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A STUDY ON EXACT DISTRIBUTIONS OF THE KAO AND CHAKRABORTI'S STATISTICS FOR COMPARING TREATMENTS WITH A CONTROL IN A RANDOMIZED COMPLETE BLOCK DESIGN

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Abstract

In this study, the exact distributions of the Kao and Chakraborti's Statistics for comparing treatments with a control in a randomized complete block design are obtained. Some exact critical values found from this distribution are also given. A great many differences for this statistics have been found between the cumulative probability values by the usual approach proposed by Kao and Chakraborti and those of the exact distributions, even when the sample size is inadequate.

Keywords: Two-way classification, Joint rank test, Simulation.

1. Introduction

In many fields such as industry, agriculture, and health care, it is more important to know whether carrying out a task with a newly-suggested method, in place of a commonly used method taken as a standard or control, causes improvements than it is to know whether it merely brings about change. For example, it is highly important for those involved in implementation in a given area of study to know whether patients treated with a newly-developed drug recover more quickly than those treated with a standard drug, or whether a newly-produced fertilizer is more productive than the standard one. Suppose that an experiment has been designed to identify, in various kinds of soil, whether a newly-developed fertilizer is better than the standard one. Such a problem may be formulated as a one-tailed hypothesis test, called a randomized complete block design, in which soil types are blocks, newly-developed fertilizers are treatments, and the standard fertilizer is control. For the test, we suppose that a one-tailed comparison of t treatments in b blocks with a control has been designed.

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2. The Model

Let X_{ijk} denotes the k th observation from the i th block for the j th treatment, where $k = 1, 2, \ldots, n_{ij}$, $j = 0, 1, \ldots, t$ and $i = 1, 2, \ldots, b$. Here, 0 represents the control. The sample sizes for the (i, j) th cell, n_{ij}, can be arbitrary, but not equal to zero. The observations X_{ijk} are assumed to be independent and identically distributed with continuous distribution functions $F_{ij}(x)$.

It is aimed to test the null hypothesis that the treatments have the same effect as the control in all blocks, against the alternative hypothesis that at least one of the treatments is better the than the control in all blocks. Thus, the null hypothesis

(1) $$H_0 : F_{i0}(x) = F_{i1}(x) = \cdots = F_{it}(x),$$

for all x, will be tested against the one-tailed alternative hypothesis

(2) $$H_1 : F_{ij}(x) \leq F_{i0}(x) \text{ or } H_2 : F_{ij}(x) \geq F_{i0}(x),$$

for at least one (i, j), with the inequality holding for at least one $j = 0, 1, \ldots, t$ and for some x. If better means higher in practice, the alternative hypothesis is H_1; otherwise, if better means smaller, the alternative hypothesis is H_2.

3. Test Statistic

The statistic, proposed by Kao and Chakraborti [5], to test H_0 against H_1 or H_2 is based on joint rank. Joint rank also forms the basis the statistics proposed by Friedman [1], Hettmansperger and Norton [4], Mack and Skillings [6] and Hettmansperger [2,3].

If ranks are assigned to all the observations (of the control and treatments) in the i th block, these are called joint ranks and their sums are called the *joint rank sums*.

Let the number of observations in the (i, j) cell be n_{ij}, the number in the i th block be $N_{i.} = \sum_{j=0}^{t} n_{ij}$, and the total number of observations be $N = \sum_{i=1}^{b} N_i$. The joint rank of X_{ijk} in the i th block, denoted by r_{ijk}, takes the values $1, 2, \ldots, N_i$. Then, for the (i, j) th cell,

(3) $$R_{ij} = \sum_{k=1}^{n_{ij}} R_{ijk} \text{ and } \bar{R}_{ij} = \frac{R_{ij}}{n_{ij}},$$

indicate the sum of the joint ranks and the mean of the joint ranks, respectively. Hence, $\bar{R}_{.j} = \sum_{i=1}^{b} \bar{R}_{ij}$ specifies the mean of the joint ranks for the j th treatment, the sum being taken over all the blocks. In order to test the hypothesis H_0 against H_1 or H_2, Kao and Chakraborti [5] proposed the following statistic

(4) $$T = \sum_{j=1}^{t} \bar{R}_{.j},$$

based on the sums of the means of the joint ranks for the treatments (excluding the control). Denoting by t the value the statistic T obtained from the data, if $t \geq t_1$, H_0 is rejected in favor of H_1. Similarly, if $t \leq t_2$, H_0 is rejected in favor of H_2. Here, t_1 and t_2 are the critical values satisfying $P(T \geq t_1) = \alpha$ and $P(T \leq t_2) = \alpha$, where α is the significance level of the test. Kao and Chakraborti [5] proposed approximating the statistic T using the normal distribution in order to obtain these critical values (and thus to find the values of P).

For such an approximation, the expected value and variance of the T statistic required for the approximation are obtained as:

$$\text{(5)} \qquad \mathrm{E}(T) = \frac{1}{2}\sum_{i=1}^{b}\sum_{j=1}^{t}(N_{i.}+1), \text{ and,}$$

$$\text{(6)} \qquad \mathrm{V}(T) = \frac{1}{12}\Big(\sum_{i=1}^{b}\sum_{j=1}^{t} N_{i.}^{2} n_{ij}^{-1} - t^{2}\sum_{i=1}^{b} N_{i.}\Big).$$

When the H_0 hypothesis is correct, the statistic $Z_T = (T - \mu_T)/\sigma_T$ exhibits standard normal distribution (Kao and Chakraborti, [5]). Then, denoting by z_α the critical value satisfying $P(Z \geq z_\alpha) = \alpha$, if $z \geq z_\alpha$, H_0 is rejected in favour of H_1. Likewise, if $z \leq -z_\alpha$, then H_0 is rejected in favour of H_2.

