Hacettepe Journal of Mathematics and Statistics
Volume 34 S (2005), 15 – 26

Doğan Çoker
Memorial Issue

A UNIFIED THEORY OF OPENNESS AND CLOSEDNESS OF FUNCTIONS

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Received 30 : 07 : 2003 : Accepted 22 : 05 : 2005

Abstract

Kandil, Kerre and Nouh unified various concepts in fuzzy topological spaces by using operations. By adapting their definition of an operation and some other definitions given by these authors to topological spaces, and by giving some new definitions, we have previously achieved some unifications related to continuity, compactness, filters and graphs. Here we will study the unification of openness and closedness properties of functions, and give some results related to $\varphi_{1,2}\psi_{1,2}$-closed functions and $\varphi_{1,2}\psi_{1,2}$-compact sets.

Keywords: Openness, Closedness, Compactness, Supratopology, Unification.

2000 AMS Classification: 54 C 10, 54 D 30.

1. Introduction

Many mathematicians have worked on the unification of properties in topological spaces and fuzzy topological spaces, as in [1–7,8,10,11]. In [3,5], some unifications for fuzzy topological spaces were studied. It was announced there, and is easily seen, that most of the definitions and results are applicable to topological spaces.

In a topological space (X, τ), $\operatorname{int}, \operatorname{cl}, \operatorname{scl}$ etc. will stand for the operations of interior, closure, semi-closure, and so on. In addition, A°, $\overline{A}$ will also denote the interior and closure of a subset A of X respectively.

1.1. Definition. Let (X, τ) be a topological space. A mapping $\varphi : P(X) \to P(X)$ is called an *operation on* (X, τ) if $A^o \subseteq \varphi(A)$ for all $A \in P(X)$ and $\varphi(\emptyset) = \emptyset$.

The class of all operations on a topological space (X, τ) will be denoted by $O(X, \tau)$.

A partial order "$\leq$" on $O(X, \tau)$ is defined by $\varphi_1 \leq \varphi_2 \iff \varphi_1(A) \subseteq \varphi_2(A)$ for each $A \in P(X)$.

An operation $\varphi \in O(X, \tau)$ is called *monotonous* if $\varphi(A) \subseteq \varphi(B)$ whenever $A \subseteq B$ for all $A, B \in P(X)$.

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1.2. Definition. Let $\varphi \in O(X,\tau)$. Then $A \subseteq X$ is called *φ-open* if $A \subseteq \varphi(A)$. Dually, $B \subseteq X$ is called *φ-closed* if $X \setminus B$ is φ-open.

Clearly, X and $\emptyset$ are both φ-open and φ-closed.

If φ is monotonous, then the family of all φ-open sets is a supratopology[†].

For a subset A of a supratopological space $(X,\mathcal{U})$ the $\mathcal{U}$-closure of A and $\mathcal{U}$-interior of A are defined as for topological spaces, i.e.

$$\begin{aligned}\mathcal{U}\text{-cl}\,A &= \bigcap\{K : A \subseteq K,\ X \setminus K \in \mathcal{U}\}\\ &= \{x : U \in \mathcal{U}, x \in U \implies U \cap A \neq \emptyset\}, \text{ and}\\ \mathcal{U}\text{-int}\,A &= \bigcup\{U : U \subseteq A, U \in \mathcal{U}\}\\ &= \{x : \exists U \in \mathcal{U}, x \in U \subseteq A\}.\end{aligned}$$

Let (X,τ) be a topological space, $\varphi \in O(X,\tau)$, $\mathcal{U} \subseteq P(X), x \in X$. We use the following notations.

$$\begin{aligned}\mathcal{U}(x) &= \{U : x \in U \in \mathcal{U}\},\\ \varphi O(X) &= \{U : U \subseteq X,\ U \text{ is } \varphi\text{-open}\},\\ \varphi C(X) &= \{K : K \subseteq X,\ K \text{ is } \varphi\text{-closed}\},\\ \varphi O(X,x) &= \{U : U \in \varphi O(X),\ x \in U\}.\end{aligned}$$

1.3. Definition. Let $\varphi_1,\ \varphi_2 \in O(X,\tau),\ A \subseteq X$. Then:

(a) $x \in \varphi_{1,2}\text{int}\,A \iff \exists U \in \varphi_1 O(X,x)$ such that $\varphi_2(U) \subseteq A$.

(b) $x \in \varphi_{1,2}\text{cl}\,A \iff \varphi_2(U) \cap A \neq \emptyset\ \forall U \in \varphi_1 O(X,x)$.

(c) $x \in \varphi_1 O(X)\text{-cl}\,A \iff x \in (\varphi_1\text{-}\imath)\text{cl}\,A$ (here $\imath$ is the identity operation).

(d) A is $\varphi_{1,2}$-open $\iff A \subseteq \varphi_{1,2}\text{int}\,A$.

(e) A is $\varphi_{1,2}$-closed $\iff \varphi_{1,2}\text{cl}\,A \subseteq A$.

We note the following:

- If $A \subseteq B$ then $\varphi_{1,2}\text{int}\,A \subseteq \varphi_{1,2}\text{int}\,B$.
- Clearly, for any set A, $X \setminus \varphi_{1,2}\text{int}\,A = \varphi_{1,2}\text{cl}\,(X \setminus A)$ and A is $\varphi_{1,2}$-open iff $X \setminus A$ is $\varphi_{1,2}$-closed.
- $\varphi_{1,2}O(X)$ ($\varphi_{1,2}C(X)$) will stand for the family of all $\varphi_{1,2}$-open subsets (the family of all $\varphi_{1,2}$-closed subsets) of X.
- $\varphi_{1,2}O(X)$ is a supratopology on X [5].

2. Openness

Let $\emptyset \neq \Gamma \subseteq P(X)$ and $f : X \to Y$. We use the following notation.

$$\begin{aligned}f(\Gamma) &= \{f(U) : U \in \Gamma\},\\ \Gamma' &= \{X \setminus U : U \in \Gamma\}.\end{aligned}$$

2.1. Theorem. *Let $\emptyset \neq \Gamma \subseteq P(X)$, $\emptyset \neq \gamma \subseteq P(Y)$ and $f : X \to Y$. Then the following statements are equivalent.*

(a) $f(\Gamma) \subseteq \gamma$.

(b) *For each $B \subseteq Y$ and for each $F \in \Gamma'$ such that $f^{-1}(B) \subseteq F$, there exists a $K \in \gamma'$ such that $B \subseteq K$ and $f^{-1}(K) \subseteq F$.*

[†]A subfamily $\mathcal{U}$ of the power set of a non-empty set X is called a *supratopology on* X if $\emptyset$, $X \in \mathcal{U}$ and $\mathcal{U}$ is closed under arbitrary unions [9].

(c) *For each* $A \in \Gamma'$, $\{y \in Y : f^{-1}(y) \subseteq A\} \in \gamma'$.
(d) *For each* $U \in \Gamma$, $\{y \in Y : f^{-1}(y) \cap U \neq \emptyset\} \in \gamma$.

Proof. (a) $\Longrightarrow$ (b) Let $B \subseteq Y$, $f^{-1}(B) \subseteq F$, $F \in \Gamma'$. Then $X \setminus F \in \Gamma$ so $f(X \setminus F) \in \gamma$. Let $K = Y \setminus f(X \setminus F) \in \gamma'$. Since $Y \setminus f(X \setminus F) = \{y \in Y : f^{-1}(y) \subseteq F\}$ we have $B \subseteq K$ and $f^{-1}(K) \subseteq F$.

(b) $\Longrightarrow$ (c) For $A \in \Gamma'$ let $B = \{y \in Y : f^{-1}(y) \subseteq A\}$. Then $f^{-1}(B) \subseteq A$, and $A \in \Gamma'$, so by (b) there exists a $K \in \gamma'$ such that $B \subseteq K$ and $f^{-1}(K) \subseteq A$. We have $K \subseteq B$ and hence $B = K \in \gamma'$.

