors (see, for example, [5, 6, 7, 8]). It is of obvious interest to investigate

*Department of Mathematics, Quchan University of Advanced Technology, Quchan, Iran, Email: t.amoozegar@yahoo.com

†Corresponding Author.

‡Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran, Email: talebi@umz.ac.ir

the dual notion of M-faithful modules. We call a right R-module $N \in \sigma[M]$ M-*cofaithful* if for every $0 \neq f \in Hom_R(N, X)$ with $X \in \sigma[M]$, $Hom_R(X, M)f \neq 0$. When M is itself M-cofaithful, M is called a *self-cofaithful* module. Example of self-cofaithful modules is quasi-injective coretractable modules (Theorem 3.1). In this paper, we investigate M-cofaithful modules.

It is known that there exists a correspondence between the closed submodules of a suitably restricted module and the closed one-side ideal of its endomorphism ring. Such a correspondence is known to hold for semisimple modules, for free modules (see [2]), and for nonsingular modules M when $End_R(E(M))$ is the maximal right quotient ring of $End_R(M)$ (see [13]), hence in particular, for nonsingular retractable modules (see [9]). Some properties of the endomorphism rings of modules, such as being Baer, extending, etc., were then obtained by means of the above lattice isomorphism. Zelmanowitz showed in [12, Theorem 1.2] that when N is an M-faithful R-module, then there exists an order-preserving correspondence between the closed R-submodules of N and the closed S-submodules of $Hom_R(M, N)$, where $S = End_R(M)$. In this paper, we give conditions under which there exists a correspondence between the coclosed R-submodules of an M-cofaithful module N and the closed S-submodules of $Hom_R(N, M)$.

In section 2, we characterize M-cofaithful modules (Proposition 2.1) and study some properties of M-cofaithful modules. For an M-cofaithful module N, we show that $u.dim({}_SU) = h.dim(N_R)$, where $U = Hom_R(N, M)$ (Theorem 2.12). We show that there is a correspondence between the coclosed R-submodules of an M-cofaithful weak supplemented module N and the closed S-submodules of $Hom_R(N, M)$ whenever $Hom_R(N, M)$ is a noetherian S-module. (Theorem 2.13). This result is used in proving that if $Hom_R(N, M)$ is a noetherian S-module, then an M-cofaithful M-cogenerated amply supplemented module N is a lifting right R-module if and only if $Hom_R(N, M)$ is a left extending S-module, where $S = End_R(M)$ (Theorem 2.15). In section 3, we show that M-coretractability characterizes M-cofaithfulness for some important families of modules and conclude that if either (i) M is an amply supplemented quasi-injective coretractable module and S is noetherian, or (ii) M is an amply supplemented $\sum$-self-cogenerator module and S is noetherian, then:

(a) There exist mutually inverse lattice correspondences between the coclosed submodules of M and the closed left ideals of $S = End_R(M)$.

(b) M is a lifting module if and only if S is a left extending ring.

We will use the notation $N \leq_e M$ to indicate that N is essential in M (i.e., $N \cap L \neq 0$ $\forall 0 \neq L \leq M$); $N \ll M$ means that N is small in M (i.e. $\forall L \lneq M, L + N \neq M$). For $K \leq N_R$ and $A \leq {}_SU$ we denote:
$An(K) = \{f \in Hom_R(N, M) \mid f(K) = 0\} (\simeq Hom_R(N/K, M))$,
$Ke(A) = \bigcap\{Keg \mid g \in A\}$.

A submodule N of M is called a *closed* submodule of M if it is not contained as a proper essential submodule of any other submodule of M. We recall that L is a *cosmall submodule of K in M* (denoted by $L \overset{cs}{\hookrightarrow} K$ in M) if $K/L \ll M/L$. Recall that a submodule L of M is called *coclosed* if L has no proper cosmall submodule (denoted by $L \overset{cc}{\hookrightarrow} M$). A *coclosure* of a submodule L of M (denoted by $\widetilde{L}$) is a cosmall submodule of L in M which is also a coclosed submodule of M.

