Hacettepe Journal of Mathematics and Statistics
Volume 31 (2002), 25–35

UNIFICATION OF SOME SEPARATION AXIOMS

T. Hatice Yalvaç*

Dedicated to the memory of Professor Doğan Çoker

Received 29. 07. 2002

Abstract

Kandil, Kerre and Nouh unified some concepts in fuzzy topological spaces by using operations. By adapting the definition of an operation and some definitions given by these authors to topological spaces, and by giving some new definitions, we have achieved some unifications related to compactness, continuity, openness and closedness of functions. Here, we will study unifications related to separation axioms, such as T_i $(i = 0, 1, 2)$ and R_2.

Key Words: Operation, Separation axiom, Continuity, Supratopology.

Mathematics Subject Classification: 54 D 10,54 C 08, 54 A 05

1. Introduction

In [4,5], some unifications for fuzzy topological spaces were studied. Many of these definitions and results were applied to topological spaces in [7-11]. Now an attempt will be made to unify concepts related to separation.

In a topological space, (X, τ), int , cl , scl , pcl , etc., will stand for the interior, closure, semi-closure, pre-closure operations, etc. Also, for a subset A of X, A^o and $\bar{A}$ will stand for the interior of A and the closure of A, respectively.

1.1. Definition. Let (X, τ) be a topological space. A mapping $\varphi : P(X) \to P(X)$ is called an *operation on* (X, τ) if $A^o \subseteq \varphi(A)$ for all $A \in P(X)$ and $\varphi(\emptyset) = \emptyset$.

The class of all operations on a topological space (X, τ) will be denoted by $O(X, \tau)$.

The operations φ, $\tilde{\varphi} \in O(X, \tau)$ are said to be *dual* if $\varphi(A) = X \setminus (\tilde{\varphi}(X \setminus A))$ (equivalently, $\tilde{\varphi}(A) = X \setminus (\varphi(X \setminus A))$) for each $A \in P(X)$.

A partial order "$\leq$" on $O(X, \tau)$ is defined by $\varphi_1 \leq \varphi_2 \Leftrightarrow \varphi_1(A) \subseteq \varphi_2(A)$ for each $A \in P(X)$.

An operation $\varphi \in O(X, \tau)$ is called *monotonous* if $\varphi(A) \subseteq \varphi(B)$ whenever $A \subseteq B$, $(A, B \in P(X))$.

*Hacettepe University, Mathematics Department, Ankara, Turkey.

1.2. Definition. Let $\varphi \in O(X,\tau)$, $\mathcal{U} \subseteq P(X)$, $x \in X$ and $\mathcal{U}(x) = \{U : x \in U \in \mathcal{U}\}$. Then φ is called:

a) *Regular with respect to* $\mathcal{U}$ if for $x \in X$ and $U, V \in \mathcal{U}(x)$, there exists $W \in \mathcal{U}(x)$ such that $\varphi(W) \subseteq \varphi(U) \cap \varphi(V)$.

b) *Weakly finite intersection preserving (W.F.I.P) with respect to* $\mathcal{U}$ if $U \cap \varphi(A) \subseteq \varphi(U \cap A)$ for each $U \in \mathcal{U}$ and for each $A \in P(X)$.

1.3. Definition. Let $\varphi \in O(X,\tau)$ and $A, B \subseteq X$. A is called *φ–open* if $A \subseteq \varphi(A)$. Likewise B is called *φ–closed* if $X \setminus B$ is φ–open.

For any operation $\varphi \in O(X,\tau)$, $\tau \subseteq \varphi O(X)$, and X, $\emptyset$ are both φ–open and φ–closed.

If φ is monotonous, then the family of all φ–open sets is a supratopology ($\mathcal{U} \subseteq \mathcal{P}(X)$ is a supratopology on X means that $\emptyset \in \mathcal{U}$, $X \in \mathcal{U}$ and $\mathcal{U}$ is closed under arbitrary unions [1]).

Let (X,τ) be a topological space, $\varphi \in O(X,\tau)$, $x \in X$. We will use the following notation.

$$\begin{aligned} \varphi O(X) &= \{U : U \subseteq X,\ U \text{ is } \varphi\text{–open}\}, \\ \varphi C(X) &= \{K : K \subseteq X,\ K \text{ is } \varphi\text{–closed}\}, \\ \varphi O(X,x) &= \{U : U \in \varphi O(X),\ x \in U\}. \end{aligned}$$

1.4. Definition. Let $\varphi_1, \varphi_2 \in O(X,\tau)$, $A \subseteq X$.

a) $x \in \varphi_{1,2}\mathrm{int}\, A \Leftrightarrow$ there exists a $U \in \varphi_1 O(X,x)$ such that $\varphi_2(U) \subseteq A$.

b) $x \in \varphi_{1,2}\mathrm{cl}\, A \Leftrightarrow$ for each $U \in \varphi_1 O(X,x)$, $\varphi_2(U) \cap A \neq \emptyset$.

c) A is $\varphi_{1,2}$–open $\Leftrightarrow A \subseteq \varphi_{1,2}\mathrm{int}\, A$.

d) A is $\varphi_{1,2}$–closed $\Leftrightarrow \varphi_{1,2}\mathrm{cl}\, A \subseteq A$.

If $A \subseteq B$ then $\varphi_{1,2}\mathrm{int}\, A \subseteq \varphi_{1,2}\mathrm{int}\, B$ and $\varphi_{1,2}\mathrm{cl}\, A \subseteq \varphi_{1,2}\mathrm{cl}\, B$. Clearly for any set A, $X \setminus \varphi_{1,2}\mathrm{int}\, A = \varphi_{1,2}\mathrm{cl}\,(X \setminus A)$ and A is $\varphi_{1,2}$–open iff $X \setminus A$ is $\varphi_{1,2}$–closed.

We will use $\varphi_{1,2}O(X)$ ($\varphi_{1,2}C(X)$) to denote the family of all $\varphi_{1,2}$–open subsets ($\varphi_{1,2}$–closed subsets) of X.