(c) $\Longrightarrow$ (d) Let $U \in \Gamma$. Then $X \setminus U \in \Gamma'$, whence $\{y \in Y : f^{-1}(y) \subseteq X \setminus U\} \in \gamma'$. But $Y \setminus f(U) = \{y \in Y : f^{-1}(y) \subseteq X \setminus U\} \in \gamma'$, so $f(U) = \{y \in Y : f^{-1}(y) \cap U \neq \emptyset\} \in \gamma$.

(d) $\Longrightarrow$ (a) Let $U \in \Gamma$. Then $\{y \in Y : f^{-1}(y) \cap U \neq \emptyset\} = f(U) \in \gamma$. Hence $f(\Gamma) \subseteq \gamma$. □

2.2. Lemma. *If* $\Gamma \subseteq P(X)$ *and* $\gamma \subseteq P(Y)$ *are supratopologies and* $\mathcal{B}$ *is a base for the supratopology* Γ, *then* $f(\Gamma) \subseteq \gamma$ *iff* $f(\mathcal{B}) \subseteq \gamma$.

2.3. Remark. If we take $\Gamma = \varphi_{1,2}O(X)$ and $\gamma = \psi_{1,2}O(Y)$ for the operations $\varphi_1, \varphi_2 \in O(X, \tau)$ and $\psi_1, \psi_2 \in O(Y, \vartheta)$ then $\Gamma' = \varphi_{1,2}C(X)$, $\gamma' = \psi_{1,2}C(Y)$ and we obtain from Theorem 2.1 equivalent conditions for a function $f : (X, \tau) \to (Y, \vartheta)$ to satisfy the property $f(\varphi_{1,2}O(X)) \subseteq \psi_{1,2}O(Y)$.

2.4. Theorem. *If* γ' *is a supratopology for* $\gamma \subseteq P(Y)$ *then* $f(\Gamma) \subseteq \gamma$ *iff for each* $y \in Y$ *and for each set* $F \in \Gamma'$ *such that* $f^{-1}(y) \subseteq F$, *there exists a* $K \in \gamma'$ *such that* $y \in K$ *and* $f^{-1}(K) \subseteq F$.

Proof. $\Longrightarrow$. Clear from Theorem 2.1.

$\Longleftarrow$. Let $B \subseteq Y$, $f^{-1}(B) \subseteq F \in \Gamma'$. For each $y \in B$, $f^{-1}(y) \subseteq F \in \Gamma'$. By hypothesis there exists a $K_y \in \gamma'$ such that $y \in K_y$ and $f^{-1}(K_y) \subseteq F$. Then $B \subseteq \bigcup K_y = K \in \gamma'$ and $f^{-1}(K) \subseteq F$. From Theorem 2.1 we obtain $f(\Gamma) \subseteq \gamma$. □

2.5. Lemma. *Let* $\varphi_1, \varphi_2 \in O(X, \tau)$. *For each* $x \in X$ *and for each* $U \in \varphi_1 O(X, x)$, $x \in \varphi_{1,2}\mathrm{int}\, \varphi_2(U)$. *Hence if* $U \in \varphi_1 O(X)$ *then* $U \subseteq \varphi_{1,2}\mathrm{int}\, \varphi_2(U)$.

Proof. If $x \in U \in \varphi_1 O(X)$, then since $\varphi_2(U) \subseteq \varphi_2(U)$ the proof is clear from Definition 1.3 (a). □

The following definition was given in [3] for fuzzy topological spaces.

2.6. Definition. Let $\varphi_1, \varphi_2 \in O(X, \tau)$, $\psi_1, \psi_2 \in O(Y, \vartheta)$ and $f : (X, \tau) \to (Y, \vartheta)$. We say that f is $\varphi_{1,2}\psi_{1,2}$-open if for each $A \subseteq X$ and for each $x \in \varphi_{1,2}\mathrm{int}\, A$ we have $f(x) \in \psi_{1,2}\mathrm{int}\, f(A)$.

From now on we accept that $\varphi_1, \varphi_2 \in O(X, \tau)$, $\psi_1, \psi_2 \in O(Y, \vartheta)$ and that $f : (X, \tau) \to (Y, \vartheta)$.

In the following theorem, almost all of the equivalent statements are given for fuzzy topological spaces in [3].

2.7. Theorem. *The following are equivalent.*

(a) f *is* $\varphi_{1,2}\psi_{1,2}$*-open.*
(b) *For each* $A \subseteq X$ *we have* $f(\varphi_{1,2}\mathrm{int}\, A) \subseteq \psi_{1,2}\mathrm{int}\, f(A)$.
(c) *For each* $B \subseteq Y$ *we have* $f^{-1}(\psi_{1,2}\mathrm{cl}\, B) \subseteq \varphi_{1,2}\mathrm{cl}\, f^{-1}(B)$.
(d) *For each* $B \subseteq Y$ *we have* $\varphi_{1,2}\mathrm{int}\, f^{-1}(B) \subseteq f^{-1}(\psi_{1,2}\mathrm{int}\, B)$.

(e) *For each* $U \in \varphi_1 O(X)$ *we have* $f(U) \subseteq \psi_{1,2}\mathrm{int}\, f(\varphi_2(U))$.

(f) *For each* $x \in X$ *and* $U \in \varphi_1 O(X, x)$, *there exists* $V \in \psi_1 O(Y, f(x))$ *such that* $\psi_2(V) \subseteq f(\varphi_2(U))$.

Proof. (a) $\Longleftrightarrow$ (b). Clear.

(b) $\Longrightarrow$ (c). Let $B \subseteq Y$. Then $f(\varphi_{1,2}\mathrm{int}\,(X \setminus f^{-1}(B))) \subseteq \psi_{1,2}\mathrm{int}\, f(X \setminus f^{-1}(B))$. Since, $\varphi_{1,2}\mathrm{int}\,(X \setminus f^{-1}(B)) = X \setminus \varphi_{1,2}\mathrm{cl}\, f^{-1}(B)$ and

$$\psi_{1,2}\mathrm{int}\, f(X \setminus f^{-1}(B)) = \psi_{1,2}\mathrm{int}\, f(f^{-1}(Y \setminus B)) \subseteq \psi_{1,2}\mathrm{int}\,(Y \setminus B),$$

then

$$\begin{aligned} X \setminus \varphi_{1,2}\mathrm{cl}\, f^{-1}(B) &\subseteq f^{-1}(f(\varphi_{1,2}\mathrm{int}\,(X \setminus f^{-1}(B)))) \\ &\subseteq f^{-1}(\psi_{1,2}\mathrm{int}\,(Y \setminus B)) \\ &= f^{-1}(Y \setminus \psi_{1,2}\mathrm{cl}\, B) \\ &= X \setminus f^{-1}(\psi_{1,2}\mathrm{cl}\, B). \end{aligned}$$

Hence we have $f^{-1}(\psi_{1,2}\mathrm{cl}\, B) \subseteq \varphi_{1,2}\mathrm{cl}\, f^{-1}(B)$.

(c) $\Longrightarrow$ (d). Let $B \subseteq Y$. Then

$$\begin{aligned} \varphi_{1,2}\mathrm{int}\, f^{-1}(B) &= X \setminus \varphi_{1,2}\mathrm{cl}\,(X \setminus f^{-1}(B)) \\ &= X \setminus \varphi_{1,2}\mathrm{cl}\, f^{-1}(Y \setminus B) \\ &\subseteq X \setminus f^{-1}(\psi_{1,2}\mathrm{cl}\,(Y \setminus B)) \\ &= X \setminus f^{-1}(Y \setminus (\psi_{1,2}\mathrm{int}\, B)) \\ &= f^{-1}(\psi_{1,2}\mathrm{int}\, B). \end{aligned}$$

(d) $\Longrightarrow$ (b). Let $A \subseteq X$. Then

$$\varphi_{1,2}\mathrm{int}\, A \subseteq \varphi_{1,2}\mathrm{int}\, f^{-1}(f(A)) \subseteq f^{-1}(\psi_{1,2}\mathrm{int}\, f(A))$$

so

$$f(\varphi_{1,2}\mathrm{int}\, A) \subseteq f(f^{-1}(\psi_{1,2}\mathrm{int}\, f(A))) \subseteq \psi_{1,2}\mathrm{int}\, f(A).$$

(a) $\Longrightarrow$ (e). Let $U \in \varphi_1 O(X)$. Then $U \subseteq \varphi_{1,2}\mathrm{int}\, \varphi_2(U)$, so

$$f(U) \subseteq f(\varphi_{1,2}\mathrm{int}\, \varphi_2(U)) \subseteq \psi_{1,2}\mathrm{int}\, f(\varphi_2(U)).$$

(e) $\Longrightarrow$ (f). Let $x \in X$ and $U \in \varphi_1 O(X, x)$. Since $f(U) \subseteq \psi_{1,2}\mathrm{int}\, f(\varphi_2(U))$, $f(x) \in \psi_{1,2}\mathrm{int}\, f(\varphi_2(U))$. Hence there exists $V \in \psi_1 O(Y, f(x))$ such that $\psi_2(V) \subseteq f(\varphi_2(U))$.