If N and L are submodules of the module M, then N is called a *supplement* (*weak supplement*) of L, if $N+L=M$ and $N\cap L\ll N$ ($N\cap L\ll M$). M is called *supplemented* (*weakly supplemented*) if each of its submodules has a supplement (weak supplement) in M. M is called *amply supplemented*, if for all submodules N and L of M with $N+L=M$, N contains a supplement of L in M.

2. M-Cofaithful Modules

A module $N\in\sigma[M]$ is called *M-cofaithful* if for every $0\neq f\in Hom_R(N,X)$ with $X\in\sigma[M]$, $Hom_R(X,M)f\neq 0$.

2.1. Proposition. *An R-module N is M-cofaithful if and only if $Hom_R(N, Ke(Hom_R(X,M)))=0$ for every $X\in\sigma[M]$.*

Proof. Let $h:N\to Ke(Hom_R(X,M))$ be a nonzero homomorphism. Composing with the natural inclusion map $i:Ke(Hom_R(X,M))\to X$ we get a nonzero homomorphism $g:N\to X$ such that $\operatorname{Im} g\subseteq Ke(Hom_R(X,M))$. Then for every $f:X\to M$, $\operatorname{Im} g\subseteq Ke(Hom_R(X,M))\subseteq \ker f$. Thus $fg=0$ which is a contradiction.

Conversely, let $\forall X\in\sigma[M]$, $Hom_R(N,Ke(Hom_R(X,M)))=0$ and $0\neq g: N\to X$ be a nonzero homomorphism. If $Hom_R(X,M)g=0$, then $\operatorname{Im} g\subseteq Ke(Hom_R(X,M))$. This gives a nonzero homomorphism $h:N\to Ke(Hom_R(X,M))$ which is a contradiction. □

2.2. Proposition. *If N is an M-cofaithful module, then $Hom_R(N,\frac{KeAn(K)}{K})=0$ for every $K\leq N$.*

Proof. It is a direct consequence of Proposition 2.1, because $\frac{KeAn(K)}{K}=Ke(Hom_R(\frac{N}{K},M))$. □

2.3. Proposition. *Let M be an R-module. If M is a cogenerator in $\sigma[M]$, then every $N\in\sigma[M]$ is M-cofaithful.*

Proof. Suppose that M is a cogenerator in $\sigma[M]$. Then for every $X\in\sigma[M]$, X is M-cogenerated. Thus $Ke(Hom_R(X,M))=0$. So $Hom_R(N,Ke(Hom_R(X,M)))=0$ for every $N\in\sigma[M]$. Hence every $N\in\sigma[M]$ is M-cofaithful. □

2.4. Proposition. *Let M be an R-module. Then every generator in $\sigma[M]$ is an M-cofaithful module if and only if every R-module in $\sigma[M]$ is an M-cofaithful module.*

Proof. Let every generator in $\sigma[M]$ is an M-cofaithful module. Suppose that $N\in\sigma[M]$ and $0\neq f\in Hom_R(N,X)$ is given with $X\in\sigma[M]$. Then there is a generator F and an epimorphism $g:F\to N$. Since F is M-cofaithful, there exists $h\in Hom_R(X,M)$ with $hfg\neq 0$. Thus $hf\neq 0$ and this proves that N is M-cofaithful. The converse is clear. □

2.5. Proposition. *Let $\{N_\alpha \mid \alpha\in I\}$ be a family of M-cofaithful modules. Then $N=\oplus_{\alpha\in I}N_\alpha$ is M-cofaithful.*

Proof. Let $0\neq f\in Hom_R(N,X)$ for $X\in\sigma[M]$. Since N_α is M-cofaithful for any $\alpha\in I$, hence there exists $h_\alpha:X\to M$ such that $h_\alpha f i_\alpha\neq 0$, where $i_\alpha:N_\alpha\to N$ is the natural injection map. Then $h_\alpha f\neq 0$ and so N is M-cofaithful. □