1.5. Theorem : [10,11]. *Let $\varphi_1, \varphi_2 \in O(X,\tau)$.*

a) *$\varphi_{1,2}O(X)$ is a supratopology on X.*

b) *If φ_2 is regular w.r.t. $\varphi_1 O(X)$, then $\varphi_{1,2}O(X)$ is a topology on X and a subset K of X is closed w.r.t. this topology iff $\varphi_{1,2}\mathrm{cl}\, K \subseteq K$. Let $\tau_{\varphi_{1,2}}$ stand for this topology $\varphi_{1,2}O(X)$.*

c) *If φ_2 is regular w.r.t. $\varphi_1 O(X)$ and ($\varphi_2 \geq \imath$ or $\varphi_2 \geq \varphi_1$), then $\varphi_{1,2}O(X)$ is a topology on X and a set K is closed w.r.t. this topology iff $\varphi_{1,2}\mathrm{cl}\, K = K$ (here $\imath$ is the identity operation).*

d) *If φ_2 is regular w.r.t. $\varphi_1 O(X)$, ($\varphi_2 \geq \imath$ or $\varphi_2 \geq \varphi_1$), and $\varphi_2(U) \in \varphi_1 O(X)$, $\varphi_2(\varphi_2(U)) \subseteq \varphi_2(U)$ for each $U \in \varphi_1 O(X)$, then $\varphi_{1,2}$–cl is a Kuratowski closure operator and $\varphi_{1,2}\mathrm{cl}\, A = \tau_{\varphi_{1,2}}\mathrm{cl}\, A$ for any $A \subseteq P(X)$.*

1.6. Example. Let the following operations be defined on a topological space (X, τ).

$\varphi_1 = \mathrm{int}\,, \quad \varphi_2 = \mathrm{cl} \circ \mathrm{int}\,, \quad \varphi_3 = \mathrm{cl}\,, \quad \varphi_4 = \mathrm{scl}\,,$

$\varphi_5 = \imath$ ($\imath$ is the idendity operation), $\quad \varphi_6 = \mathrm{int} \circ \mathrm{cl}\,, \quad \varphi_7 = \mathrm{int} \circ \mathrm{cl} \circ \mathrm{int}\,.$

Clearly,

$\varphi_1 \leq \varphi_7 \leq \varphi_2 \leq \varphi_3, \quad \varphi_1 \leq \varphi_5 \leq \varphi_4 \leq \varphi_3$ and $\varphi_1 \leq \varphi_7 \leq \varphi_6 \leq \varphi_4.$

Now we have:

$\varphi_1 O(X) = \tau,$

$\varphi_2 O(X) = SO(X) =$ the family of all semi–open sets.

$\varphi_3 O(X) = \varphi_5 O(X) = \varphi_4 O(X) = \mathcal{P}(X) =$ power set of X.

$\varphi_6 O(X) = PO(X) =$ the family of all pre-open sets.

$\varphi_7 O(X) = \tau^\alpha =$ the topology of all α–open sets in (X, τ).

$\varphi_{1,3} O(X) = \tau_\theta =$ the topology of all θ–open sets.

$\varphi_{2,4} O(X) = S\theta O(X) =$ the family of all semi–θ-open sets.

$\varphi_{1,6} O(X) = \tau_s =$ the semi regularization topology of X. It is the topology with base $RO(X)$ which consisting of regular open sets = the family of all δ–open sets.

$\varphi_{2,3} O(X) = \theta SO(X) =$ the family of all θ–semi-open sets.

Note that φ_1 and φ_3 (φ_2 and φ_6) are pairs of dual operations. All operations in this example are regular w.r.t. $\varphi_1 O(X)$, w.r.t. $\varphi_7 O(X)$ and w.r.t. $\varphi_5 O(X)$.

It will be accepted that φ_1, $\varphi_2 \in O(X, \tau)$, ψ_1, $\psi_2 \in O(Y, \vartheta)$ and $f : (X, \tau) \to (Y, \vartheta)$ whenever they are used. $G_f = \{(x, f(x)) : x \in X\}$ will stand for the graph of f.

1.7. Definition.

a) f is called $\varphi_{1,2}\psi_{1,2}$–continuous if for each $x \in X$ and for each $V \in \psi_1 O(Y, f(x))$, there exists a $U \in \varphi_1 O(X, x)$ such that $f(\varphi_2(U)) \subseteq \psi_2(V)$ [4,7].

b) f is called $\varphi_{1,2}\psi_{1,2}$–open if for each $A \subseteq X$ and for each $x \in \varphi_{1,2}\mathrm{int}\, A$, $f(x) \in \psi_{1,2}\mathrm{int}\, f(A)$ ([4,8]).

c) G_f is called $\varphi_{1,2}\psi_{1,2}$–closed if for each $(x, y) \in X \times Y \setminus G_f$, there exist $U \in \varphi_1 O(X, x)$ and $V \in \psi_1 O(Y, y)$ such that $f(\varphi_2(U)) \cap \psi_2(V) = \emptyset$ [9].

1.8. Note. For any supratopological space $(Z, \mathcal{U})$, the separation axioms T_0, T_1 and T_2 will be defined as for topological spaces, and for any subset A of Z, $\mathcal{U}$–int A and $\mathcal{U}$–cl A will stand for the closure and the interior of A in this supratopological space $(Z, \mathcal{U})$. i.e. with a similar meaning as in topological spaces.

Let (X,τ) be a topological space and $\mathcal{U}$ a supratopology containing τ. Then the mappings $\varphi : P(X) \to P(X)$, $\tilde{\varphi} : P(X) \to P(X)$ defined by $\varphi(A) = \mathcal{U}$–$\operatorname{int} A = \{x :$ there exists a $U \in \mathcal{U}$ s.t. $x \in U \subseteq A\} = \bigcup\{U : U \subseteq A, U \in \mathcal{U}\}$ and $\tilde{\varphi}(A) = \mathcal{U}$–$\operatorname{cl} A = \{x : x \in U \in \mathcal{U} \Rightarrow U \cap A \neq \emptyset\} = \bigcap\{K : A \subseteq K, X \setminus K \in \mathcal{U}\}$ are operations on (X,τ) and they are dual operations to each other.

