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ON THE CONSTRUCTION OF ORTHOGONAL ARRAYS

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Abstract

In this study, the geometric representation of an Orthogonal Array is obtained using finite analytic projective geometry of the Galois field GF(s) of t-dimensions, which can be denoted by PG(t, s), where s is a prime or a power of a prime number. We give relations between the parameters of Orthogonal Arrays and properties of the projective geometry and of related geometries. We offer some geometrical examples.

Key Words: Orthogonal Array, Projective Geometry, Net.

1. Introduction

Orthogonal arrays, first introduced by Rao [7, 8] have been used extensively in factorial designs. Specifically, an orthogonal array of size N, k constraints, s levels and strength t, denoted by OA(N, k, s, t), is a $k \times N$ matrix X of s symbols such that all the ordered t-tuples of the symbols occur equally often as column vectors of any $t \times N$ submatrix of X. It is clear that N must be of the form λs^t, where λ is usually called the index of the orthogonal array. In applications to factorial designs, each row corresponds to a factor, the symbols are factor levels and each column represents a combination of the factor levels. Thus every OA(N, k, s, t) defines an N-run factorial design for k factors each having s levels [6], where $\lambda = 1$, and we refer to such arrays as "orthogonal arrays of index unity" [5].

A connection between orthogonal arrays and the theory of confounding in symmetrical factorial designs (based on the use of finite projective geometries) was first established by Bose and Kishen [3], and this was later amplified by Bose [1, 2].

Bose [1] proved that the maximum number of factors, which can be accommodated in symmetrical problems of the design of experiments, could be attacked fruitfully by interpreting the statistical terms involved in terms of finite geometries [10].

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The reader may refer to Raghavarao [9] for the definition of an orthogonal array, various construction methods and the maximum the number of constraints. Further, Raghavarao shows that projective geometries are related to orthogonal arrays, discusses the product of orthogonal arrays and gives the relationship between orthogonal arrays and partially balanced arrays.

2. The use of Projective Geometry in the Construction of Orthogonal Arrays

There is a GF$(s = p^n)$ of order p^n for every prime number p and every positive integer n. A point in the t-dimensional finite projective geometry PG(t, s) is an ordered set of $t+1$ elements of GF(s), not all of which are equal to zero. If we delete any line of PG(t, s), together with the points on it, we obtain a finite Euclidean Geometry EG(t, s), constructed on GF(s). The idea of deleting a line can be considered as grouping s varieties according to a treatment or combination of treatments, which are the points of the deleted line. The geometric representation of an Orthogonal Array can be obtained using a finite analytic projective geometry over the Galois field GF(s) of t-dimensions. Rao [7] considered constructions of hypercubes from PG(t, s). The orthogonal array OA$(s^t, (s^t - 1)/(s - 1), s, 2)$ of index $\lambda = s^{t-2}$, which is the same as the hypercube $[(s^t - 1)/(s - 1), s, t, 2]$ of strength 2, can be constructed from PG(t, s) in the following manner: in PG(t, s) the equation

$$x_0 = 0 \tag{1}$$

represents the $(t-1)$-flat at infinity, and any other $(t-1)$-flat can be represented by an equation of the from

$$a_0x_0 + a_1x_1 + \cdots + a_tx_t = 0, \tag{2}$$

so a $(t-2)$-flat at infinity can be represented by

$$x_0 = 0; a_1x_1 + a_2x_2 + \cdots + a_tx_t = 0,\ a_i \in GF(s),\ i = 1, 2, \ldots, t. \tag{3}$$

We observe that there are $(s^t-1)/(s-1)$, $(t-2)$-flats at infinity. From the $(t-2)$-flat given by (3), there originates a pencil of s, $(t-1)$-flats given by the following equation:

$$\alpha_ix_0 + a_1x_1 + a_2x_2 + \cdots + a_tx_t = 0,\ \alpha_i \in \mathrm{GF}(s) \tag{4}$$

The $(t-2)$-flat (3) is called the vertex of the pencil (4), and corresponds to a factor identified by $(a_1, a_2, \ldots, a_t)$. The $(t-1)$-flat (4), with α_i as the coefficient of x_0, may be identified with α_i, which may be taken to correspond to the i^{th} level $(i = 0, 1, \ldots, s-1)$ of a factor, identified by $(a_1, a_2, \ldots, a_t)$, defining the vertex (3) of the pencil (4). Thus there are $m = (s^t-1)/(s-1)$ factors, each at s levels.

There are s^t finite points $(1, x_1, x_2, \ldots, x_t)$, and through each of the finite points there passes exactly one $(t-1)$-flat from each of the $(s^t-1)/(s-1)$ pencils of $(t-1)$-flats. We now identify the s^t finite points and the factors with the columns and rows of an array, and fill its $(i,j)^{th}$ position by k if the α_k^{th} flat of the i^{th} pencil passes through the j^{th} finite point ($k = 0, 1, \ldots, s-1$; $i = 1, 2, \ldots, (s^t - 1)/(s-1)$; $j = 1, 2, \ldots, s^t$). The array thus constructed can easily be verified to be the orthogonal array $\mathrm{OA}(s^t, (s^t-1)/(s-1), s, 2)$ of index $\lambda = s^{t-2}$ [9].