It is necessary to use large-sized samples for the approach proposed by Kao and Chakraborti to yield satisfactory results. It is observed in this study, too, that this approach with small-sized samples does not produce satisfactory results.

4. Exact Distributions of the T Statistic

In this section, the exact distributions of the statistic T, which were not studied by Kao and Chakraborti [5], are given. For the case $n_{ij} = n$, samples with cells of equal size, for various values of b, t and n, the exact distributions of the statistic T has been obtained using a program developed by the present authors. Using this program, $\left\{\binom{tn}{n}\times\binom{tn-n}{n}\times\cdots\times\binom{n}{n}\right\}^{b}$ different orderings of the ordinal numbers were obtained for the different values of t, b, and n, and the values of the test statistics for these orderings were calculated and converted into a probability distribution. The program is available on request. Because 0 symbolizes the control, the critical values providing some cumulative probability values are given in Table 1 and Table 2 for $t = 1, 2, 3$; $n = 2, 3, 4, 5$ and $b = 2, 3, 4$.

Table 1. Selected Critical Values of T for the Balanced Design

$[\,b = 2, 3, 4, t = 1, 2, 3 \text{ and } n = 2, 3, 4, 5\,]$

$P(T \leq t'_\alpha)$ is the probability that the value of T found is less than or equal to t'_α

$t = 1 \quad b = 2$

$n = 2$: t'_α	$P(T \leq t'_\alpha)$	$n = 3$: t'_α	$P(T \leq t'_\alpha)$	$n = 4$: t'_α	$P(T \leq t'_\alpha)$	$n = 5$: t'_α	$P(T \leq t'_\alpha)$
3.00	0.028	4.00	0.003	5.25	0.001	7.00	0.001
3.50	0.083	4.33	0.008	5.50	0.002	7.20	0.002
4.00	0.222	4.67	0.020	5.75	0.004	7.40	0.004
		5.00	0.045	6.00	0.008	7.60	0.006
		5.33	0.085	6.25	0.014	7.80	0.010
		5.67	0.145	6.50	0.025	8.00	0.015
		6.00	0.228	6.75	0.041	8.20	0.022
				7.00	0.064	8.40	0.032
				7.25	0.095	8.60	0.045
				7.50	0.135	8.80	0.061
				7.75	0.184	9.00	0.082
						9.20	0.107
						9.40	0.137
						9.60	0.172

Table 1. (Continued)

$t = 1 \quad b = 3$

$n = 2$

t'_α	$P(T \leq t'_\alpha)$
4.50	0.005
5.00	0.019
5.50	0.060
6.00	0.134
6.50	0.259

$n = 3$

t'_α	$P(T \leq t'_\alpha)$
6.67	0.002
7.00	0.004
7.33	0.010
7.67	0.021
8.00	0.038
8.33	0.066
8.67	0.106
9.00	0.160

$n = 4$

t'_α	$P(T \leq t'_\alpha)$
9.00	0.001
9.25	0.002
9.50	0.004
9.75	0.007
10.00	0.011
10.25	0.018
10.50	0.027
10.75	0.040
11.00	0.057
11.25	0.079
11.50	0.108
11.75	0.142
12.00	0.183

$n = 5^*$

t'_α	$P(T \leq t'_\alpha)$
11.60	0.001
11.80	0.002
12.00	0.003
12.20	0.004
12.40	0.006
12.60	0.009
12.80	0.013
13.00	0.018
13.20	0.025
13.40	0.033
13.60	0.044
13.80	0.057
14.00	0.073
14.20	0.091
14.40	0.113
14.60	0.139
14.80	0.167

$t = 1 \quad b = 4$

$n = 2$

t'_α	$P(T \leq t'_\alpha)$
6.00	0.001
6.50	0.004
7.00	0.015
7.50	0.039
8.00	0.090
8.50	0.170

$n = 3$

t'_α	$P(T \leq t'_\alpha)$
9.33	0.001
9.67	0.002
10.00	0.005
10.33	0.010
10.67	0.018
11.00	0.031
11.33	0.051
11.67	0.079
12.00	0.117
12.33	0.166

$n = 4$

t'_α	$P(T \leq t'_\alpha)$
12.75	0.001
13.00	0.002
13.25	0.003
13.30	0.005
13.45	0.008
14.00	0.012
14.25	0.017
14.50	0.025
14.75	0.035
15.00	0.049
15.25	0.065
15.50	0.086
15.75	0.111
16.00	0.142
16.25	0.176

$n = 5^*$

t'_α	$P(T \leq t'_\alpha)$
16.40	0.001
16.60	0.002
16.80	0.003
17.00	0.004
17.20	0.006
17.40	0.008
17.60	0.011
17.80	0.014
18.00	0.019
18.20	0.025
18.40	0.032
18.60	0.040
18.80	0.051
19.00	0.063
19.20	0.078
19.40	0.094
19.60	0.113
19.80	0.135
20.00	0.160

$t = 2 \quad b = 2$

$n = 2$

t'_α	$P(T \leq t'_\alpha)$
10.00	0.004
10.50	0.013
11.00	0.036
11.50	0.071
12.00	0.133
12.50	0.213

$n = 3$

t'_α	$P(T \leq t'_\alpha)$
14.67	0.001
15.00	0.003
15.33	0.005
15.67	0.009
16.00	0.016
16.33	0.026
16.67	0.041
17.00	0.061
17.33	0.088
17.67	0.121
18.00	0.163