2. Separation Axioms

2.1. Definition. Let φ_1, $\varphi_2 \in O(X,\tau)$.

a) (X,τ) is called $\varphi_{1,2}$–R_2 if for each $x \in X$ and for each $U \in \varphi_1 O(X,x)$, there exists $V \in \varphi_1 O(X,x)$ such that $\varphi_2(V) \subseteq U$, [5,9].

b) (X,τ) is called $\varphi_{1,2}$–T_2 if $x, y \in X$ and $x \neq y$, then there exist $U \in \varphi_1 O(X,x)$ and $V \in \varphi_1 O(X,y)$ such that $\varphi_2(U) \cap \varphi_2(V) = \emptyset$, [5,9].

c) (X,τ) is called $\varphi_{1,2}$–T_1 if $x, y \in X$ and $x \neq y$, then there exist $U \in \varphi_1 O(X,x)$ and $V \in \varphi_1 O(X,y)$ such that $x \notin \varphi_2(V)$ and $y \notin \varphi_2(U)$ (given for fuzzy topological spaces in [5]).

d) (X,τ) is called $\varphi_{1,2}$–T_0 if $x, y \in X$ and $x \neq y$, then there exists a φ_1–open set U such that $x \in U$, $y \notin \varphi_2(U)$ or $y \in U$, $x \notin \varphi_2(U)$.

Clearly each $\varphi_{1,2}$–T_1 space is $\varphi_{1,2}$–T_0 and if ($\varphi_2 \geq \varphi_1$ or $\varphi_2 \geq \imath$) then each $\varphi_{1,2}$–T_2 space is $\varphi_{1,2}$–T_1. (X,τ) is $\varphi_{1,2}$–R_2 iff $\varphi_1 O(X) \subseteq \varphi_{1,2} O(X)$.

The following definition was given for fuzzy topological spaces in [4] at the same time.

2.2. Definition. Let $A \subseteq X$. A is said to be $\varphi_{1,2}$–dense in (X,τ) if $\varphi_{1,2}\operatorname{cl} A = X$.

The following three theorems given for fuzzy topological spaces in [4] are valid in our case. Only the $\varphi_{1,2}$–T_0 separation axiom has been added.

2.3. Theorem: *If $f, g : X \to Y$ $\varphi_{1,2}\psi_{1,2}$–continuous, Y is $\psi_{1,2}$–T_2, φ_2 is regular w.r.t. $\varphi_1 O(X)$, then $E = \{x : f(x) = g(x)\}$ is $\varphi_{1,2}$–closed and if E is $\varphi_{1,2}$–dense in X, then $f = g$.*

2.4. Theorem: *The axioms $\varphi_{1,2}$–T_i $(i = 0, 1, 2)$ are inverse invariant under a $\varphi_{1,2}\psi_{1,2}$–continuous injective mapping.*

Proof. As an example, we prove the inverse invariance of $\varphi_{1,2}$–T_0.

Let $x, x' \in X$, $x \neq x'$. $f(x) \neq f(x')$. There exists a ψ_1–open set V s.t. $(f(x) \in V, f(x') \notin \psi_2(V))$ or $(f(x') \in V, f(x) \notin \psi_2(V))$.

Let $f(x) \in V$, $f(x') \notin \psi_2(V)$. Since f is $\varphi_{1,2}\psi_{1,2}$–continuous, there exists a φ_1–open set U such that $x \in U$, $f(\varphi_2(U)) \subseteq \psi_2(V)$. Since $f(x') \notin \psi_2(V)$, $x' \notin \varphi_2(U)$.

If $f(x') \in V$, $f(x) \notin \psi_2(V)$, then the proof proceeds in the same way. So, (X,τ) is $\varphi_{1,2}$–T_0. □

2.5. Theorem: *(X,τ) is $\varphi_{1,2}$–T_1 iff each singleton set is $\varphi_{1,2}$–closed.*

2.6. Theorem: *If f is $\varphi_{1,2}\psi_{1,2}$–open bijection and X is $\varphi_{1,2}$–T_i, then Y is $\psi_{1,2}$–T_i, $(i = 0, 1, 2)$.*

Proof. We give the proof for $i = 2$. The other cases can be proved similarly.

Let $y, y' \in Y$ and $y \neq y'$. There exist $x, x' \in X$ such that $f(x) = y$, $f(x') = y'$. Since $x \neq x'$, there are $U \in \varphi_1 O(X, x)$ and $U' \in \varphi_1 O(X, x')$ such that $\varphi_2(U) \cap \varphi_2(U') = \emptyset$. Since $U \subseteq \varphi_{1,2}\text{int}\,\varphi_2(U)$, $U' \subseteq \psi_{1,2}\text{int}\,\varphi_2(U')$ and f is $\varphi_{1,2}$–open we have $f(U) \subseteq f(\varphi_{1,2}\text{int}\,\varphi_2(U)) \subseteq \psi_{1,2}\text{int}\,f(\varphi_2(U))$ and $f(U') \subseteq f(\varphi_{1,2}\text{int}\,\varphi_2(U')) \subseteq \psi_{1,2}\text{int}\,f(\varphi_2(U'))$. $y = f(x) \in \psi_{1,2}\text{int}\,f(\varphi_2(U))$ and $y' = f(x') \in \psi_{1,2}\text{int}\,f(\varphi_2(U'))$. There are $V \in \psi_1 O(Y, y)$ and $V' \in \psi_1 O(Y, y')$ such that $\psi_2(V) \subseteq f(\varphi_2(U))$ and $\psi_2(V') \subseteq f(\varphi_2(U'))$. Since $f(\varphi_2(U) \cap \varphi_2(U')) = f(\varphi_2(U) \cap f(\varphi_2(U')) = \emptyset$, we have $\psi_2(V) \cap \psi_2(V') = \emptyset$. □

2.7. Theorem: *If f is an injection with $\varphi_{1,2}\psi_{1,2}$–closed graph and ($\psi_2 \geq \psi_1$ or $\psi_2 \geq \imath$), then X is $\varphi_{1,2}$–T_1.*