2.6. Proposition. *Let N be an M-cofaithful R-module. Then every supplement submodule of N is M-cofaithful.*

Proof. Let K be a supplement submodule of N and $0 \neq g \in Hom_R(K, X)$ for $X \in \sigma[M]$. Then there exists $L \leq N$ such that $K + L = N$ and $K \cap L \ll K$. Put $X' = g(K \cap L) \ll X$ and let g' denote the composition $N \xrightarrow{\pi} (K+L)/K \cong K/(K \cap L) \xrightarrow{g} X/X'$. Then $0 \neq g' : N \to X/X'$. By assumption, there exists $0 \neq h \in Hom_R(X/X', M)$ with $hg' \neq 0$. Then $hg \neq 0$ and so $Hom_R(X, M)g \neq 0$ because $h\pi' \neq 0$, where $\pi' : X \to X/X'$ denotes the natural map. $\square$

2.7. Corollary. *Let N be an M-cofaithful R-module. Then:*

(i) *Every direct summand of N is M-cofaithful.*

(ii) *Every weak supplement coclosed submodule of N is M-cofaithful.*

Proof. (i) By Proposition 2.6.

(ii) Since every weak supplement coclosed submodule is a supplement submodule, it follows by Proposition 2.6. $\square$

2.8. Lemma. *Let N be an M-cofaithful R-module. Then for every proper submodule K of N, $K \overset{cs}{\hookrightarrow} KeAn(K)$ in N and $KeAn(K) \lneq N$; in particular, $Hom_R(N/K, M) \neq 0$.*

Proof. If $K \lneq N$ and $\pi : N \to N/K$ is the natural epimorphism, then $Hom_R(N/K, M)\pi \neq 0$ since $N/K \in \sigma[M]$. Thus $KeAn(K) \lneq N$. Let $K \leq L \lneq N$, then $KeAn(K) + L \leq KeAn(L) \lneq N$. Therefore $K \overset{cs}{\hookrightarrow} KeAn(K)$ in N. $\square$

2.9. Proposition. *Assume that N is an M-cofaithful R-module. Let $K \leq N$ and L be a weak supplement coclosed submodule of N such that $L \subseteq KeAn(K)$. Then $L \subseteq K$. In particular, if K is a weak supplement coclosed submodule of N, then K is the unique coclosure of $KeAn(K)$ in N.*

Proof. Let $K \leq N$ and L be a weak supplement coclosed submodule of N such that $L \subseteq KeAn(K)$. Suppose that g denotes the composition $L \xrightarrow{\subseteq} KeAn(K) \xrightarrow{\pi} \frac{KeAn(K)}{K}$. Then by Proposition 2.2 and Corollary 2.7, $g = 0$, and so $L \subseteq K$. $\square$

2.10. Proposition. *Let N be an M-cofaithful R-module. Then the following conditions hold:*

(1) *For every finitely generated S-submodule $A \leq {}_SU$, $A \leq_e Hom_R(N/Ke(A), M)$ (equivalently, $A \leq_e AnKe(A)$).*

(2) *Let $L \leq K \leq N$. If $An(K) \leq_e An(L)$, then $L \overset{cs}{\hookrightarrow} K$ in N. The converse holds if $Hom_R(N, M)$ is a noetherian S-module.*

(3) *Let $A \leq B \leq {}_SU$ and $Hom_R(N, M)$ be a noetherian S-module. Then $A \leq_e B$ if and only if $Ke(B) \overset{cs}{\hookrightarrow} Ke(A)$ in N.*

Proof. (1) Let $0 \neq f \in Hom_R(N/Ke(A), M)$. Set $A = Sg_1 + Sg_2 + ... + Sg_k$ with $g_i \in Hom_R(N, M)$. Then $Ke(A) = \bigcap_{i \leq k} Keg_i$. Let $P = \{(f(n + Ke(A)), (\prod_{i=1}^{k} g_i)(n + Ke(A))) \mid n \in N\}$ and let $\bar{i}_1 : M^{(k)} \to M \oplus M^{(k)} \to \frac{M \oplus M^{(k)}}{P}$ and $\bar{i}_2 : M \to M \oplus M^{(k)} \to \frac{M \oplus M^{(k)}}{P}$ be the canonical maps. We have the following commutative diagram:

$$\begin{array}{ccc} 0 \longrightarrow N/Ke(A) & \overset{\prod_{i=1}^{k} g_i}{\longrightarrow} & M^{(k)} \\ \downarrow f & & \downarrow -\bar{i_1} \\ M & \overset{\bar{i_2}}{\longrightarrow} & (M \oplus M^{(k)})/P. \end{array}$$