(f) $\Longrightarrow$ (a). Let $A \subseteq X$ and $x \in \varphi_{1,2}\mathrm{int}\, A$. There exists $U \in \varphi_1 O(X, x)$ such that $\varphi_2(U) \subseteq A$. There exists $V \in \psi_1 O(Y, f(x))$ such that $\psi_2(V) \subseteq f(\varphi_2(U)) \subseteq f(A)$. So $f(x) \in \psi_{1,2}\mathrm{int}\, f(A)$. □

2.8. Theorem. *If f is 1-1 and onto then the following are equivalent.*

(a) f *is* $\varphi_{1,2}\psi_{1,2}$*-open.*

(b) *For each* $A \subseteq X$, $\psi_{1,2}\mathrm{cl}\, f(A) \subseteq f(\varphi_{1,2}\mathrm{cl}\, A)$.

Proof. (a) $\Longrightarrow$ (b). Let f be $\varphi_{1,2}\psi_{1,2}$-open and $A \subseteq X$. Then for $f(A) \subseteq Y$ we have $f^{-1}(\psi_{1,2}\mathrm{cl}\, f(A)) \subseteq \varphi_{1,2}\mathrm{cl}\, f^{-1}(f(A)) = \varphi_{1,2}\mathrm{cl}\, A$, so $\psi_{1,2}\mathrm{cl}\, f(A) \subseteq f(\varphi_{1,2}\mathrm{cl}\, A)$.

(b) $\Longrightarrow$ (a). Let $B \subseteq Y$. Then

$$\psi_{1,2}\mathrm{cl}\, B = \psi_{1,2}\mathrm{cl}\, f(f^{-1}(B)) \subseteq f(\varphi_{1,2}\mathrm{cl}\, f^{-1}(B))$$

so

$$f^{-1}(\psi_{1,2}\mathrm{cl}\, B) \subseteq \varphi_{1,2}\mathrm{cl}\, f^{-1}(B).$$

The result is now clear from Theorem 2.7. □

The following theorem given in [3] clearly also holds for topological spaces.

2.9. Theorem. *If f is $\varphi_{1,2}\psi_{1,2}$-open then $f(\varphi_{1,2}O(X)) \subseteq \psi_{1,2}O(Y)$.*

2.10. Theorem. *If ($\varphi_2 \geq \varphi_1$ or $\varphi_2 \geq \imath$) and $\varphi_{1,2}\mathrm{int}\, A \subseteq \varphi_{1,2}\mathrm{int}\, (\varphi_{1,2}\mathrm{int}\, A)$ for each $A \subseteq X$, then f is $\varphi_{1,2}\psi_{1,2}$-open iff $f(\varphi_{1,2}O(X)) \subseteq \psi_{1,2}O(Y)$.*

Proof. $\Longleftarrow$. Let $f(\varphi_{1,2}O(X)) \subseteq \psi_{1,2}O(Y)$ and $A \subseteq X$. By the hypotheses, $\varphi_{1,2}\mathrm{int}\, A \in \varphi_{1,2}O(X)$, $f(\varphi_{1,2}\mathrm{int}\, A) \in \psi_{1,2}O(Y)$ and $\varphi_{1,2}\mathrm{int}\, A \subseteq A$. Hence,

$$f(\varphi_{1,2}\mathrm{int}\, A) \subseteq \psi_{1,2}\mathrm{int}\, f(\varphi_{1,2}\mathrm{int}\, A) \subseteq \psi_{1,2}\mathrm{int}\, f(A).$$

Now from Theorem 2.7, f is $\varphi_{1,2}\psi_{1,2}$-open. □

2.11. Lemma.

(a) *If $\varphi_2(U) \in \varphi_{1,2}O(X)$ $\forall U \in \varphi_1 O(X)$ then $\varphi_{1,2}\mathrm{int}\, A \subseteq \varphi_{1,2}\mathrm{int}\, (\varphi_{1,2}\mathrm{int}\, A)$ for each $A \subseteq X$.*

(b) *If $\varphi_2 \leq \imath$, then $\varphi_2(U) \in \varphi_{1,2}O(X)$ for each $U \in \varphi_1 O(X)$.*

(c) *If $\varphi_2 = \imath$, then the conditions of Theorem 2.10 are satisfied.*

(d) *If $\varphi_2(U) \in \varphi_1 O(X)$ and $\varphi_2(\varphi_2(U)) \subseteq \varphi_2(U)$ for each $U \in \varphi_1 O(X)$, then $\varphi_2(U) \in \varphi_{1,2}O(X)$ for each $U \in \varphi_1 O(X)$* [11, Lemma 2.6].

(e) *If ($\varphi_2 \geq \varphi_1$ or $\varphi_2 \geq \imath$) and $\varphi_2(U) \in \varphi_{1,2}O(X)$ for each $U \in \varphi_1 O(X)$, then $\mathcal{B} = \{\varphi_2(U) : U \in \varphi_1 O(X)\}$ is a base for the supratopology $\varphi_{1,2}O(X)$* [11, Theorem 2.7].

Proof. (a) Let $A \subseteq X$ and $x \in \varphi_{1,2}\mathrm{int}\, A$. Then there exists $U \in \varphi_1 O(X, x)$ such that $\varphi_2(U) \subseteq A$. Hence, $\varphi_2(U) \subseteq \varphi_{1,2}\mathrm{int}\, \varphi_2(U) \subseteq \varphi_{1,2}\mathrm{int}\, A$, so $x \in \varphi_{1,2}\mathrm{int}\, (\varphi_{1,2}\mathrm{int}\, A)$. □

2.12. Remark. If $\varphi_1, \varphi_2 \in O(X, \tau)$ satisfy the conditions of Theorem 2.10, then we get equivalent conditions for a function $f : (X, \tau) \to (Y, \vartheta)$ to be $\varphi_{1,2}\psi_{1,2}$-open by using Remark 2.3 and Theorems 2.1, 2.7 and 2.10.

For a topological space (X, τ), $RO(X)$ ($SO(X)$) will stand for the family of regular open sets (semi-open sets) of X, respectively.

2.13. Example. Let $\varphi_1 = \mathrm{int}$, $\varphi_2 = \mathrm{int} \circ \mathrm{cl}$, $\psi_1 = \mathrm{int}$ and $\psi_2 = \mathrm{cl}$. Then:

$\varphi_1 O(X) = \tau$,

$x \in \varphi_{1,2}\mathrm{int}\, A \iff x \in \delta\text{-int}\, A$,

$\varphi_{1,2}O(X) = \tau_s =$ the topology generated by the base $RO(X)$.

Clearly $\varphi_2 \geq \varphi_1$. For each $U \in \varphi_1 O(X) = \tau$ we have $\varphi_2(U) = \overline{U}^o \in \varphi_1 O(X)$ and $\varphi_2(\varphi_2(U)) = \varphi_2(U)$. Also,

$\mathcal{B} = \{\varphi_2(U) : U \in \varphi_1 O(X)\} = RO(X)$.

$\psi_1 O(Y) = \vartheta$,

$x \in \psi_{1,2}\mathrm{int}\, B \iff x \in \Theta\text{-int}B$.

$\psi_{1,2}O(Y) = \vartheta_\Theta =$ the topology of Θ-open sets.