Proof. Let $x, x' \in X$, $x \neq x'$. $f(x) \neq f(x')$. Then $(x, f(x')) \notin G_f$ and $(x', f(x)) \notin G_f$. There exist $U \in \varphi_1 O(X, x)$ and $V \in \psi_1 O(Y, f(x'))$ such that $f(\varphi_2(U)) \cap \psi_2(V) = \emptyset$. Hence, $f(x') \in V \subseteq \psi_2(V)$ and $x' \notin \varphi_2(U)$. There exist $U' \in \varphi_1 O(X, x')$ and $V' \in \psi_1 O(Y, f(x))$ such that $f(\varphi_2(U')) \cap \psi_2(V') = \emptyset$. $f(x) \in V' \subseteq \psi_2(V')$, $x \notin \varphi_2(U')$, so, X is $\varphi_{1,2}$–T_1. □

Let ψ_0 be the identity operation $\imath$ on (Y, ϑ).

2.8. Theorem:

a) *If ($\psi_2 \geq \psi_1$ or $\psi_2 \geq \imath$) then each $\varphi_{1,2}\psi_{1,0}$–continuous function is $\varphi_{1,2}\psi_{1,2}$–continuous.*

b) *If (Y, ϑ) is a $\psi_{1,2}$–R_2 space, then each $\varphi_{1,2}\psi_{1,2}$–continuous function is $\varphi_{1,2}\psi_{1,0}$–continuous.*

Proof. b). Let $x \in X$ and $V \in \psi_1 O(Y, f(x))$. Since (Y, ϑ) is $\psi_{1,2}$–R_2, there exists a $W \in \psi_1 O(Y, f(x))$ such that $\psi_2(W) \subseteq V$. Since f is $\varphi_{1,2}\psi_{1,2}$–continuous, there exists a $U \in \varphi_1 O(X, x)$ such that $f(\varphi_2(U)) \subseteq \psi_2(W)$. We have $f(\varphi_2(U)) \subseteq V = \psi_0(V)$, so, f is $\varphi_{1,2}\psi_{1,0}$–continuous. □

2.9. Corollary. *If (Y, ϑ) is $\psi_{1,2}$–R_2 and ($\psi_2 \geq \psi_1$ or $\psi_2 \geq \imath$), then the following are equivalent:*

a) *f is $\varphi_{1,2}\psi_{1,0}$–continuous,*

b) *f is $\varphi_{1,2}\psi_{1,2}$–continuous,*

c) *$f^{-1}(V) \in \varphi_{1,2} O(X)$ for each $V \in \psi_1 O(Y)$.*

Proof. ($b \Rightarrow c$). Since Y is $\psi_{1,2}$–R_2, each ψ_1–open set is $\psi_{1,2}$–open. Since f is $\varphi_{1,2}\psi_{1,2}$–continuous, the inverse image of each $\psi_{1,2}$–open set under f is $\varphi_{1,2}$–open [7].

($c \Rightarrow a$). Let $x \in X$ and $V \in \psi_1 O(Y, f(x))$. $x \in f^{-1}(V) \in \varphi_{1,2}O(X)$. There exists a φ_1–open set U such that $x \in U$ and $\varphi_2(U) \subseteq f^{-1}(V)$. $f(\varphi_2(U)) \subseteq V = \psi_0(V) = \imath(V)$. □

2.10. Lemma. *If ($\varphi_2 \geq \varphi_1$ or $\varphi_2 \geq \imath$) and φ_1 is monotonous, then (X, τ) is $\varphi_{1,2}$–R_2 iff $\varphi_1 O(X) = \varphi_{1,2}O(X)$.*

Proof. One part is clear from the definitions of $\varphi_{1,2}$–R_2 space and $\varphi_{1,2}$–open set. Let $U \in \varphi_{1,2}O(X)$. Then $U \subseteq \varphi_{1,2}\text{int}\, U$. For each $x \in U$, there exists an $W \in \varphi_1 O(X, x)$ such that $\varphi_2(W) \subseteq U$. Since $\varphi_1 O(X) \subseteq \varphi_2 O(X)$, $W \subseteq \varphi_2(W) \subseteq U$. U can be written as a union of φ_1–open sets. Since φ_1 is monotonous, U will be a φ_1–open set. □

2.11. Example. Let $\varphi_1 = \text{int} \circ \text{cl} \circ \text{int}$, $\varphi_2 = \text{cl}$, $\varphi_3 = \tau_s$–cl, $\varphi_4 = \tau^\alpha$–cl be operations on (X, τ). Let $\psi_1 = \text{int}$, $\psi_2 = \text{cl}$, $\psi_3 = \Theta$–cl, $\psi_4 = \upsilon_s$–cl, $\psi_5 = \upsilon^\alpha$–cl, $\psi_6 = \text{pcl}$ be operations on (Y, ϑ). Then:

$\varphi_1 O(X) = \tau^\alpha$, $\psi_1 O(Y) = \vartheta$.

In [7], by using operations the following equalities were shown. For each $U \in \tau^\alpha = \varphi_1 O(X)$, $\varphi_2(U) = \varphi_3(U) = \varphi_4(U)$. For each $V \in \vartheta = \psi_1 O(Y)$, $\psi_2(V) = \psi_3(V) = \psi_4(V) = \psi_5(V) = \psi_6(V)$.

$\varphi_{1,2}O(X) = \varphi_{1,3}O(X) = \varphi_{1,4}O(X) = \tau_\Theta = \tau^\alpha_\Theta$.

$\psi_{1,2}O(X) = \psi_{1,3}O(X) = \psi_{1,4}O(X) = \psi_{1,5}O(X) = \psi_{1,6}O(X) = \vartheta_\Theta$.

(Y, ϑ) is $\psi_{1,2}$–R_2 iff $\psi_{1,2}O(X) = \psi_1 O(X)$ iff (Y, ϑ) is regular.