Then $0 \neq \bar{i_2} f = -\bar{i_1}(\prod_{i=1}^{k} g_i) : N/Ke(A) \to (M \oplus M^{(k)})/P$. By hypothesis, there exists $h \in Hom_R(\frac{M \oplus M^{(k)}}{P}, M)$ with $h\bar{i_2} f \neq 0$. We may consider $h(-\bar{i_1})$ as $\sum_{i=1}^{k} s_i$ for some $s_i \in S$. Thus $0 \neq h\bar{i_2} f = h(-\bar{i_1})(\prod_{i=1}^{k} g_i) = \sum_{i=1}^{k} s_i g_i \in A$. Therefore $A \leq_e Hom_R(N/Ke(A), M)$.

(2) Let $An(K) \leq_e An(L)$ for $L \leq K \leq N$. Suppose that $K/L + X/L = N/L$, where $L \leq X \leq N$. If $X \neq N$, then by hypothesis, there exists $0 \neq f \in U$ with $f(X) = 0$. Thus $f(L) = 0$ and so $0 \neq f \in An(L)$. As $An(K) \leq_e An(L)$, there exists $g \in S$ such that $0 \neq gf \in An(K)$. Hence $gf(N) = gf(K + X) = 0$, which is a contradiction. Therefore $L \overset{cs}{\hookrightarrow} K$ in N. Conversely, assume that $L \overset{cs}{\hookrightarrow} K$ in N and let $0 \neq A \leq An(L)$. Then $L \leq Ke(A) \lneq N$ and so $K + Ke(A) \lneq N$. Thus $0 \neq An(K + Ke(A)) = An(K) \cap AnKe(A)$. But $A \leq_e AnKe(A)$ from (1), so $An(K) \cap A \neq 0$. Therefore $An(K) \leq_e An(L)$.

(3) It is clear that A is essential in B if and only if $AnKe(A)$ is essential in $AnKe(B)$, by (1) (because A and B are finitely generated, so (1) can be applied). By using (2), the claimed property holds. □

Recall that a module M is said to have *uniform* (*or Goldie*) *dimension* n, denoted by $u.dim(M) = n$ for some $n \in \mathbb{N}$, if $sup\{k \in \mathbb{N} \mid M$ contains k independent submodules $\} = n$ [4]. A module M is said to have *hollow dimension* n, denoting this by $h.dim(M) = n$ for some $n \in \mathbb{N}$, if $sup\{k \in \mathbb{N} \mid M$ has k coindependent submodules $\} = n$ [3].

2.11. Lemma. *Let $N \in \sigma[M]$ be a nonzero R-module and $K, L \leq N$. If $K + L = N$, then $An(K \cap L) = An(K) + An(L)$.*