$n = 4^*$

t'_α	$P(T \leq t'_\alpha)$
19.50	0.001
19.75	0.002
20.00	0.002
20.25	0.004
20.50	0.005
20.75	0.008
21.00	0.011
21.25	0.015
21.50	0.020
21.75	0.027
22.00	0.035
22.25	0.046
22.50	0.059
22.75	0.074
23.00	0.092
23.25	0.113
23.50	0.137
23.75	0.164

$n = 5^*$

t'_α	$P(T \leq t'_\alpha)$
25.20	0.001
25.40	0.002
25.60	0.002
25.80	0.003
26.00	0.004
26.20	0.005
26.40	0.007
26.60	0.009
26.80	0.012
27.00	0.015
27.20	0.019
27.40	0.024
27.60	0.029
27.80	0.036
28.00	0.044
28.20	0.053
28.40	0.063
28.60	0.075
28.80	0.089
29.00	0.104

Table 1. (Continued)

$t = 2 \quad b = 3$

$n = 2$

t'_α	$P(T \le t'_\alpha)$
15.50	0.001
16.00	0.004
16.50	0.009
17.00	0.021
17.50	0.041
18.00	0.072
18.50	0.118
19.00	0.180

$n = 3^*$

t'_α	$P(T \le t'_\alpha)$
23.00	0.001
23.33	0.002
23.67	0.004
24.00	0.006
24.33	0.009
24.67	0.013
25.00	0.019
25.33	0.028
25.67	0.038
26.00	0.052
26.33	0.069
26.67	0.090
27.00	0.116
27.33	0.146
27.67	0.181

$n = 4^*$

t'_α	$P(T \le t'_\alpha)$
31.00	0.001
31.25	0.002
31.50	0.002
31.75	0.003
32.00	0.004
32.25	0.005
32.50	0.007
32.75	0.009
33.00	0.012
33.25	0.016
33.50	0.020
33.75	0.025
34.00	0.032
34.25	0.039
34.50	0.048
34.75	0.058
35.00	0.071
35.25	0.085
35.50	0.101
35.75	0.119
36.00	0.139
36.25	0.161

$n = 5^*$

t'_α	$P(T \le t'_\alpha)$
39.80	0.001
40.00	0.002
40.20	0.002
40.40	0.003
40.60	0.004
40.80	0.005
41.00	0.006
41.20	0.007
41.40	0.009
41.60	0.011
41.80	0.014
42.00	0.017
42.20	0.020
42.40	0.024
42.60	0.028
42.80	0.033
43.00	0.039
43.20	0.046
43.40	0.053
43.60	0.062
43.80	0.071
44.00	0.082
44.20	0.093
44.40	0.106

$t = 2 \quad b = 4$

$n = 2^*$

t'_α	$P(T \le t'_\alpha)$
21.50	0.001
22.00	0.002
22.50	0.004
23.00	0.009
23.50	0.019
24.00	0.034
24.50	0.058
25.00	0.093
25.50	0.141
26.00	0.202

$n = 3^*$

t'_α	$P(T \le t'_\alpha)$
32.33	0.001
32.67	0.002
33.00	0.003
33.33	0.004
33.67	0.006
34.00	0.009
34.33	0.013
34.67	0.019
35.00	0.027
35.33	0.036
35.67	0.049
36.00	0.064
36.33	0.083
36.67	0.105
37.00	0.132
37.33	0.162

$n = 4^*$

t'_α	$P(T \le t'_\alpha)$
43.50	0.001
43.75	0.002
44.00	0.003
44.25	0.004
44.50	0.005
44.75	0.006
45.00	0.008
45.25	0.010
45.50	0.013
45.75	0.016
46.00	0.020
46.25	0.025
46.50	0.031
46.75	0.038
47.00	0.046
47.25	0.055
47.50	0.065
47.75	0.077
48.00	0.091
48.25	0.106
48.50	0.122
48.75	0.141
49.00	0.161

$n = 5^*$

t'_α	$P(T \le t'_\alpha)$
55.00	0.002
55.20	0.003
55.40	0.004
55.60	0.004
55.80	0.005
56.00	0.006
56.20	0.008
56.40	0.009
56.60	0.011
56.80	0.013
57.00	0.015
57.20	0.018
57.40	0.021
57.60	0.025
57.80	0.028
58.00	0.033
58.20	0.038
58.40	0.043
58.60	0.050
58.80	0.056
59.00	0.064
59.20	0.072
59.40	0.081
59.60	0.091
59.80	0.102

Table 1. (Continued)

$t=3 \quad b=2$

$n=2$

t'_α	$P(T \le t'_\alpha)$
21.00	0.001
21.50	0.004
22.00	0.010
22.50	0.020
23.00	0.038
23.50	0.064
24.00	0.102
24.50	0.150

$n=3^*$

t'_α	$P(T \le t'_\alpha)$
31.67	0.001
32.00	0.002
32.33	0.003
32.67	0.005
33.00	0.008
33.33	0.013
33.67	0.018
34.00	0.026
34.33	0.036
34.67	0.049
35.00	0.064
35.33	0.084
35.67	0.107
36.00	0.133
36.33	0.164

$n=4^*$

t'_α	$P(T \le t'_\alpha)$
42.50	0.001
42.75	0.002
43.00	0.002
43.25	0.003
43.50	0.004
43.75	0.006
44.00	0.007
44.25	0.010
44.50	0.012
44.75	0.016
45.00	0.020
45.25	0.025
45.50	0.031
45.75	0.037
46.00	0.045
46.25	0.055
46.50	0.065
46.75	0.077
47.00	0.091
47.25	0.106
47.50	0.123
47.75	0.142
48.00	0.163