Now in case (Y, ϑ) is regular, and $f : X \to Y$, the following are equivalent:

a) For each $V \in \vartheta$, $f^{-1}(V)$ is Θ–open.

b) For each Θ–open set V, $f^{-1}(V)$ is Θ–open.

c) f is $\varphi_{1,i}\psi_{1,j}$–continuous. Here $i = 2, 3, 4$, $j = 2, 3, 4, 5, 6$.

By referring to [7], we may obtain other statements equivalent to the above.

2.12. Theorem:

a) *If $(X, \varphi_{1,2}O(X))$ is T_i, then (X, τ) is $\varphi_{1,2}$–T_i, $i = 0, 1, 2$.*

b) *If ($\varphi_2 \geq \varphi_1$ or $\varphi_2 \geq \imath$), φ_1 is monotonous and (X, τ) is $\varphi_{1,2}$–R_2, then $(X, \varphi_1 O(X))$ is T_i iff $(X, \varphi_{1,2}O(X))$ is T_i, $i = 0, 1, 2$.*

c) *If $\tilde{\varphi}_2$ is the dual operation of φ_2, then (X, τ) is $\varphi_{1,2}$–T_2 iff for distinct points x and y in X, there exist a $U \in \varphi_1 O(X)$ and a $K \in \varphi_1 C(X)$ such that $x \in U$, $y \notin K$ and $\varphi_2(U) \subseteq \tilde{\varphi}_2(K)$.*

Proof. a). We give the proof for $i = 1$. Let $x, y \in X$, $x \neq y$. There are $\varphi_{1,2}$–open sets U and V such that $x \in U$, $y \in V$ and $x \notin V$, $y \notin U$. Since $x \in \varphi_{1,2}$–int$U$ and $y \in \varphi_{1,2}$–int$V$, there are $U_x \in \varphi_1 O(X, x)$ and $V_y \in \varphi_1 O(X, y)$ such that $\varphi_2(U_x) \subseteq U$, $\varphi_2(V_y) \subseteq V$. We have $y \notin \varphi_2(U_x)$ and $x \notin \varphi_2(V_y)$. Hence (X, τ) is $\varphi_{1,2}$–T_1.

b). From Lemma 2.10, we have $\varphi_1 O(X) = \varphi_{1,2} O(X)$. The proofs are now clear.

c). Let (X, τ) be $\varphi_{1,2}$–T_2, $x, y \in X$ and $x \neq y$. There are φ_1–open sets U and V such that $x \in U$, $y \in V$ and $\varphi_2(U) \cap \varphi_2(V) = \emptyset$. Now $\varphi_2(U) \subseteq X \setminus \varphi_2(V) = \tilde{\varphi}_2(X \setminus V)$. Let $K = X \setminus V$. Then $K \in \varphi_1 C(X)$ and $y \notin K$.

Conversely, let $x, y \in X$ and $x \neq y$. There is a φ_1–open set U and φ_1–closed set K such that $x \in U$, $y \notin K$ and $\varphi_2(U) \subseteq \tilde{\varphi}_2(K)$. Let $V = X \setminus K$. Then $V \in \varphi_1 O(X)$, $y \in V$ and $\tilde{\varphi}_2(K) = X \setminus \varphi_2(X \setminus K) = X \setminus \varphi_2(V)$. We now have $\varphi_2(U) \subseteq X \setminus \varphi_2(V)$ and $\varphi_2(U) \cap \varphi_2(V) = \emptyset$. □

2.13. Theorem: *Let $\mathcal{B} = \{\varphi_2(U) : U \in \varphi_1 O(X)\}$. If ($\varphi_2 \geq \varphi_1$ or $\varphi_2 \geq \imath$) and $\varphi_2(U) \in \varphi_1 O(X)$, $\varphi_2(\varphi_2(U)) \subseteq \varphi_2(U)$ for each $U \in \varphi_1 O(X)$, then the following are valid:*

a) *(X, τ) is $\varphi_{1,2}$–T_i iff $(X, \varphi_{1,2} O(X))$ is T_i, $i = 0, 1, 2$.*

b) *$(X, \varphi_{1,2} O(X))$ is T_2 iff for $x, y \in X$ and $x \neq y$, there are $\mathcal{B}_x, \mathcal{B}_y \in \mathcal{B}$ such that $x \in \mathcal{B}_x$, $y \in \mathcal{B}_y$ and $\mathcal{B}_x \cap \mathcal{B}_y = \emptyset$.*

c) *$(X, \varphi_{1,2} O(X))$ is T_1 iff for $x, y \in X$ and $x \neq y$, there are $\mathcal{B}_x, \mathcal{B}_y \in \mathcal{B}$ such that $x \in \mathcal{B}_x$, $y \in \mathcal{B}_y$ and $x \notin \mathcal{B}_y$, $y \notin \mathcal{B}_x$.*

d) *$(X, \varphi_{1,2} O(X))$ is T_0 iff for $x, y \in X$ and $x \neq y$, there is a $B \in \mathcal{B}$ such that ($x \in B$, $y \notin B$) or ($y \in B$, $x \notin B$).*

Proof. Under the given conditions, $\mathcal{B}$ is a base for the supratopology $\varphi_{1,2} O(X)$, $\mathcal{B} \subseteq \varphi_{1,2} O(X) \cap \varphi_1 O(X)$ and $U \subseteq \varphi_2(U)$ for each $U \in \varphi_1 O(X)$, [10].

a). One part of the proof is clear from Theorem 2.12.(a). For the remaining parts, we give the proof only for $i = 2$.