Finally, f is $\varphi_{1,2}\psi_{1,2}$-open iff for each $A \subseteq X$ and for each $x \in \delta\text{-int}A$ we have $f(x) \in \Theta\text{-int}\, f(A)$.

Now, using Lemma 2.11, Theorem 2.10, Theorem 2.7, Remark 2.3, Lemma 2.2 and Theorem 2.1, we get the following theorem.

2.14. Theorem. *Let $f : (X,\tau) \to (Y,\vartheta)$. The following are equivalent.*

(a) $f(\tau_s) \subseteq \vartheta_\Theta$.

(b) *For each $A \subseteq X$ and for each $x \in \delta\text{-int}\, A$, $f(x) \in \Theta\text{-int}\, f(A)$.*

(c) $f(RO(X)) \subseteq \vartheta_\Theta$.

(d) *For each $B \subseteq Y$ and for each τ_s-closed set (i.e. δ-closed set) such that $f^{-1}(B) \subseteq F$, there exists a Θ-closed set K with $B \subseteq K$ and $f^{-1}(K) \subseteq F$.*

(e) *For each $B \subseteq Y$ and for each regular-closed set F such that $f^{-1}(B) \subseteq F$, there exists a Θ-closed set K such that $B \subseteq K$ and $f^{-1}(K) \subseteq F$.*

(f) *For each δ-closed set A in X, $\{y \in Y : f^{-1}(y) \subseteq A\}$ is Θ-closed.*

(g) *For each regular-closed set A in X, $\{y \in Y : f^{-1}(y) \subseteq A\}$ is Θ-closed.*

(h) *For each δ-open set U in X, $\{y \in Y : f^{-1}(y) \cap U \neq \emptyset\}$ is Θ-open.*

(i) *For each regular open set U in X, $\{y \in Y : f^{-1}(y) \cap U \neq \emptyset\}$ is Θ-open.*

(j) *For each $A \subseteq X$, $f(\delta\text{-int}\, A) \subseteq \Theta\text{-int}\, f(A)$.*

(k) *For each $B \subseteq Y$, $f^{-1}(\Theta\text{-cl}\, B) \subseteq \delta\text{-cl}\, f^{-1}(B)$.*

(l) *For each $B \subseteq Y$, $\delta\text{-int}\, f^{-1}(B) \subseteq f^{-1}(\Theta\text{-int}\, B)$.*

(m) *For each $U \in \tau$, $f(U) \subseteq \Theta\text{-int}\, f(\overline{U}^o)$.*

(n) *For each $x \in X$ and for each $U \in \tau(x)$, there exists a $V \in \vartheta(f(x))$ such that $\overline{V} \subseteq f(\overline{U}^o)$.*

2.15. Example. Let $\varphi_1 = \text{cl} \circ \text{int}$ and $\varphi_2 = \text{scl}$. Then:

$\varphi_1 O(X) = SO(X)$ and $\varphi_2 \geq \imath$.

For $U \in \varphi_1 O(X) = SO(X)$, $\varphi_2(U) = \text{scl}\, U \in SO(X)$ and $\varphi_2(\varphi_2(U)) = \varphi_2(U)$,

$x \in \varphi_{1,2}\text{int}\, A \iff x \in \text{semi-}\Theta\text{-int}\, A$.

$U \in \varphi_{1,2} O(X) \iff U$ is semi-Θ-open.

$\mathcal{B} = \{\varphi_2(U) : U \in \varphi_1 O(X)\} = SR(X) =$ the family of semi-regular sets.

$SR(X)$ is a base for the supratopology $\varphi_{1,2} O(X) = S\Theta O(X)$.

Now, by using the same Lemma, Remarks and Theorems as in Example 2.13, we could obtain a similar theorem for $f : (X,\tau) \to (Y,\vartheta)$ to be $\varphi_{1,2}\psi_{1,2}$-open for any operations $\psi_1, \psi_2 \in O(Y,\vartheta)$.

2.16. Remark. Let φ_1 and ψ_1 be monotonous. Then $\Gamma = \varphi_1 O(X)$ and $\gamma = \psi_1 O(Y)$ are supratopologies.

If we take $\varphi_2 = \imath$, $\psi_2 = \imath$, we get $\varphi_{1,2} O(X) = \varphi_1 O(X)$ and $\psi_{1,2} O(Y) = \psi_1 O(Y)$.

In this case we can say, $(\varphi_1\text{-}\imath)\text{cl}\, A = \varphi_1 O(X)\text{-cl}\, A$, $(\psi_1\text{-}\imath)\text{cl}\, B = \psi_1 O(Y)\text{-cl}\, B$, $\varphi_{1,2}\text{int}\, A = \varphi_1 O(X)\text{-int}\, A$, $\psi_{1,2}\text{int}\, B = \psi_1 O(Y)\text{-int}\, B$ for $A \subseteq X$, $B \subseteq Y$.

F is φ_1-closed iff $\varphi_1 O(X)\text{-cl}\, F \subseteq F$ and K is ψ_1-closed iff $\psi_1 O(Y)\text{-cl}\, K \subseteq K$.

f is $\varphi_{1,2}\psi_{1,2}$-open iff $f(\varphi_{1,2} O(X)) \subseteq \psi_{1,2} O(Y)$ iff $f(\varphi_1 O(X)) \subseteq \psi_1 O(Y)$.

Hence we obtain the following theorem.

2.17. Theorem. *For monotonous operations φ_1, ψ_1, and $f : (X,\tau) \to (Y,\vartheta)$, the following are equivalent.*

(a) *For each $U \in \varphi_1 O(X)$, $f(U) \in \psi_1 O(Y)$.*

(b) *For each $B \subseteq Y$ and for each φ_1-closed set F such that $f^{-1}(B) \subseteq F$, there exists a ψ_1-closed set K such that $B \subseteq K$ and $f^{-1}(K) \subseteq F$.*

(c) *For each φ_1-closed set A, the set $\{y \in Y : f^{-1}(y) \subseteq A\}$ is ψ_1-closed.*

(d) *For each φ_1-open set U, the set $\{y \in Y : f^{-1}(y) \cap U \neq \emptyset\}$ is ψ_1-open.*

(e) *For each* $A \subseteq X$, $f^{-1}(\varphi_1 O(X)\text{-int}\, A) \subseteq \psi_1 O(Y)\text{-int}\, f(A)$.

(f) *For each* $B \subseteq Y$, $f^{-1}(\psi_1 O(Y)\text{-cl}\, B) \subseteq \varphi_1 O(X)\text{-cl}\, f^{-1}(B)$.

(g) *For each* $B \subseteq Y$, $\varphi_1 O(X)\text{-int}\, f^{-1}(B)) \subseteq f^{-1}(\psi_1 O(Y)\text{-int}\, B)$.

(h) *For each* $U \in \varphi_1 O(X)$, $f(U) \subseteq \psi_1 O(Y)\text{-int}\, f(U)$.

(i) *For each* $x \in X$ *and* $U \in \varphi_1 O(X, x)$, *there exists* $V \in \psi_1 O(Y, f(x))$ *such that* $V \subseteq f(U)$.

2.18. Example. (a) Let $\varphi_1 = \text{int}$, $\varphi_2 = \imath$, $\psi_1 = \text{int}$ and $\psi_2 = \imath$. Then:

$\varphi_1 O(X)\text{-cl}\, A = \overline{A}$, $\varphi_1 O(X)\text{-int}\, A = A^o$ and $\psi_1 O(Y)\text{-cl}\, B = \overline{B}$, $\psi_1 O(Y)\text{-int}\, B = B^o$.

Hence, F is φ_1-closed $\iff$ F is τ-closed and K is ψ_1-closed $\iff$ K is ϑ-closed. Using the above theorem one could obtain equivalent conditions for a function to be open.

(b) Let $\varphi_1 = \text{int}$, $\varphi_2 = \imath$, $\psi_1 = \text{cl} \circ \text{int}$ and $\psi_2 = \imath$. Then:

$\varphi_1 O(X)\text{-int}\, A = A^o$, $\varphi_1 O(X)\text{-cl}\, A = \overline{A}$, and

$\psi_1 O(Y)\text{-int}\, B = \text{semi-int}\, B$, $\psi_1 O(Y)\text{-cl}\, B = \text{scl}\, B$.