$n=5^*$

t'_α	$P(T \le t'_\alpha)$
53.60	0.001
53.80	0.002
54.00	0.002
54.20	0.003
54.40	0.003
54.60	0.004
54.80	0.005
55.00	0.006
55.20	0.007
55.40	0.009
55.60	0.011
55.80	0.013
56.00	0.015
56.20	0.018
56.40	0.021
56.60	0.024
56.80	0.028
57.00	0.033
57.20	0.038
57.40	0.043
57.60	0.050
57.80	0.056
58.00	0.064
58.20	0.072
58.40	0.082
58.60	0.092
58.80	0.102
59.00	0.114
59.20	0.127
59.40	0.140
59.60	0.154

$t=3 \quad b=3$

$n=2^*$

t'_α	$P(T \le t'_\alpha)$
33.00	0.001
33.50	0.002
34.00	0.005
34.50	0.009
35.00	0.017
35.50	0.028
36.00	0.045
36.50	0.069
37.00	0.100
37.50	0.140
38.00	0.190

$n=3^*$

t'_α	$P(T \le t'_\alpha)$
49.33	0.001
49.67	0.002
50.00	0.002
50.33	0.004
50.67	0.005
51.00	0.007
51.33	0.010
51.67	0.014
52.00	0.018
52.33	0.024
52.67	0.031
53.00	0.040
53.33	0.051
53.67	0.064
54.00	0.079
54.33	0.097
54.67	0.117
55.00	0.140
55.33	0.166

$n=4^*$

t'_α	$P(T \le t'_\alpha)$
66.75	0.002
67.00	0.003
67.25	0.004
67.50	0.005
67.75	0.006
68.00	0.008
68.25	0.010
68.50	0.012
68.75	0.014
69.00	0.017
69.25	0.021
69.50	0.025
69.75	0.029
70.00	0.035
70.25	0.041
70.50	0.047
70.75	0.055
71.00	0.064
71.25	0.073
71.50	0.084
71.75	0.095
72.00	0.108
72.25	0.122
72.50	0.137
72.75	0.153

$n=5^*$

t'_α	$P(T \le t'_\alpha)$
83.80	0.003
84.00	0.004
84.20	0.004
84.40	0.005
84.60	0.006
84.80	0.007
85.00	0.008
85.20	0.009
85.40	0.010
85.60	0.012
85.80	0.014
86.00	0.015
86.20	0.018
86.40	0.020
86.60	0.023
86.80	0.026
87.00	0.029
87.20	0.033
87.40	0.037
87.60	0.041
87.80	0.046
88.00	0.051
88.20	0.057
88.40	0.063
88.60	0.070
88.80	0.077
89.00	0.085
89.20	0.093
89.40	0.102

Table 1. (Continued)

$t = 3 \quad b = 4$

$n = 2^*$

t'_α	$P(T \leq t'_\alpha)$
45.50	0.001
46.00	0.003
46.50	0.005
47.00	0.008
47.50	0.014
48.00	0.023
48.50	0.034
49.00	0.050
49.50	0.072
50.00	0.099
50.50	0.133
51.00	0.174

$n = 3^*$

t'_α	$P(T \leq t'_\alpha)$
67.67	0.001
68.00	0.002
68.33	0.003
68.67	0.004
69.00	0.005
69.33	0.007
69.67	0.009
70.00	0.012
70.33	0.016
70.67	0.020
71.00	0.026
71.33	0.032
71.67	0.040
72.00	0.049
72.33	0.060
72.67	0.072
73.00	0.086
73.33	0.102
73.67	0.120
74.00	0.140
74.33	0.162

$n = 4^*$

t'_α	$P(T \leq t'_\alpha)$
90.50	0.002
90.75	0.003
91.00	0.003
91.25	0.004
91.50	0.005
91.75	0.006
92.00	0.007
92.25	0.008
92.50	0.010
92.75	0.012
93.00	0.014
93.25	0.016
93.50	0.019
93.75	0.022
94.00	0.025
94.25	0.030
94.50	0.034
94.75	0.039
95.00	0.045
95.25	0.051
95.50	0.058
95.75	0.066
96.00	0.074
96.25	0.083
96.50	0.093
96.75	0.104
97.00	0.115
97.25	0.128
97.50	0.141
97.75	0.156

$n = 5^*$

t'_α	$P(T \leq t'_\alpha)$
113.80	0.003
114.00	0.004
114.20	0.004
114.40	0.005
114.60	0.006
114.80	0.007
115.00	0.008
115.20	0.009
115.40	0.010
115.60	0.011
115.80	0.012
116.00	0.014
116.20	0.015
116.40	0.017
116.60	0.019
116.80	0.022
117.00	0.024
117.20	0.027
117.40	0.030
117.60	0.033
117.80	0.036
118.00	0.040
118.20	0.044
118.40	0.049
118.60	0.053
118.80	0.058
119.00	0.064
119.20	0.069
119.40	0.076
119.60	0.082
119.80	0.089
120.00	0.097
120.20	0.104

* The table values appearing in columns headed by an asterisk were obtained by simulation

Table 2. Selected Critical Values of T for the Balanced Design

$[\, b = 2, 3, 4, t = 1, 2, 3$ and $n = 2, 3, 4, 5\,]$

$P(T \geq t_\alpha")$ is the probability that the value of T found is greater than or equal to $t_\alpha"$

$t = 1 \quad b = 2$

$n = 2$

$t_\alpha"$	$P(T \geq t_\alpha")$
7.00	0.028
6.50	0.083
6.00	0.222

$n = 3$

$t_\alpha"$	$P(T \geq t_\alpha")$
10.00	0.003
9.67	0.008
9.33	0.020
9.00	0.045
8.67	0.085
8.33	0.145
8.00	0.228