Let (X, τ) be $\varphi_{1,2}$–T_2, $x, y \in X$ and $x \neq y$. There are φ_1–open sets U and V such that $x \in U$, $y \in V$ and $\varphi_2(U) \cap \varphi_2(V) = \emptyset$. $x \in U \subseteq \varphi_2(U) \in \varphi_{1,2} O(X)$, $y \in V \subseteq \varphi_2(V) \in \varphi_{1,2} O(X)$, $\varphi_2(U) \cap \varphi_2(V) = \emptyset$. Hence (X, $\varphi_{1,2} O(X)$) is T_2.

b)–d). Straightforward. □

2.14. Theorem:

a) *If φ_2 is W.F.I.P. w.r.t. $\varphi_1 O(X)$, then $U \cap \varphi_2(A) = \emptyset$ whenever $U \in \varphi_1 O(X)$, $A \in P(X)$ and $U \cap A = \emptyset$.*

b) *If φ_2 is W.F.I.P. w.r.t. $\varphi_1 O(X)$ and $\varphi_2(U) \in \varphi_1 O(X)$ for each $U \in \varphi_1 O(X)$, then $\varphi_2(U) \cap \varphi_2(V) = \emptyset$ whenever $U, V \in \varphi_1 O(X)$ and $U \cap V = \emptyset$.*

Proof. a). Since φ_2 is W.F.I.P. w.r.t. $\varphi_1 O(X)$, for $U \in \varphi_1 O(X)$ and $A \in P(X)$ we have $U \cap \varphi_2(A) \subseteq \varphi_2(U \cap A)$. If $U \cap A = \emptyset$, then $\varphi_2(U \cap A) = \emptyset$ and $U \cap \varphi_2(A) = \emptyset$.

b). Let $U, V \in \varphi_1 O(X)$ and $U \cap V = \emptyset$. From (a), $U \cap \varphi_2(V) = \emptyset$. Since $\varphi_2(U) \in \varphi_1 O(X)$, again from (a), $\varphi_2(U) \cap \varphi_2(V) = \emptyset$. □

2.15. Theorem: *Let φ_1 be monotonous, $i = 0, 1, 2$.*

a) *If (X, τ) is T_i, then $(X, \varphi_1 O(X))$ is T_i.*

b) *If ($\varphi_2 \geq \varphi_1$ or $\varphi_2 \geq \imath$), then $(X, \varphi_1 O(X))$ is T_i when (X, τ) is $\varphi_{1,2}$–T_i.*

c) *If $\varphi_2 \leq \imath$, then(X, τ) is $\varphi_{1,2}$–T_i when $(X, \varphi_1 O(X))$ is T_i.*

Proof. a). Since $\varphi_1 O(X)$ is a supratopology and $\tau \subseteq \varphi_1 O(X)$, the proofs are clear.

b). For $U \in \varphi_1 O(X)$ and $x \in U$, we have $x \in U \subseteq \varphi_2(U)$, whence the results follow.

c). Since $\varphi_2(U) \subseteq U$ for $U \in \varphi_1 O(X)$, the proofs are clear. □

2.16. Theorem: *Let φ_1 be monotonous.*

a) *If φ_2 is W.F.I.P. w.r.t. $\varphi_1 O(X)$, and $\varphi_2(U) \in \varphi_1 O(X)$ for each $U \in \varphi_1 O(X)$, then (X, τ) is $\varphi_{1,2}$–T_2 when $(X, \varphi_1 O(X))$ is T_2.*

b) *If φ_2 is W.F.I.P. w.r.t. $\varphi_1 O(X)$, ($\varphi_2 \geq \varphi_1$ or $\varphi_2 \geq \imath$), and $\varphi_2(U) \in \varphi_1 O(X)$ for each $U \in \varphi_1 O(X)$, then (X, τ) is $\varphi_{1,2}$–T_2 iff $(X, \varphi_1 O(X))$ is T_2.*

c) *If φ_2 is W.F.I.P. w.r.t. $\varphi_1 O(X)$, ($\varphi_2 \geq \varphi_1$ or $\varphi_2 \geq \imath$), and $\varphi_2(U) \in \varphi_1 O(X)$, $\varphi_2(\varphi_2(U)) \subseteq \varphi_2(U)$ for each $U \in \varphi_1 O(X)$, then (X, τ) is $\varphi_{1,2}$–T_2 iff $(X, \varphi_1 O(X))$ is T_2 iff $(X, \varphi_{1,2} O(X))$ is T_2.*

Proof. a). Let $x, y \in X$, $x \neq y$. There are φ_1–open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. From Theorem 2.14 (b)., $\varphi_2(U) \cap \varphi_2(V) = \emptyset$. Hence (X, τ) is $\varphi_{1,2}$–T_2.

b). This follows from (a) and Theorem 2.15 (b).

c). This follows from (b) and Theorem 2.13 (a). □

2.17. Theorem:

a) *(X, τ) is $\varphi_{1,2}$–T_0 iff for $x, y \in X$, $x \neq y$, $x \notin \varphi_{1,2}cl\{y\}$ or $y \notin \varphi_{1,2}cl\{x\}$.*

b) *If $\varphi_2(U) \in \varphi_1 O(X)$, $\varphi_2(\varphi_2(U)) \subseteq \varphi_2(U)$ for each $U \in \varphi_1 O(X)$, and ($\varphi_2 \geq \varphi_1$ or $\varphi_2 \geq \imath$) then, (X, τ) is $\varphi_{1,2}$–T_0 iff for $x, y \in X$, $x \neq y$, $\varphi_{1,2}\mathrm{cl}\,\{x\} \neq \varphi_{1,2}\mathrm{cl}\,\{y\}$.*

c) *(X, τ) is $\varphi_{1,2}$–T_1 iff for $x, y \in X$, $x \neq y$, $x \notin \varphi_{1,2}\mathrm{cl}\,\{y\}$ and $y \notin \varphi_{1,2}\mathrm{cl}\,\{x\}$.*

Proof. a). Let (X,τ) be $\varphi_{1,2}$–T_0, $x,y\in X$ and $x\neq y$. There is a φ_1–open set U such that $(x\in U,\ y\notin\varphi_2(U))$ or $(y\in U,\ x\notin\varphi_2(U))$. Hence $(x\in U,\ \varphi_2(U)\cap\{y\}=\emptyset)$ or $(y\in U,\ \varphi_2(U)\cap\{x\}=\emptyset)$. Thus, $x\notin\varphi_{1,2}\mathrm{cl}\,\{y\}$ or $y\notin\varphi_{1,2}\mathrm{cl}\,\{x\}$. The other part may be proved in a similar way.

b). Under the given conditions, $\varphi_1O(X)\subseteq\varphi_2O(X)$, and if $U\in\varphi_1O(X,x)$, then $x\in\varphi_2(U)$. So, for each $x\in X$, $\{x\}\subseteq\varphi_{1,2}\mathrm{cl}\,\{x\}$. Now, if (X,τ) is $\varphi_{1,2}$–T_0, the proof is clear from (a).