Proof. It follows from [1, Lemma 4.9]. □

2.12. Theorem. *Let N be an M-cofaithful module. Then $u.dim(_SU) = h.dim(N_R)$.*

Proof. Assume first that $Sf_1, Sf_2, ..., Sf_n$ is an independent family of submodules of $_SU$ and $0 \neq f_i \in {_SU}$ for all $1 \leq i \leq n$. Since $Sf_i \cap Sf_j = 0$ for any $i \neq j$, and $Sf_i \leq_e AnKe(Sf_i)$ for all $1 \leq i \leq n$, $AnKe(Sf_i) \cap AnKe(Sf_j) = 0$. Thus $An(Ke(Sf_i) + Ke(Sf_j)) = 0$. Since N is M-cofaithful, $Ke(Sf_i) + Ke(Sf_j) = N$. By Lemma 2.11, $An(Ke(Sf_i) \cap Ke(Sf_j)) = AnKe(Sf_i) + AnKe(Sf_j)$ for each $i \neq j$. Let $i, j, k \in \{1, 2, ..., n\}$ be distinct. Since $Sf_i \cap (Sf_j + Sf_k) = 0$ and $Sf_i \cap (Sf_j + Sf_k) \leq_e AnKe(Sf_i) \cap (AnKe(Sf_j) + AnKe(Sf_k))$, hence $0 = AnKe(Sf_i) \cap (AnKe(Sf_j) + AnKe(Sf_k)) = AnKe(Sf_i) \cap An(Ke(Sf_j) \cap Ke(Sf_k)) = An(Ke(Sf_i) + (Ke(Sf_j) \cap Ke(Sf_k)))$. Therefore $Ke(Sf_i) + (Ke(Sf_j) \cap Ke(Sf_k)) = N$. It is easy to see by induction that for every $1 \leq i \leq n$, $Ke(Sf_i) + (\bigcap_{j \neq i} Ke(Sf_j)) = N$. Hence $\{Ke(Sf_i), ..., Ke(Sf_n)\}$ is coindependent. Thus $u.dim(_SU) \leq h.dim(N_R)$. On the other hand, from [1, Proposition 4.10], $u.dim(_SU) \geq h.dim(N_R)$ and the proof is completed. □

2.13. Theorem. *Assume that N is an M-cofaithful weak supplemented module and $Hom_R(N,M)$ is a noetherian S-module. Then for every $A \leq^c {}_SU = Hom_R(N,M)$, $Ke(A)$ has a unique coclosure $\widetilde{Ke(A)}$ in N and the maps $K \to An(K)$ and $A \to \widetilde{Ke(A)}$ determine mutually inverse correspondences between the coclosed R-submodules of N and the closed S-submodules of $U = Hom_R(N,M)$.*

Proof. Let $K \overset{cc}{\hookrightarrow} N$ and $An(K) \leq_e A \leq {}_SU$. By Zorn's Lemma, we may assume that A is closed in ${}_SU$. From Proposition 2.10, $Ke(A) \overset{cs}{\hookrightarrow} KeAn(K)$ in N. By Proposition 2.9, $K \overset{cs}{\hookrightarrow} Ke(A)$ in N. Hence $A \subseteq AnKe(A) \subseteq An(K)$. Thus $A = An(K)$; that is, $An(K) \leq^c {}_SU$. Also, $K = \widetilde{KeAn}(K)$.

Assume that $A \leq^c {}_SU$. We show that $Ke(A)$ has a unique coclosure in N. Let $K \overset{cc}{\hookrightarrow} N$ and $K \overset{cs}{\hookrightarrow} Ke(A)$ in N. By using Proposition 2.10, $A \leq_e AnKe(A) \leq_e An(K)$, and so $A = An(K)$. Thus $Ke(A) = KeAn(K)$. Therefore K is a unique coclosure of $Ke(A)$ (by Proposition 2.9). So $A = An(K) = An(\widetilde{Ke(A)})$. □

2.14. Corollary. *Let N be an M-cofaithful module and $Hom_R(N,M)$ be a noetherian S-module. Then, $\widetilde{Ke(A)} = Ke(A)$ for every $A \leq^c {}_SU$ if and only if every $K \overset{cc}{\hookrightarrow} N$ is M-cogenerated.*

Proof. Assume that for every $A \leq^c {}_SU$, $\widetilde{Ke(A)} = Ke(A)$ and let $K \overset{cc}{\hookrightarrow} N$. Then $An(K) \leq^c {}_SU$. From Theorem 2.13 and hypothesis, $K = \widetilde{KeAn(K)} = KeAn(K)$. Thus K is M-cogenerated. Conversely, suppose that every $K \overset{cc}{\hookrightarrow} N$ is M-cogenerated and $A \leq^c {}_SU$. By Theorem 2.13, $A = An(\widetilde{Ke(A)})$. On the other hand, by hypothesis, $\widetilde{Ke(A)} = KeAn(\widetilde{Ke(A)})$. Therefore $\widetilde{Ke(A)} = Ke(A)$. □

Recall that an R-module M is an *extending* module if for every submodule K of M there exists a direct summand L of M such that $K \leq_e L$, or equivalently, every closed submodule of M is a direct summand. A left extending ring is a ring which is a extending module over itself. Dually, a module M is called a *lifting* module if, every submodule N of M can be written in the form $N = K \oplus D$ where K is a direct summand of M and $D \ll M$. By [10, 4.8], M is lifting if and only if it is amply supplemented and its coclosed submodules are direct summands.