$n = 4$

$t_\alpha"$	$P(T \geq t_\alpha")$
12.75	0.001
12.50	0.002
12.25	0.004
12.00	0.008
11.75	0.014
11.50	0.025
11.25	0.041
11.00	0.064
10.75	0.095
10.50	0.135
10.25	0.184

$n = 5$

$t_\alpha"$	$P(T \geq t_\alpha")$
15.00	0.001
14.80	0.002
14.60	0.004
14.40	0.006
14.20	0.010
14.00	0.015
13.80	0.022
13.60	0.032
13.40	0.045
13.20	0.061
13.00	0.082
12.80	0.107
12.60	0.137
12.40	0.172

Table 2. (Continued)

$t = 1 \quad b = 3$

$n = 2$

t_α"	$P(T \geq t_\alpha$")
10.50	0.005
10.00	0.019
9.50	0.060
9.00	0.134
8.50	0.259

$n = 3$

t_α"	$P(T \geq t_\alpha$")
14.33	0.002
14.00	0.004
13.67	0.010
13.33	0.021
13.00	0.038
12.67	0.066
12.33	0.106
12.00	0.160

$n = 4$

t_α"	$P(T \geq t_\alpha$")
18.00	0.001
17.75	0.002
17.50	0.004
17.25	0.007
17.00	0.011
16.75	0.018
16.50	0.027
16.25	0.040
16.00	0.057
15.75	0.079
15.50	0.108
15.25	0.142
15.00	0.183

$n = 5^*$

t_α"	$P(T \geq t_\alpha$")
21.40	0.001
21.20	0.002
21.00	0.003
20.80	0.004
20.60	0.006
20.40	0.009
20.20	0.013
20.00	0.018
19.80	0.025
19.60	0.033
19.40	0.044
19.20	0.057
19.00	0.073
18.80	0.091
18.60	0.113
18.40	0.139
18.20	0.167

$t = 1 \quad b = 4$

$n = 2$

t_α"	$P(T \geq t_\alpha$")
14.00	0.001
13.50	0.004
13.00	0.015
12.50	0.039
12.00	0.090
11.50	0.170

$n = 3$

t_α"	$P(T \geq t_\alpha$")
18.67	0.001
18.33	0.002
18.00	0.005
17.67	0.010
17.33	0.018
17.00	0.031
16.67	0.051
16.33	0.079
16.00	0.117
15.67	0.166

$n = 4$

t_α"	$P(T \geq t_\alpha$")
23.25	0.001
23.00	0.002
22.75	0.003
22.50	0.005
22.25	0.008
22.00	0.012
21.75	0.017
21.50	0.025
21.25	0.035
21.00	0.049
20.75	0.065
20.50	0.086
20.25	0.111
20.00	0.142
19.75	0.176

$n = 5^*$

t_α"	$P(T \geq t_\alpha$")
27.60	0.001
27.40	0.002
27.20	0.003
27.00	0.004
26.80	0.006
26.60	0.008
26.40	0.011
26.20	0.014
26.00	0.019
25.80	0.025
25.60	0.032
25.40	0.040
25.20	0.051
25.00	0.063
24.80	0.078
24.60	0.094
24.40	0.113
24.20	0.135
24.00	0.160

$t = 2 \quad b = 2$

$n = 2$

t_α"	$P(T \geq t_\alpha$")
18.00	0.004
17.50	0.013
17.00	0.036
16.50	0.071
16.00	0.133
16.50	0.213

$n = 3$

t_α"	$P(T \geq t_\alpha$")
25.33	0.001
25.00	0.003
24.67	0.005
24.33	0.009
24.00	0.016
23.67	0.026
23.33	0.041
23.00	0.061
22.67	0.088
22.33	0.121
22.00	0.163

$n = 4^*$

t_α"	$P(T \geq t_\alpha$")
32.50	0.001
32.25	0.002
32.00	0.002
31.75	0.004
31.50	0.005
31.25	0.008
31.00	0.011
30.75	0.015
30.50	0.020
30.25	0.027
30.00	0.035
29.75	0.046
29.50	0.059
29.25	0.074
29.00	0.092
28.75	0.113
28.50	0.137
28.25	0.164

$n = 5^*$

t_α"	$P(T \geq t_\alpha$")
38.80	0.001
38.60	0.002
38.40	0.002
38.20	0.003
38.00	0.004
37.80	0.005
37.60	0.007
37.40	0.009
37.20	0.012
37.00	0.015
36.80	0.019
36.60	0.024
36.40	0.029
36.20	0.036
36.00	0.044
35.80	0.053
35.60	0.063
35.40	0.075
35.20	0.089
35.00	0.104
34.80	0.121
34.60	0.139
34.40	0.160

Table 2. (Continued)

$t = 2 \quad b = 3$

$n = 2$

t_α"	$P(T \geq t_\alpha")$
26.50	0.001
26.00	0.004
25.50	0.009
25.00	0.021
24.50	0.041
24.00	0.072
23.50	0.118
23.00	0.180

$n = 3^*$

t_α"	$P(T \geq t_\alpha")$
37.00	0.001
36.67	0.002
36.33	0.004
36.00	0.006
35.67	0.009
35.33	0.013
35.00	0.019
34.67	0.028
34.33	0.038
34.00	0.052
33.67	0.069
33.33	0.090
33.00	0.116
32.67	0.146
32.33	0.181

$n = 4^*$

t_α"	$P(T \geq t_\alpha")$
47.00	0.001
46.75	0.002
46.50	0.002
46.25	0.003
46.00	0.004
45.75	0.005
45.50	0.007
45.25	0.009
45.00	0.012
44.75	0.016
44.50	0.020
44.25	0.025
44.00	0.032
43.75	0.039
43.50	0.048
43.25	0.058
43.00	0.071
42.75	0.085
42.50	0.101
42.25	0.119
42.00	0.139
41.75	0.161