Suppose that for $x,y\in X$, $x\neq y$, we have $\varphi_{1,2}\mathrm{cl}\,\{x\}\neq\varphi_{1,2}\mathrm{cl}\,\{y\}$. There is $z\in X$ such that $(z\in\varphi_{1,2}\mathrm{cl}\,\{x\},\ z\notin\varphi_{1,2}\mathrm{cl}\,\{y\})$ or $(z\in\varphi_{1,2}\mathrm{cl}\,\{y\},\ z\notin\varphi_{1,2}\mathrm{cl}\,\{x\})$. Let $z\in\varphi_{1,2}\mathrm{cl}\,\{x\}$ but $z\notin\varphi_{1,2}\mathrm{cl}\,\{y\}$. Now there is a $U\in\varphi_1O(X,z)$ such that $\varphi_2(U)\cap\{y\}=\emptyset$. But since $z\in\varphi_{1,2}\mathrm{cl}\,\{x\}$, we have $\varphi_2(U)\cap\{x\}\neq\emptyset$. Hence, $x\in\varphi_2(U)$, $\varphi_2(U)\in\varphi_1O(X)$ and $\varphi_2(\varphi_2(U))\subseteq\varphi_2(U)$. So, $y\notin\varphi_2(\varphi_2(U))$.

In the case where $z\in\varphi_{1,2}\mathrm{cl}\,\{y\}$, $z\notin\varphi_{1,2}\mathrm{cl}\,\{x\}$, the proof proceeds in the same way. Hence (X,τ) is $\varphi_{1,2}$–T_0.

c). Let (X,τ) be a $\varphi_{1,2}$–T_1 space, $x,y\in X$ and $x\neq y$. Now, $\{x\}$ and $\{y\}$ are $\varphi_{1,2}$–closed sets, so $\varphi_{1,2}\mathrm{cl}\,\{x\}\subseteq\{x\}$ and $\varphi_{1,2}\mathrm{cl}\,\{y\}\subseteq\{y\}$. Hence $y\notin\varphi_{1,2}\mathrm{cl}\,\{x\}$ and $x\notin\varphi_{1,2}\mathrm{cl}\,\{y\}$.

Conversely, let $x\notin\varphi_{1,2}\mathrm{cl}\,\{y\}$, $y\notin\varphi_{1,2}\mathrm{cl}\,\{x\}$ for $x,y\in X$, $x\neq y$. There are φ_1–open sets U and V such that $x\in U$, $\varphi_2(U)\cap\{y\}=\emptyset$ and $y\in V$, $\varphi_2(V)\cap\{x\}=\emptyset$. Now, $x\in U$, $y\notin\varphi_2(U)$ and $y\in V$, $x\notin\varphi_2(V)$. Hence (X,τ) is $\varphi_{1,2}$–T_1. □

2.18. Theorem: *Let φ_1, φ_2, φ_3, φ_1', $\varphi_2'\in O(X,\tau)$.*

a) *If $\varphi_1\leq\varphi_1'$, $\varphi_2'\leq\varphi_2$ and (X,τ) is $\varphi_{1,2}$–T_i, then $(X,\tau)$is $\varphi_{1,2}'$–T_i, $(i=0,1,2)$.*

b) *If $\varphi_2(U)=\varphi_3(U)$ for each $\varphi_1O(X)$, then (X,τ) is $\varphi_{1,2}$–T_i iff (X,τ) is $\varphi_{1,3}$–T_i, $i=0,1,2$.*

2.19. Theorem: *Let φ_1 be monotonous and $\varphi_2=\varphi_1O(X)$–cl.*

a) *$U\cap\varphi_2(V)=\emptyset$ if $U,V\in\varphi_1O(X)$ and $U\cap V=\emptyset$.*

b) *If $\varphi_2(U)\in\varphi_1O(X)$ for each $U\in\varphi_1O(X)$, then $(X,\varphi_1O(X))$ is T_2 iff (X,τ) is $\varphi_{1,2}$–T_2.*

Proof. Note that $\varphi_1O(X)$ is a supratopology, $\tau\subseteq\varphi_1O(X)$, $\varphi_2\in O(X,\tau)$ and $\varphi_2\geq\imath$.

a). Let $U,V\in\varphi_1O(X)$, $U\cap V=\emptyset$, but $U\cap\varphi_2(V)\neq\emptyset$. Then there is a point $x\in X$ such that $x\in U$ and $x\in\varphi_2(V)$. Now $x\in U\in\varphi_1O(X)$, $x\in\varphi_1O(X)$–cl$V$, so $U\cap V=\emptyset$. This contradiction completes the proof.

b). Let $(X,\varphi_1O(X))$ be T_2, $x,y\in X$, $x\neq y$. There are φ_1–open sets U and V such that $x\in U$, $y\in V$ and $U\cap V=\emptyset$. Hence $U\cap\varphi_2(V)=\emptyset$ from (a). Since $U,\ \varphi_2(V)\in\varphi_1O(X)$, we have $\varphi_2(U)\cap\varphi_2(V)=\emptyset$.

Now, let (X,τ) be $\varphi_{1,2}$–T_2, $x,y\in X$ with $x\neq y$. There are φ_1–open sets U and V such that $x\in U$, $y\in V$, $\varphi_2(U)\cap\varphi_2(V)=\emptyset$. Now $x\in U\subseteq\varphi_2(U)$, $y\in V\subseteq\varphi_2(V)$, and $\varphi_2(U)\in\varphi_1O(X)$, $\varphi_2(V)\in\varphi_1O(X)$. Hence $(X,\varphi_1O(X))$ is T_2. □

2.20. Example. Let $\varphi_1 = \text{int}$, $\varphi_2 = \text{int} \circ \text{cl}$.

φ_2 is W.F.I.P. w.r.t. $\varphi_1 O(X) = \tau$. Also φ_2 is regular w.r.t. $\varphi_1 O(X)$ and $\varphi_2 \geq \varphi_1$. For $U \in \varphi_1 O(X)$, we have $\varphi_2(U) \in \varphi_1 O(X)$ and $\varphi_2(\varphi_2(U)) \subseteq \varphi_2(U)$.