Hence, F is φ_1-closed $\iff$ F is τ-closed, and K is ψ_1-closed $\iff$ K is semi-closed. With the aid of Theorem 2.17 one can then obtain equivalent conditions for a function to be semi-open.

The following theorem is given in [3].

2.19. Theorem. *If for the operations* $\varphi_1, \varphi_2, \varphi_3, \varphi_4 \in O(X, \tau)$ *and* $\psi_1, \psi_2, \psi_3, \psi_4 \in O(Y, \vartheta)$ *satisfying* $\varphi_3 \leq \varphi_1$, $\varphi_2 \leq \varphi_4$ *and* $\psi_1 \leq \psi_3$, $\psi_4 \leq \psi_2$, *each* $\varphi_{1,2}\psi_{1,2}$*-open function is* $\varphi_{3,4}\psi_{3,4}$*-open.*

3. Closedness

The following definition was given in [3] for fuzzy topological spaces.

3.1. Definition. Let $\varphi_1, \varphi_2 \in O(X, \tau)$, $\psi_1, \psi_2 \in O(Y, \vartheta)$ and $f : (X, \tau) \to (Y, \vartheta)$. Then f is said to be $\varphi_{1,2}\psi_{1,2}$-closed if $f(K)$ is $\psi_{1,2}$-closed for each $\varphi_{1,2}$-closed set K. That is, f is $\varphi_{1,2}\psi_{1,2}$-closed iff $f(\varphi_{1,2}C(X)) \subseteq \psi_{1,2}C(Y)$.

3.2. Theorem. *The following are equivalent.*

(a) f *is* $\varphi_{1,2}\psi_{1,2}$*-closed.*

(b) *For each* $B \subseteq Y$ *and for each* $\varphi_{1,2}$*-open set* U *such that* $f^{-1}(B) \subseteq U$, *there exists a* $\psi_{1,2}$*-open set* V *such that* $B \subseteq V$ *and* $f^{-1}(V) \subseteq U$.

(c) *For each* $\varphi_{1,2}$*-open set* U *in* X, *the set* $\{y \in Y : f^{-1}(y) \subseteq U\}$ *is* $\psi_{1,2}$*-open.*

(d) *For each* $\varphi_{1,2}$*-closed set* K *in* X, *the set* $\{y \in Y : f^{-1}(y) \cap K \neq \emptyset\}$ *is* $\psi_{1,2}$*-closed.*

(e) *For each* $y \in Y$ *and for each* $\varphi_{1,2}$*-open set* U *such that* $f^{-1}(y) \subseteq U$, *there exists a* $V \in \psi_{1,2}O(Y, y)$ *such that* $f^{-1}(V) \subseteq U$.

(f) *For each* $y \in Y$ *and* $\varphi_{1,2}$*-open set* U *such that* $f^{-1}(y) \subseteq U$, *there exists* $V \in \psi_1 O(Y, y)$ *such that* $f^{-1}(\psi_2(V)) \subseteq U$.

Proof. The equivalence of (a)-(e) is clear from Theorem 2.1 and Theorem 2.4.

(e) $\Longrightarrow$ (f). Take $y \in Y$ and $f^{-1}(y) \subseteq U \in \varphi_{1,2}O(X)$. Then there exists $W \in \psi_{1,2}O(Y, y)$ such that $f^{-1}(W) \subseteq U$. Since $y \in W \subseteq \psi_{1,2}\text{int}\, W$, there exists $V \in \psi_1 O(Y, y)$ such that $\psi_2(V) \subseteq W$ and $f^{-1}(\psi_2(V)) \subseteq f^{-1}(W) \subseteq U$.

(f) $\Longrightarrow$ (a). Let K be a $\varphi_{1,2}$-closed set. We show that $\psi_{1,2}\text{cl}\, f(K) \subseteq f(K)$.

Take $y \in \psi_{1,2}\text{cl}\, f(K)$. For each $W \in \psi_1 O(Y, y)$, $\psi_2(W) \cap f(K) \neq \emptyset$. Suppose that $y \notin f(K)$. Then $f^{-1}(y) \subseteq X \setminus K$ and $U = X \setminus K$ is a $\varphi_{1,2}$-open set. Hence there exists $V \in \psi_1 O(Y,y)$ such that $f^{-1}(\psi_2(V)) \subseteq U$. We have $f^{-1}(\psi_2(V)) \cap K = \emptyset$, so $f(K) \cap \psi_2(V) = \emptyset$ which is a contradiction. Hence, $y \in \psi_{1,2}\mathrm{cl}\, f(K) \implies y \in f(K)$, so f is $\varphi_{1,2}\psi_{1,2}$-closed. □

3.3. Corollary. *If for each $B \subseteq Y$ and for each $\varphi_{1,2}$-open set U such that $f^{-1}(B) \subseteq U$, there exists $V \in \psi_1 O(Y)$ such that $B \subseteq V$ and $f^{-1}(\psi_2(V)) \subseteq U$, then f is $\varphi_{1,2}\psi_{1,2}$-closed.*

3.4. Theorem. *If $\varphi_2 \geq \varphi_1$ or $\varphi_2 \geq \imath$, and if $\varphi_2(U) \in \varphi_{1,2}O(X)$ for each $U \in \varphi_1 O(X)$, then f is $\varphi_{1,2}\psi_{1,2}$-closed iff $\psi_{1,2}\mathrm{cl}\, f(A) \subseteq f(\varphi_{1,2}\mathrm{cl}\, A)$ for all $A \in P(X)$.*

Proof. Under the given conditions we have $A \subseteq \varphi_{1,2}\mathrm{cl}\, A$, $\varphi_{1,2}\mathrm{cl}\, (\varphi_{1,2}\mathrm{cl}\, A) \subseteq \varphi_{1,2}\mathrm{cl}\, A$ and $\varphi_{1,2}\mathrm{cl}\, A \in \varphi_{1,2}C(X)$ for any $A \in P(X)$.

$\implies$. Let f be $\varphi_{1,2}\psi_{1,2}$-closed and $A \subseteq X$. Since $\varphi_{1,2}\mathrm{cl}\, A \in \varphi_{1,2}C(X)$, $f(\varphi_{1,2}\mathrm{cl}\, A) \in \psi_{1,2}C(Y)$ and so, $\psi_{1,2}\mathrm{cl}\, f(\varphi_{1,2}\mathrm{cl}\, A) \subseteq f(\varphi_{1,2}\mathrm{cl}\, A)$. Also, as $A \subseteq \varphi_{1,2}\mathrm{cl}\, A$ we have $f(A) \subseteq f(\varphi_{1,2}\mathrm{cl}\, A)$. Hence $\psi_{1,2}\mathrm{cl}\, f(A) \subseteq \psi_{1,2}\mathrm{cl}\, f(\varphi_{1,2}\mathrm{cl}\, A) \subseteq f(\varphi_{1,2}\mathrm{cl}\, A)$.

$\impliedby$. Let us prove that $f(A) \in \psi_{1,2}C(Y)$ for any $A \in \varphi_{1,2}C(X)$. Since $A \in \varphi_{1,2}C(X)$ we have $\varphi_{1,2}\mathrm{cl}\, A \subseteq A$. From the hypotheses, $\psi_{1,2}\mathrm{cl}\, f(A) \subseteq f(\varphi_{1,2}\mathrm{cl}\, A)$. Hence $\psi_{1,2}\mathrm{cl}\, f(A) \subseteq f(A)$, so $f(A) \in \psi_{1,2}C(Y)$ as required. □

3.5. Definition. (a) Let $\varphi \in O(X,\tau)$, $X \in \mathcal{A} \subseteq P(X)$ and $A \in P(X)$. If whenever $A \subseteq \bigcup \mathcal{U}$, for $\mathcal{U} \subseteq \mathcal{A}$, there exists $U_1, U_2, \dots, U_n \in \mathcal{U}$ such that $A \subseteq \bigcup_{i=1}^{n} \varphi(U_i)$, then A is called $(\mathcal{A}\text{-}\varphi)$-*compact relative to* X (for short, an $(\mathcal{A}\text{-}\varphi)$-*compact set*).