$n = 5^*$

t_α"	$P(T \geq t_\alpha")$
56.20	0.001
56.00	0.002
55.80	0.002
55.60	0.003
55.40	0.004
55.20	0.005
55.00	0.006
54.80	0.007
54.60	0.009
54.40	0.011
54.20	0.014
54.00	0.017
53.80	0.020
53.60	0.024
53.40	0.028
53.20	0.033
53.00	0.039
52.80	0.046
52.60	0.053
52.40	0.062
52.20	0.071
52.00	0.082
51.80	0.093
51.60	0.106
51.40	0.120
51.20	0.135
51.00	0.151

$t = 2 \quad b = 4$

$n = 2^*$

t_α"	$P(T \geq t_\alpha")$
34.50	0.001
34.00	0.002
33.50	0.004
33.00	0.009
32.50	0.019
32.00	0.034
31.50	0.058
31.00	0.093
30.50	0.141
30.00	0.202

$n = 3^*$

t_α"	$P(T \geq t_\alpha")$
47.67	0.001
47.33	0.002
47.00	0.003
46.67	0.004
46.33	0.006
46.00	0.009
45.67	0.013
45.33	0.019
45.00	0.027
44.67	0.036
44.33	0.049
44.00	0.064
43.67	0.083
43.33	0.105
43.00	0.132
42.67	0.162

$n = 4^*$

t_α"	$P(T \geq t_\alpha")$
60.50	0.001
60.25	0.002
60.00	0.003
59.75	0.004
59.50	0.005
59.25	0.006
59.00	0.008
58.75	0.010
58.50	0.013
58.25	0.016
58.00	0.020
57.75	0.025
57.50	0.031
57.25	0.038
57.00	0.046
56.75	0.055
56.50	0.065
56.25	0.077
56.00	0.091
55.75	0.106
55.50	0.122
55.25	0.141
55.00	0.161

$n = 5^*$

t_α"	$P(T \geq t_\alpha")$
73.00	0.002
72.80	0.003
72.60	0.004
72.40	0.004
72.20	0.005
72.00	0.006
71.80	0.008
71.60	0.009
71.40	0.011
71.20	0.013
71.00	0.015
70.80	0.018
70.60	0.021
70.40	0.025
70.20	0.028
70.00	0.033
69.80	0.038
69.60	0.043
69.40	0.050
69.20	0.056
69.00	0.064
68.80	0.072
68.60	0.081
68.40	0.091
68.20	0.102
68.00	0.113
67.80	0.126
67.60	0.139
67.40	0.153

Table 2. (Continued)

$t = 3 \quad b = 2$

$n = 2$

t_α"	$P(T \geq t_\alpha")$
33.00	0.001
32.50	0.004
32.00	0.010
31.50	0.020
31.00	0.038
30.50	0.064
30.00	0.102
29.50	0.150

$n = 3^*$

t_α"	$P(T \geq t_\alpha")$
46.33	0.001
46.00	0.002
45.67	0.003
45.33	0.005
45.00	0.008
44.67	0.013
44.33	0.018
44.00	0.026
43.67	0.036
43.33	0.049
43.00	0.064
42.67	0.084
42.33	0.107
42.00	0.133
41.67	0.164

$n = 4^*$

t_α"	$P(T \geq t_\alpha")$
59.50	0.001
59.25	0.002
59.00	0.002
58.75	0.003
58.50	0.004
58.25	0.006
58.00	0.007
57.75	0.010
57.50	0.012
57.25	0.016
57.00	0.020
56.75	0.025
56.50	0.031
56.25	0.037
56.00	0.045
55.75	0.055
55.50	0.065
55.25	0.077
55.00	0.091
54.75	0.106
54.50	0.123
54.25	0.142
54.00	0.163

$n = 5^*$

t_α"	$P(T \geq t_\alpha")$
72.40	0.001
72.20	0.002
72.00	0.002
71.80	0.003
71.60	0.003
71.40	0.004
71.20	0.005
71.00	0.006
70.80	0.007
70.60	0.009
70.40	0.011
70.20	0.013
70.00	0.015
69.80	0.018
69.60	0.021
69.40	0.024
69.20	0.028
69.00	0.033
68.80	0.038
68.60	0.043
68.40	0.050
68.20	0.056
68.00	0.064
67.80	0.072
67.60	0.082
67.40	0.092
67.20	0.102
67.00	0.114
66.80	0.127
66.60	0.140
66.40	0.154

$t = 3 \quad b = 3$

$n = 2^*$

t_α"	$P(T \geq t_\alpha")$
48.00	0.001
47.50	0.002
47.00	0.005
46.50	0.009
46.00	0.017
45.50	0.028
45.00	0.045
44.50	0.069
44.00	0.100
43.50	0.140
43.00	0.190

$n = 3^*$

t_α"	$P(T \geq t_\alpha")$
67.67	0.001
67.33	0.002
67.00	0.002
66.67	0.004
66.33	0.005
66.00	0.007
65.67	0.010
65.33	0.014
65.00	0.018
64.67	0.024
64.33	0.031
64.00	0.040
63.67	0.051
63.33	0.064
63.00	0.079
62.67	0.097
62.33	0.117
62.00	0.140
61.67	0.166

$n = 4^*$

t_α"	$P(T \geq t_\alpha")$
86.25	0.002
86.00	0.003
85.75	0.004
85.50	0.005
85.25	0.006
85.00	0.008
84.75	0.010
84.50	0.012
84.25	0.014
84.00	0.017
83.75	0.021
83.50	0.025
83.25	0.029
83.00	0.035
82.75	0.041
82.50	0.047
82.25	0.055
82.00	0.064
81.75	0.073
81.50	0.084
81.25	0.095
81.00	0.108
80.75	0.122
80.50	0.137
80.25	0.153