$\mathcal{B} = \{\varphi_2(U) : U \in \varphi_1 O(X)\} = RO(X)$. $\varphi_{1,2} O(X) = \tau_s$. (X,τ) is $\varphi_{1,2}$–R_2 iff $\tau = \varphi_1 O(X) = \tau_s$ iff (X,τ) is semi-regular. From Theorem 2.13, we have, (X,τ) is $\varphi_{1,2}$–T_i iff (X,τ_s) is T_i, $i = 0,1,2$. From Theorem 2.16 (c), (X,τ) is $\varphi_{1,2}$–T_2 iff (X,τ) is T_2 iff (X,τ_s) is T_2.

2.21. Example. Let $\varphi_1 = \text{int} \circ \text{cl} \circ \text{int}$ and $\varphi_2 = \text{scl}$. Then φ_2 is W.F.I.P. w.r.t. $\varphi_1 O(X) = \tau^\alpha$ [7]. φ_2 is regular w.r.t. $\varphi_1 O(X)$ and $\varphi_2 \geq \imath$.

For each $U \in \varphi_1 O(X) = \tau^\alpha$, we have $\varphi_2(U) = \text{scl}\, U = U \cup \overset{\circ}{\overline{U}} = U \cup \overset{\circ}{\overline{\overset{\circ}{U}}} = \overset{\circ}{\overline{\overset{\circ}{U}}} \in \varphi_1 O(X)$ and $\varphi_2(\varphi_2(U)) \subseteq \varphi_2(U)$. Then $\varphi_{1,2} O(X)$ is a topology and $\mathcal{B} = \{\varphi_2(U) : U \in \varphi_1 O(X)\} = \{\text{scl}\, U : U \in \tau^\alpha\} = \{\overset{\circ}{\overline{\overset{\circ}{U}}} : U \in \tau^\alpha\} = RO(X)$ is a base for $\varphi_{1,2} O(X)$. Also, $\varphi_{1,2} O(X) = \tau_s$. Hence (X,τ) is $\varphi_{1,2}$–T_2 iff (X,τ^α) is T_2 iff (X,τ_s) is T_2.

2.22. Example. Let $\varphi_1 = \text{cl} \circ \text{int}$. Then φ_1 is monotonous, $\varphi_1 O(X) = SO(X)$ is a supratopology on X and $\varphi_2 = \varphi_1 O(X)$–cl$=$ scl. For $U \in \varphi_1 O(X) = SO(X)$, $\varphi_2(U) = \text{scl}\, U \in SO(X) = \varphi_1 O(X)$. So, (X,τ) is $\varphi_{1,2}$–T_2 iff $(X, \varphi_1 O(X))$ is T_2 iff for $x, y \in X$, $x \neq y$, there are semi-open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$ iff for $x, y \in X$, $x \neq y$, there are semi-open sets U and V such that $x \in U$, $y \in V$ and $\text{scl}\, U \cap \text{scl}\, V = \emptyset$.

2.23. Example. Let $\varphi_1 = \text{int} \circ \text{cl}$, $\varphi_2 = \text{pcl}$, $\psi_1 = \text{int}$, $\psi_2 = \imath$.

$f : X \to Y$ is $\varphi_{1,2}\psi_{1,2}$–continuous iff f is strongly Θ–pre-continuous. Also, G_f is $\varphi_{1,2}\psi_{1,2}$–closed iff G_f is strongly pre-closed [6]. Hence, (Y,ϑ) is $\psi_{1,2}$–T_2 iff (Y,ϑ) is Hausdorff. Finally, (X,τ) is $\varphi_{1,2}$–T_2 iff (X,τ) is pre-Urysohn [6]. Many of the results in [6] have been obtained in [7-11] and here.

It is now possible to obtain many known results, some of which can be seen in [2,3,6], by choosing particular operations. In addition, by combining the unifications in [7-9] with those in the present paper, we may obtain many other results.

References

[1] Abd El-Monsef, M. E. and Lashien, E. F. Local discrete extensions of supratopologies. Tamkang J. Math. **21 (1)**, 1–6, 1990.

[2] Jafari, S. and Noiri T. Functions with strongly P-closed graphs, Univ. Bacāu Stu. Cer. Ştıntifice, Seria. Matematica, **9**, 103–110, 1999.

[3] Jafari, S. and Noiri T. Some properties of almost s-continuous functions, Rendiconti Del Circolo Matematico Di Palermo, Series II, Tomo XLVIII, 571–582, 1999.

[4] Kandil, A., Kerre, E. E. and Nouh, A. A. Operations and mappings on fuzzy topological spaces, Ann. Soc. Sci. Bruxelles **105 (4)**, 165–188, 1991.

[5] Kerre, E. E., Nouh, A. A. and Kandil, A. Operations on the class of fuzzy sets on a universe endowed with a fuzzy topology, J. Math. Anal. Appl. **180**, 325–341, 1993.

[6] Noiri, T. Strongly Θ-precontinuous functions, Acta Math. Hungar. **90 (4)**, 307–316, 2001.

[7] Yalvaç, T. H. A unified theory for continuities, submitted.

[8] Yalvaç, T. H. A unified theory on openness and closedness of functions, submitted.

[9] Yalvaç, T. H. A unified approach to compactness and filters, Hacettepe Bull. Nat. Sci. Eng. Series B **29**, 63–75, 2000.

[10] Yalvaç, T. H. On some unifications, The First Turkish International Conference on Topology and its Applications, August, 2000; Hacettepe Bull. Nat. Sci. Eng. Series B **30**, 27–38, 2001.

[11] Yalvaç, T. H. Unification of some topological concepts, International Conference on Topology and its Applications, September, 2000, Macedonia.