A will be called an $\mathcal{A}$-*compact set* if A is a $(\mathcal{A}\text{-}\imath)$-compact set.

(b) A subset A of X is called an $\varphi_{1,2}$-*compact set* if A is an $(\varphi_1 O(X)\text{-}\varphi_2)$-compact set (for $\varphi_1, \varphi_2 \in O(X,\tau)$).

Each $\varphi_{1,2}$-compact set is an $\varphi_{1,2}O(X)$-compact set [13].

3.6. Remark. (a) In the case where φ_1 is monotonous and $\varphi_2 = \imath$, we have $\varphi_1 O(X) = \varphi_{1,2}O(X)$, so A is φ_1-closed $\iff$ A is $\varphi_{1,2}$-closed. Hence:

f is $\varphi_{1,2}\psi_{1,2}$-closed iff $f(\varphi_1\text{-closed}(X)) \subseteq \psi_{1,2}C(Y)$.

A is a $\varphi_{1,2}$-compact set iff A is a $(\varphi_1 O(X)\text{-}\imath)$-compact set iff A is a $\varphi_1 O(X)$-compact set.

(b) If ψ_1 is monotonous and $\psi_2 = \imath$, then we have similar properties to those in (a). Specifically,

f is $\varphi_{1,2}\psi_{1,2}$-closed iff $f(\varphi_{1,2}C(X)) \subseteq \psi_1\text{-closed}(Y)$.

B is an $\psi_{1,2}$-compact set iff B is an $(\psi_1 O(Y)\text{-}\imath)$-compact set iff B is an $\psi_1 O(Y)$-compact set.

(c) If φ_1, ψ_1 are monotonous and $\varphi_2 = \imath$, $\psi_2 = \imath$, then $\varphi_{1,2}O(X) = \varphi_1 O(X)$, $\varphi_{1,2}C(X) = \varphi_1\text{-closed}(X) = \varphi_1 C(X)$ and $\psi_{1,2}O(Y) = \psi_1 O(Y)$, $\psi_{1,2}C(Y) = \psi_1\text{-closed}(Y) = \psi_1 C(Y)$. Hence, f is $\varphi_{1,2}\psi_{1,2}$-closed iff $f(\varphi_1 C(X)) \subseteq \psi_1 C(Y)$.

3.7. Theorem. *Let $\varphi_2(U) \in \varphi_{1,2}O(X)$ for each $U \in \varphi_1 O(X)$. Then*

$$\{f(X \setminus \varphi_2(U)) : U \in \varphi_1 O(X)\} \subseteq \psi_{1,2}C(Y)$$

if and only if for each $y \in Y$ and for each $U \in \varphi_1 O(X)$ such that $f^{-1}(y) \subseteq \varphi_2(U)$, there exists $V \in \psi_1 O(Y,y)$ such that $f^{-1}(\psi_2(V)) \subseteq \varphi_2(U)$.

Proof. $\Longrightarrow$. Let $\beta = \{\varphi_2(U) : U \in \varphi_1 O(X)\}$ and $\gamma = \psi_{1,2}C(Y)$. Now, $\gamma' = \psi_{1,2}O(Y)$ is a supratopology and $f(\beta') \subseteq \gamma$. If $y \in Y$, $f^{-1}(y) \subseteq \varphi_2(U)$ and $U \in \varphi_1 O(X)$, so that from Theorem 2.4 there exists $W \in \psi_{1,2}O(Y, y)$ such that $f^{-1}(W) \subseteq \varphi_2(U)$, then there exists $V \in \psi_1 O(Y, y)$ such that $\psi_2(V) \subseteq W$, whence $f^{-1}(\psi_2(V)) \subseteq \varphi_2(U)$.

$\Longleftarrow$. We show that for each $U \in \varphi_1 O(X)$ we have $\psi_{1,2}\mathrm{cl}\, f(X \setminus \varphi_2(U)) \subseteq f(X \setminus \varphi_2(U))$. Assume that $y \in \psi_{1,2}\mathrm{cl}\, f(X \setminus \varphi_2(U))$ but $y \notin f(X \setminus \varphi_2(U))$. Then for each $W \in \psi_1 O(Y, y)$, $\psi_2(W) \cap f(X \setminus \varphi_2(U)) \neq \emptyset$. Since $y \notin f(X \setminus \varphi_2(U))$ we have $f^{-1}(y) \cap (X \setminus \varphi_2(U)) = \emptyset$, so $f^{-1}(y) \subseteq \varphi_2(U)$. From the hypothesis there exists $V \in \varphi_1 O(Y, y)$ such that $f^{-1}(\psi_2(V)) \subseteq \varphi_2(U)$. Hence, $f^{-1}(\psi_2(V)) \cap (X \setminus \varphi_2(U)) = \emptyset$. This contradiction gives the result. □

3.8. Theorem. *If ($\varphi_2 \geq \varphi_1$ or $\varphi_2 \geq \imath$) and for each $U \in \varphi_1 O(X)$, $\varphi_2(U) \in \varphi_1 O(X)$ and $\varphi_2(\varphi_2(U)) \subseteq \varphi_2(U)$, then the following are equivalent:*

(a) $\{f(X \setminus \varphi_2(U)) : U \in \varphi_1 O(X)\} \subseteq \psi_{1,2}\text{-}C(Y)$.

(b) *For each $y \in Y$ and for each φ_1-open set U such that $f^{-1}(y) \subseteq \varphi_2(U)$, there exists $V \in \psi_{1,2}O(Y, y)$ such that $f^{-1}(V) \subseteq \varphi_2(U)$.*

(c) *For each $y \in Y$ and for each φ_1-open set U such that $f^{-1}(y) \subseteq U$, there exists $V \in \psi_{1,2}O(Y, y)$ such that $f^{-1}(V) \subseteq \varphi_2(U)$.*

(d) *For each $y \in Y$ and for each φ_1-open set U such that $f^{-1}(y) \subseteq U$, there exists $V \in \psi_1 O(Y, y)$ such that $f^{-1}(\psi_2(V)) \subseteq \varphi_2(U)$.*

(e) *For each $y \in Y$ and for each φ_1-open set U such that $f^{-1}(y) \subseteq \varphi_2(U)$, there exists $V \in \psi_1 O(Y, y)$ such that $f^{-1}(\psi_2(V)) \subseteq \varphi_2(U)$.*

Proof. (a) $\Longrightarrow$ (b). For $\mathcal{B} = \{\varphi_2(U) : U \in \varphi_1 O(X)\}$ we have $\mathcal{B}' = \{X \setminus \varphi_2(U) : U \in \varphi_1 O(X)\}$. Hence $f(\mathcal{B}') \subseteq \varphi_{1,2}C(Y)$. If $y \in Y$ and $f^{-1}(y) \subseteq \varphi_2(U)$ with $U \in \varphi_1 O(X)$, then $f^{-1}(y) \subseteq \varphi_2(U) \in \mathcal{B}$. From Theorem 2.1 there exists $V \in \psi_{1,2}O(Y, y)$ such that $f^{-1}(V) \subseteq \varphi_2(U)$.

(b) $\Longrightarrow$ (c). Let $y \in Y$, $f^{-1}(y) \subseteq U \in \varphi_1 O(X)$. Then $f^{-1}(y) \subseteq \varphi_2(U)$, from which the result is clear.

(c) $\Longrightarrow$ (d). Let $y \in Y$, $f^{-1}(y) \subseteq U \in \varphi_1 O(X)$. Then there exists $W \in \psi_{1,2}O(Y, y)$ such that $f^{-1}(W) \subseteq \varphi_2(U)$. There exists $V \in \psi_1 O(Y, y)$ such that $\psi_2(V) \subseteq W$, so $f^{-1}(\psi_2(V)) \subseteq \varphi_2(U)$.

(d) $\Longrightarrow$ (e). Since $\varphi_2(U) \in \varphi_1 O(X)$ and $\varphi_2(\varphi_2(U)) \subseteq \varphi_2(U)$ for $U \in \varphi_1 O(X)$, the proof is clear.