$n = 5^*$

t_α"	$P(T \geq t_\alpha")$
105.20	0.003
105.00	0.004
104.80	0.004
104.60	0.005
104.40	0.006
104.20	0.007
104.00	0.008
103.80	0.009
103.60	0.010
103.40	0.012
103.20	0.014
103.00	0.015
102.80	0.018
102.60	0.020
102.40	0.023
102.20	0.026
102.00	0.029
101.80	0.033
101.60	0.037
101.40	0.041
101.20	0.046
101.00	0.051
100.80	0.057
100.60	0.063
100.40	0.070
100.20	0.077
100.00	0.085
99.80	0.093
99.60	0.102

Table 2. (Continued)

$t = 3 \quad b = 4$

$n = 2^*$

t_{α}"	$P(T \geq t_{\alpha}")$
62.50	0.001
62.00	0.003
61.50	0.005
61.00	0.008
60.50	0.014
60.00	0.023
59.50	0.034
59.00	0.050
58.50	0.072
58.00	0.099
57.50	0.133
57.00	0.174

$n = 3^*$

t_{α}"	$P(T \geq t_{\alpha}")$
88.33	0.001
88.00	0.002
87.67	0.003
87.33	0.004
87.00	0.005
86.67	0.007
86.33	0.009
86.00	0.012
85.67	0.016
85.33	0.020
85.00	0.026
84.67	0.032
84.33	0.040
84.00	0.049
83.67	0.060
83.33	0.072
83.00	0.086
82.67	0.102
82.33	0.120
82.00	0.140
81.67	0.162

$n = 4^*$

t_{α}"	$P(T \geq t_{\alpha}")$
113.50	0.002
113.25	0.003
113.00	0.003
112.75	0.004
112.50	0.005
112.25	0.006
112.00	0.007
111.75	0.008
111.50	0.010
111.25	0.012
111.00	0.014
110.75	0.016
110.50	0.019
110.25	0.022
110.00	0.025
109.75	0.030
109.50	0.034
109.25	0.039
109.00	0.045
108.75	0.051
108.50	0.058
108.25	0.066
108.00	0.074
107.75	0.083
107.50	0.093
107.25	0.104
107.00	0.115
106.75	0.128
106.50	0.141
106.25	0.156

$n = 5^*$

t_{α}"	$P(T \geq t_{\alpha}")$
138.20	0.003
138.00	0.004
137.80	0.004
137.60	0.005
137.40	0.006
137.20	0.007
137.00	0.008
136.80	0.009
136.60	0.010
136.40	0.011
136.20	0.012
136.00	0.014
135.80	0.015
135.60	0.017
135.40	0.019
135.20	0.022
135.00	0.024
134.80	0.027
134.60	0.030
134.40	0.033
134.20	0.036
134.00	0.040
133.80	0.044
133.60	0.049
133.40	0.053
133.20	0.058
133.00	0.064
132.80	0.069
132.60	0.076
132.40	0.082
132.20	0.089
132.00	0.097
131.80	0.104

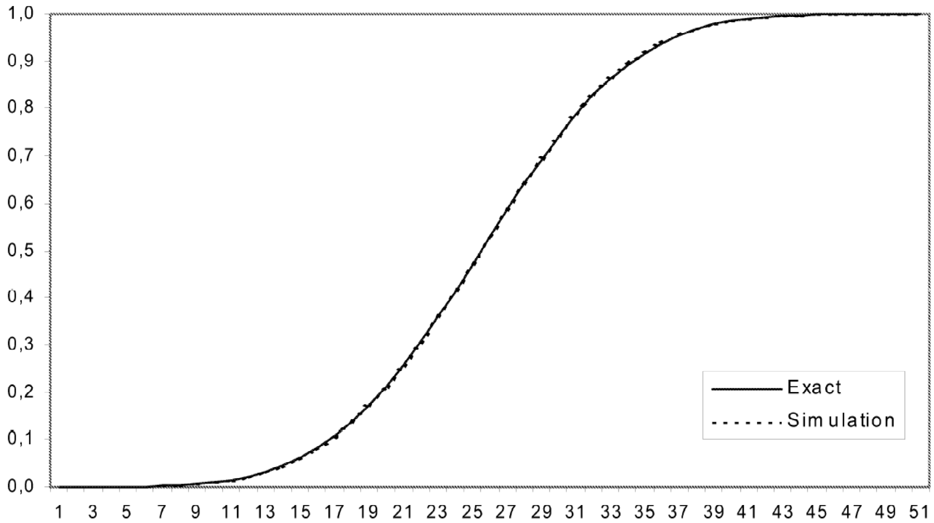
* The table values appearing in columns headed by an asterisk were obtained by simulation

It is quite clear that the critical values in Table 1 to test H_0 against H_2 and those in Table 2 to test H_0 against H_1 will be useful, because, when exact distributions are formed, the number of processes increases rapidly. For instance, when $b = 4$, $t = 3$ and $n = 4$, $158160674247936 \times 10^{17}$ processes are required.

The cumulative probability values in columns headed by a starred value in Table 1 and Table 2 have been obtained using distributions obtained through simulation, with 10000000 repetitions. It has been verified that the distribution obtained through simulation is identical to the exact distribution. For example, when $b = 2$, $t = 1$ and $n = 5$, the value of $P(T \leq 9)$ obtained from the exact distribution is 0.082 and while that from the distribution obtained through simulation is 0.08.

Similarly, the value of $P(T \leq 15)$ is 0.999 when obtained from the exact distribution and 0.9995 when obtained from the one through simulation. For $b = 2$, $t = 1$ and $n = 5$, the distribution functions found in the two ways are shown in Figure 1.