2.15. Theorem. *Let N be an M-cofaithful module and $Hom_R(N,M)$ be a noetherian S-module. If N_R is a lifting module, then ${}_SU$ is an extending module; and the converse holds when N is M-cogenerated and amply supplemented.*

Proof. Let N be a lifting module and let $A \leq^c {}_SU$. Then, by Theorem 2.13, $N = \widetilde{Ke(A)} \oplus D$ for some $D \leq N$. Thus $U = An(\widetilde{Ke(A)}) \oplus An(D) = A \oplus An(D)$. Conversely, suppose that N is M-cogenerated and amply supplemented and let ${}_SU$ be an extending module and $K \overset{cc}{\hookrightarrow} N$. Then, by Theorem 2.13 again, $U = An(K) \oplus B$ for some $B \leq {}_SU$. Thus $0 = Ke(U) = KeAn(K) \cap Ke(B) = K \cap Ke(B)$. On the other hand, $N = K + Ke(B)$ since if $K + Ke(B) \lneq N$, then $0 \neq An(K + Ke(B)) = An(K) \cap AnKe(B)$, whence $An(K) \cap B \neq 0$, which is a contradiction. Therefore $N = K \oplus Ke(B)$ and so N is lifting. □

3. Applications to coretractable modules

Recall that an R-module N is called *M-coretractable* if, for any proper submodule K of N, there exists a nonzero homomorphism $f : N \to M$ with $f(K) = 0$, that is, $Hom_R(N/K, M) \neq 0$. An R-module M is called *coretractable* if M is itself M-coretractable [3]. By Lemma 2.8, every M-cofaithful module N is M-coretractable.

3.1. Theorem. *Let M be a quasi-injective R-module. Then $N \in \sigma[M]$ is M-cofaithful if and only if N is M-coretractable.*

Proof. By Lemma 2.8, every M-cofaithful module N is M-coretractable. Conversely, suppose that N is M-coretractable. It suffices to show that for every $X \in \sigma[M]$, $Hom_R(N, Ke(Hom_R(X, M))) = 0$. Assume that there exists a nonzero homomorphism $f : N \to Ke(Hom_R(X, M))$. Then $0 \neq if : N \to X$, where $i : Ke(Hom_R(X, M)) \to X$ is the inclusion map. Since $if \neq 0$, there exists $Z = \mathrm{Im}\,(if) \neq 0$. Now, $Hom_R(Z, M) \neq 0$ because Z is a quotient of N and N is M-coretractable. But every homomorphism $g : Z \to M$ can be extended to a homomorphism $h : X \to M$ because M is quasi-injective and $X \in \sigma[M]$ (by [11, 16.3]). Since $Z \subseteq Ke(Hom_R(X, M)) \subseteq X$, $h(Z) = 0$, which is a contradiction. □

3.2. Corollary. *Let $S = End_R(M)$ be a noetherian ring and M an amply supplemented module with one of the following properties:*

(i) M is a quasi-injective coretractable module;

(ii) M is a $\sum$-self-cogenerator module (that is, any direct sum of copies of M is a self-cogenerator). Then:

(a) There exist mutually inverse lattice correspondences between the coclosed submodules of M and the closed left ideals of $S = End_R(M)$.

(b) M is a lifting module if and only if S is a left extending ring.

Proof. By combining Theorems 2.13, 2.15, 3.1, and in the special case when $N = M$ and $U = S$. □

3.3. Corollary. *Let M be an R-module. If any of the following conditions is satisfied, then the hollow dimension of M is equal to n if and only if the right uniform dimension of S is n:*

(i) M is a quasi-injective coretractable module.

(ii) M is a $\sum$-self-cogenerator.

Proof. Using Theorems 2.12 and 3.1 in the special case when $N = M$ and $U = S$. □

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