(e) $\Longrightarrow$ (a). Clear from Lemma 2.11 and Theorem 3.7. □

3.9. Remark. Since $\{X \setminus \varphi_2(U) : U \in \varphi_1 O(X)\} \subseteq \varphi_{1,2}\text{-}C(X)$ when $\varphi_2(U) \in \varphi_{1,2}O(X)$ for each $U \in \varphi_1 O(X)$, if f is $\varphi_{1,2}\psi_{1,2}$-closed then $\{f(X \setminus \varphi_2(U)) : U \in \varphi_1 O(X)\} \subseteq \psi_{1,2}\text{-}C(Y)$.

3.10. Example. Let $\varphi_1 = \mathrm{int}$, $\varphi_2 = \mathrm{int} \circ \mathrm{cl}$, $\psi_1 = cl \circ \mathrm{int}$ and $\psi_2 = \imath$. Then:

F is $\psi_{1,2}$-closed iff F is semi-cosed, so $\psi_{1,2}O(Y) = \psi_1 O(Y) = SO(Y)$. As can be seen in Example 2.15,

$\mathcal{B} = \{\varphi_2(U) : U \in \varphi_1 O(X)\} = RO(X)$ is a base for the topology τ_s on X.

Now, using Theorems 2.1, 2.4 and 3.8 we obtain the following theorem.

3.11. Theorem. *Let $f : (X, \tau) \to (Y, \vartheta)$. The following are equivalent.*

(a) *$f(F)$ is semi-closed for each regular closed set F.*

(b) *For each* $y \in Y$ *and for each open set* U *such that* $f^{-1}(y) \subseteq \overline{U}^o$*, there exists a* $V \in SO(Y,y)$ *such that* $f^{-1}(V) \subseteq \overline{U}^o$.

(c) *For each* $y \in Y$ *and for each open set* U *such that* $f^{-1}(y) \subseteq U$*, there exists* $V \in SO(Y,y)$ *such that* $f^{-1}(V) \subseteq \overline{U}^o$.

(d) *For each* $y \in Y$ *and for each regular-open set* U *such that* $f^{-1}(y) \subseteq U$*, there exists a* $V \in SO(Y,y)$ *such that* $f^{-1}(V) \subseteq U$.

(e) *For each* $B \subseteq Y$ *and for each regular-open set* U *such that* $f^{-1}(B) \subseteq U$*, there exists a semi-open set* V *such that* $B \subseteq V$ *and* $f^{-1}(V) \subseteq U$.

(f) *For each* $U \in RO(X)$*, the set* $\{y \in Y : f^{-1}(y) \subseteq U\}$ *is semi-open.*

(g) *For each regular-closed set* F *in* X*, the set* $\{y \in Y : f^{-1}(y) \cap F \neq \emptyset\}$ *is semi-closed.*

In the following theorem, (1) and (2) were given in [3] for fuzzy topological spaces.

3.12. Theorem.

(1) *For* $\varphi_1, \varphi_2, \varphi_3, \varphi_4 \in O(X,\tau)$ *and* $\psi_1, \psi_2, \psi_3, \psi_4 \in O(Y,\vartheta)$*, if* $\varphi_3 \leq \varphi_1, \varphi_2 \leq \varphi_4$ *and* $\psi_1 \leq \psi_3$*,* $\psi_4 \leq \psi_2$*, then each* $\varphi_{1,2}\psi_{1,2}$*-closed function is* $\varphi_{3,4}\psi_{3,4}$*-closed.*

(2) *If* f *is 1-1 and onto, then* f *is* $\varphi_{1,2}\psi_{1,2}$*-closed iff* $f(\varphi_{1,2}O(X)) \subseteq \psi_{1,2}O(Y)$.

(3) *If* f *is 1-1 and onto,* ($\varphi_2 \geq \varphi_1$ *or* $\varphi_2 \geq \imath$) *and* $\varphi_{1,2}\mathrm{int}\, A$ *is* $\varphi_{1,2}$*-open for each* $A \subseteq X$*, then* f *is* $\varphi_{1,2}\psi_{1,2}$*-closed iff* $f(\varphi_{1,2}O(X)) \subseteq \psi_{1,2}O(Y)$ *iff* f *is* $\varphi_{1,2}\psi_{1,2}$*-open.*

3.13. Theorem. *If* f *is* $\varphi_{1,2}\psi_{1,2}$*-closed and* $f^{-1}(y)$ *is a* $\varphi_{1,2}O(X)$*-compact set for each* $y \in Y$*, then* $f^{-1}(K)$ *is a* $\varphi_{1,2}O(X)$*-compact set for each* $\psi_{1,2}O(Y)$*-compact set* K.

Proof. Let K be a $\psi_{1,2}O(Y)$-compact set, $f^{-1}(K) \subseteq \bigcup \mathcal{U}$, and $\mathcal{U} \subseteq \varphi_{1,2}O(X)$. For each $y \in K$ we have $f^{-1}(y) \subseteq \bigcup \mathcal{U}$. Hence, for each $y \in K$, there exists $U_1, \ldots, U_{n_y} \in \mathcal{U}$ such that $f^{-1}(y) \subseteq \bigcup_{i=1}^{n_y} U_i = U_y \in \varphi_{1,2}O(X)$. Since f is $\varphi_{1,2}\psi_{1,2}$-closed, there exists a $V_y \in \psi_{1,2}O(Y,y)$ such that $f^{-1}(V_y) \subseteq U_y$. Since $K \subseteq \bigcup V_y$, and K is a $\psi_{1,2}O(Y)$-compact set, there exists $y_1, \ldots, y_m \in K$ such that $K \subseteq \bigcup_{j=1}^{m} V_{y_j}$. Hence

$$f^{-1}(K) \subseteq f^{-1}(\bigcup_{j=1}^{m} V_{y_j}) = \bigcup_{j=1}^{m} f^{-1}(V_{y_j}) \subseteq \bigcup_{j=1}^{m} U_{y_j} = \bigcup_{j=1}^{m} (\bigcup_{i=1}^{n_{y_j}} U_i).$$

So, $f^{-1}(K)$ is a $\varphi_{1,2}O(X)$-compact set. □

3.14. Example. Let $\varphi_1 = \mathrm{int}$, $\varphi_2 = \imath$, $\psi_1 = \mathrm{int}$, and $\psi_2 = \imath$. Then:

f is $\varphi_{1,2}\psi_{1,2}$-closed iff $f(F)$ is closed for each closed set F iff f is closed.

We obtain the following well known result:

If f is closed and has compact point inverses, then the inverse image of each compact set is compact.

3.15. Theorem. *Let* ($\varphi_2 \geq \varphi_1$ *or* $\varphi_2 \geq \imath$) *and* $\varphi_2(U) \in \varphi_{1,2}O(X)$ *for each* $U \in \varphi_1 O(X)$*. Then if* f *is a* $\varphi_{1,2}\psi_{1,2}$*-closed with* $\varphi_{1,2}O(X)$*-compact point inverses, the inverse image of each* $\psi_{1,2}O(Y)$*-compact set is a* $\varphi_{1,2}$*-compact set.*

Proof. It follows that $\varphi_1 O(X) \subseteq \varphi_2 O(X)$ and $\varphi_2(U) \in \varphi_{1,2}O(X)$ for each $U \in \varphi_1 O(X)$. Under these conditions, $\varphi_{1,2}$-compactness relative to X is equivalent to $\varphi_{1,2}O(X)$-compactness relative to X [13, Lemma 2.7]. The result is now clear from Theorem 3.13. □

3.16. Corollary. *Let* ($\varphi_2 \geq \varphi_1$ *or* $\varphi_2 \geq \imath$) *and* $\varphi_2(U) \in \varphi_1 O(X)$*,* $\varphi_2(\varphi_2(U)) \subseteq \varphi_2(U)$ *for each* $U \in \varphi_1 O(X)$*. Then if* f *is* $\varphi_{1,2}\psi_{1,2}$*-closed and* $f^{-1}(y)$ *is a* $\mathcal{B}$*-compact set*

($\mathcal{B} = \{\varphi_2(U) : U \in \varphi_1 O(X)\}$) *for each* $y \in Y$*, then the inverse image of each* $\psi_{1,2}O(Y)$*-compact set is a* $\varphi_{1,2}$*-compact set.*

Proof. Since $\mathcal{B}$ is a base for $\varphi_{1,2}O(X)$ (Lemma 2.11), a set A is a $\varphi_{1,2}O(X)$-compact set iff A is a $\mathcal{B}$-compact set. Now, the proof is clear from Lemma 2.11 and Theorem 3.15. □

3.17. Example. 1) Let $\varphi_1 = \mathrm{cl} \circ \mathrm{int}$, $\varphi_2 = \mathrm{scl}$, $\psi_1 = \mathrm{int}$, and $\psi_2 = \mathrm{cl}$. Then:

A is a $\varphi_{1,2}$-compact set iff A is an s-set.