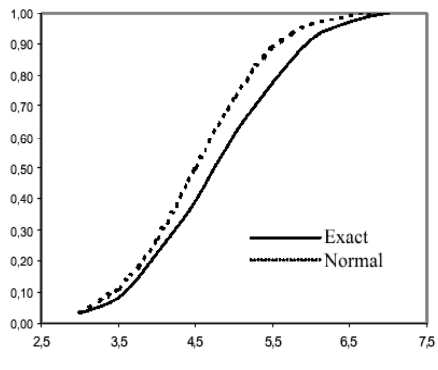
Figure 1: For b=2, t=1 and n=5, the distribution functions obtained from exact distribution and through simulation



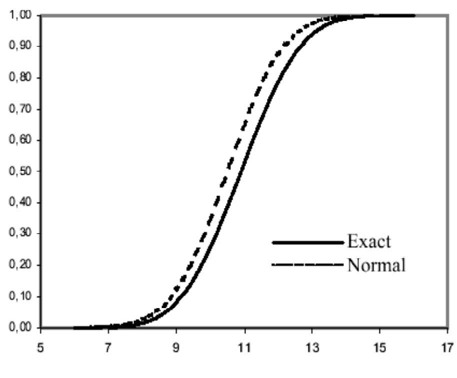
A comparison of the cumulative probability values found using the normal distribution approach of Kao and Chakraborti [5], and those obtained with the exact distribution, has produced interesting results. Significant differences have been observed between the results found in the two ways, especially when n is small. For example, when $b = 2$, $t = 1$, and $n = 2$, the value of $P(T \leq 4.5)$ as obtained from the normal distribution is 0.500, while that from the exact distribution is 0.389.

When $b = 4$, $t = 3$ and $n = 2$, the value of $P(T \leq 52)$ from the normal distribution is 0.298 and that from the exact distribution is 0.276. Likewise, when $b = 3$, $t = 2$ and $n = 5$, the value of $P(T \leq 44)$ obtained from the normal distribution is 0.101, while that from the exact distribution is 0.082. When $b = 4$, $t = 3$ and $n = 5$, the value of $P(T \leq 120)$ from the normal distribution is 0.109 and that from the exact distribution is 0.097.

Figure 2: For b=2, t=1, the distribution functions obtained from exact distribution and normal distribution

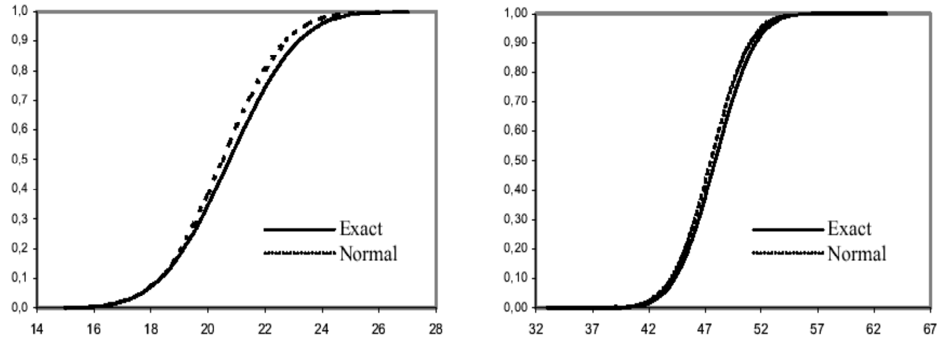


(a) n=2



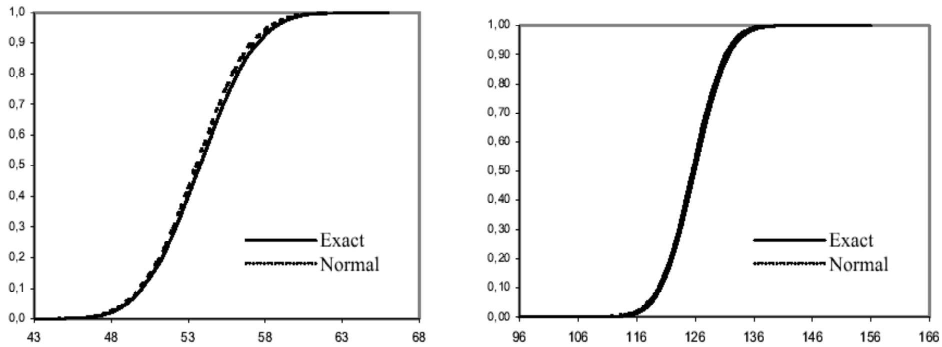
(b) n=5

Figure 3: For b=3, t=2, the distribution functions obtained from exact distribution and normal distribution



(a) n=2 (b) n=5

Figure 4: For b=4, t=3, the distribution functions obtained from exact distribution and normal distribution



(a) n=2 (b) n=5

The distribution functions for the T statistic for some of the values of b, t and n are demonstrated in Figure 2, 3 and 4.

Upon examining these figures, it will be clear that the results obtained for these two approaches are not close to one another if b, t and n are not sufficiently large. However, when $b = 4$, $t = 3$ and $n = 5$, the distribution functions obtained through the two ways become quite close to one another.

5. Conclusion

In practice the value of n, the number of units in each cell, rarely reaches 5. In such cases, using the normal distribution approximation may produce incorrect results. This is also so when $b < 4$ and $t < 3$, even if $n \geq 5$. Therefore, for such cases, it will clearly be more appropriate to use the critical values in Table 1 and Table 2, or the cumulative probability values, rather than the normal distribution approximation, when finding the value of P needed to test the required hypothesis.

References

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