A is a $\varphi_{1,2}O(X)$-compact set iff A is a semi-Θ-open(X)-compact set.

K is a $\psi_{1,2}$O(Y)-compact set iff K is a ϑ_Θ-compact set.

f is $\varphi_{1,2}\psi_{1,2}$-closed iff $f(F)$ is Θ-closed for each semi-Θ-closed set F.

It can be seen from Example 2.15 that φ_1 and φ_2 satisfy the properties of Corollary 3.16. So if $f(F)$ is Θ-closed for each semi-Θ-closed set F, and $f^{-1}(y)$ is an $SR(X)$-compact set for each $y \in Y$, then $f^{-1}(K)$ is an s-set for each ϑ_Θ-compact set K.

2) Let $\varphi_1 = \mathrm{cl} \circ \mathrm{int}$, $\varphi_2 = \mathrm{cl}$, $\psi_1 = \mathrm{int}$ and $\psi_2 = \mathrm{cl}$. Then:

φ_1 and φ_2 satisfy the properties of Corollary 3.15.

$\mathcal{B} = \{\overline{U} : U \in SO(X)\} = RC(X)$, $\varphi_{1,2}O(X) = \Theta$-semi-open$(X)$.

A is a $\varphi_{1,2}$-compact set iff A is an S-set iff A is a Θ-$SO(X)$-compact set iff A is a $RC(X)$-compact set.

If $f(F)$ is Θ-closed for each Θ-semi-closed set F, and $f^{-1}(y)$ is a $RC(X)$-compact set for each $y \in Y$, then $f^{-1}(K)$ is an S-set for each ϑ_Θ-compact set K.

3.18. Theorem. *Let* ($\varphi_2 \geq \varphi_1$ *or* $\varphi_2 \geq \imath$)*, let* φ_1 *and* φ_2 *be monotonous,* $\varphi_2(U) \in \varphi_{1,2}O(X)$ *for each* $U \in \varphi_1 O(X)$ *and* $\mathcal{B} = \{\varphi_2(U) : U \in \varphi_1 O(X)\}$.

If $f(\mathcal{B}') = \{f(X \setminus \varphi_2(U)) : U \in \varphi_1 O(X)\} \subseteq \psi_{1,2}$*-closed*$(Y)$*,* $f^{-1}(y)$ *is* $\mathcal{B}$*-compact for each* $y \in Y$ *and* K *is a* $\psi_{1,2}O(Y)$*-compact set in* Y*, then*

(a) *For each* φ_1*-open cover* $\mathcal{U}$ *of* $f^{-1}(K)$ *there exists* $U_1, \ldots, U_n \in \mathcal{U}$ *such that* $f^{-1}(K) \subseteq \varphi_2(\bigcup_{i=1}^n U_i)$.

(b) *If for* $\varphi_3 \in O(X,\tau)$*, we have* $\varphi_3 \geq \varphi_2$ *and* $\varphi_3(U_1 \cup U_2) = \varphi_3(U_1) \cup \varphi_3(U_2)$ *for* $U_1, U_2 \in \varphi_1 O(X)$*, then* $f^{-1}(K)$ *is a* $\varphi_{1,3}$*-compact set.*

Proof. (a) Let $f^{-1}(K) \subseteq \bigcup \mathcal{U}$ and $\mathcal{U} \subseteq \varphi_1 O(X)$. For each $y \in K$, $f^{-1}(y) \subseteq \bigcup \mathcal{U} \subseteq \bigcup\{\varphi_2(U) : U \in \mathcal{U}\}$. Since $f^{-1}(y)$ is a $\mathcal{B}$-compact set there exists $U_1, \ldots, U_{t_y} \in \mathcal{U}$ such that $f^{-1}(y) \subseteq \bigcup_{i=1}^{t_y} \varphi_2(U_i) \subseteq \varphi_2(\bigcup_{i=1}^{t_y} U_i)$. Since φ_1 is monotonous, $\bigcup_{i=1}^{t_y} U_i \in \varphi_1 O(X)$, and hence $\varphi_2(\bigcup_{i=1}^{t_y} U_i) \in \mathcal{B}$.

From Theorem 2.1, there exists $V_y \in \psi_{1,2}O(Y,y)$ with $f^{-1}(V_y) \subseteq \varphi_2(\bigcup_{i=1}^{t_y} U_i)$. We have $K \subseteq \bigcup_{y \in K} V_y$. Hence, since K is a $\psi_{1,2}O(Y)$ compact set, there exists $y_1, \ldots, y_m \in K$ such that $K \subseteq \bigcup_{j=1}^m V_{y_j}$. Hence,

$$f^{-1}(K) \subseteq \bigcup_{j=1}^{m} f^{-1}(V_{y_j}) \subseteq \bigcup_{j=1}^{m} \left(\varphi_2\left(\bigcup_{i=1}^{t_{y_j}} U_i\right)\right) \subseteq \varphi_2\left(\bigcup_{j=1}^{m}\left(\bigcup_{i=1}^{t_{y_j}} U_i\right)\right).$$

Now, by taking $n = \Sigma_{j=1}^m t_{y_j}$, we can write $f^{-1}(K) \subseteq \varphi_2(\bigcup_{i=1}^n U_i)$.

(b) This is now clear as $f^{-1}(K) \subseteq \varphi_2(\bigcup_{i=1}^n U_i) \subseteq \varphi_3(\bigcup_{i=1}^n U_i) = \bigcup_{i=1}^n \varphi_3(U_i)$. □

3.19. Example. Let $\varphi_1 = \mathrm{int}$, $\varphi_2 = \mathrm{int} \circ \mathrm{cl}$, $\varphi_3 = \mathrm{cl}$, $\psi_1 = \mathrm{cl} \circ \mathrm{int}$ and $\psi_2 = \imath$. Then:

φ_1, φ_2 are monotonous and $\varphi_3 \geq \varphi_2$.

For $U_1, U_2 \in \varphi_1 O(X) = \tau$,

$$\varphi_3(U_1 \cup U_2) = \overline{U_1 \cup U_2} = \overline{U}_1 \cup \overline{U}_2 = \varphi_3(U_1) \cup \varphi_3(U_2).$$

$\varphi_{1,2}O(X) = \tau_s$, $\psi_{1,2}O(Y) = SO(Y)$ $\psi_{1,2}C(Y) = SC(Y)$, $\mathcal{B} = RO(X)$ and $\mathcal{B}' = RC(X)$.

$f^{-1}(y)$ is a $RO(X)$ compact set iff $f^{-1}(y)$ is a τ_s-compact set iff $f^{-1}(y)$ is an N-set.

K is a $\psi_{1,2}O(Y)$-compact set iff K is a $SO(Y)$-compact set.

Hence, if $f(F)$ is semi-closed for each regular-closed set F and $f^{-1}(y)$ is an N-set for each $y \in Y$, then $f^{-1}(K)$ is an H-set, further, for each open cover $\mathcal{U}$ of $f^{-1}(K)$ there exist $U_1, \ldots, U_n \in \mathcal{U}$ such that $f^{-1}(K) \subseteq \left(\bigcup_{i=1}^{n} U_i\right)^{o}$ for each $SO(Y)$-compact set K.

Clearly, many more results may be obtained by combining the unifications of continuities, compactness, filters, graphs, openness and closedness made here and in [12,13].

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