2.1. Illustration. We illustrate this method of construction by constructing the orthogonal array $\mathrm{OA}(8, 7, 2, 2)$ of index 2. This orthogonal array can be constructed from $\mathrm{PG}(3, 2)$. In $\mathrm{PG}(3, 2)$ each of the 15 points can be represented by (x_0, x_1, x_2, x_3), where $x_i = 0$ or 1 ($i = 0, 1, 2, 3$), and not all the x_i coordinates are equal to zero. The vertices corresponding to the pencil (4) are (1,0,0), (0,1,0), (0,0,1), (1,1,0), (1,0,1), (0,1,1) and (1,1,1), and we call them the factors A, B, C, D, E, F and G, respectively. The eight finite points are (1,0,0,0), (1,1,0,0), (1,0,1,0), (1,0,0,1), (1,1,1,0), (1,1,0,1), (1,0,1,1) and (1,1,1,1), numbered in the serial order $1, 2, \ldots, 8$. Let us form an array of seven rows and eight columns, the rows corresponding to the seven factors, and the columns corresponding to the eight finite points, and fill the $(i,j)^{th}$ entry by the flat number of the i^{th} pencil passing through the j^{th} finite point ($i = A, B, \ldots, G$; $j = 1, 2, \ldots, 8$). For example, the second point lies on the flat number 1 passing through the vertex D, and hence the entry in the second column under row D is 1. The completed arrangement constituting the orthogonal array $\mathrm{OA}(8, 7, 2, 2)$ of index 2 will then be as follows:

Table 1. Orthogonal Array $\mathbf{OA(8, 7, 2, 2)}$ of index 2

	Finite Points							
Factor	1	2	3	4	5	6	7	8
A	0	1	0	0	1	1	0	1
B	0	0	1	0	1	0	1	1
C	0	0	0	1	0	1	1	1
D	0	1	1	0	0	1	1	0
E	0	1	0	1	1	0	1	0
F	0	0	1	1	1	1	0	0
G	0	1	1	1	0	0	0	1

To construct a hypercube of strength 3, or equivalently an orthogonal array of index $\lambda = s^{t-3}$ from $\mathrm{PG}(t, s)$, we need to select the factors suitably. To get the orthogonal array of index $\lambda = s^{t-3}$, it is necessary that three $(t-1)$-flats belonging to three different pencils from the set of $(s^t-1)/(s-1)$ pencils intersect at s^{t-3} finite points. Any two vertices corresponding to two pencils intersect at a $(t-3)$-flat at infinity. A third vertex has to be chosen so that it does not pass through the intersection of the previous two vertices; a fourth vertex has to be chosen so that it does not pass through the intersection of any two vertices already chosen; and so on. It is now easy to see that the conditions for an orthogonal array of index $\lambda = s^{t-3}$ are satisfied if the selected vertices are identified with factors [9].

Thus, Table 1 defines the finite $\mathrm{EG}(3, 2)$.

2.2. Illustration. Let us construct the orthogonal array $\mathrm{OA}(9,4,3,2)$ of index unity, where $s=3$ and $t=2$. This orthogonal array can be constructed from $\mathrm{PG}(2,3)$. In $\mathrm{PG}(2,3)$ each of the 13 points can be represented by (x_0,x_1,x_2), where $x_i=0,1$ or 2 $(i=0,1)$ and not all the coordinates x_i are equal to zero. The lines and the points on those lines in $\mathrm{PG}(2,3)$ are as follows:

Table 2. Lines and Points in PG(2, 3)

Lines	Points on Lines
$[0,0,1]$	(1,0,0), (0,1,0), (1,1,0), (1,2,0)
$[0,1,0]$	(1,0,0), (0,0,1), (1,0,1), (1,0,2)
$[0,2,1]$	(1,0,0), (0,1,1), (1,1,1), (1,2,2)
$[0,1,1]$	(1,0,0), (0,2,1), (1,1,2), (1,2,1)
$[1,0,0]$	(0,1,0), (0,0,1), (0,1,1), (0,2,1)
$[1,0,2]$	(0,1,0), (1,0,1), (1,1,1), (1,2,1)
$[1,0,1]$	(0,1,0), (1,0,2), (1,1,2), (1,2,2)
$[1,2,0]$	(1,1,0), (0,0,1), (1,1,1), (1,1,2)
$[2,1,1]$	(1,1,0), (1,0,2), (0,1,1), (1,2,1)
$[1,2,1]$	(1,2,0), (0,0,1), (1,2,1), (1,2,2)
$[1,1,0]$	(1,1,0), (1,0,1), (0,2,1), (1,2,2)
$[1,1,2]$	(1,2,0), (1,0,1), (0,1,1), (1,1,2)
$[1,1,1]$	(1,2,0), (1,0,2), (0,2,1), (1,1,1)

We selected the finite points $(1,x_1,x_2,\ldots,x_t)$ to correspond to s^t, those points whose first coordinate is zero and the first line will be removed. The points and corresponding factors and factor combinations are given as follows:

Table 3.

Lines	Points on Lines	Factor and Factor Combinations			
$[1,0,0]$	(0,1,0) (0,0,1) (0,1,1) (0,2,1)	B	C	BC	BC^2
$[0,0,1]$	(1,0,0) (0,1,0) (1,1,0) (1,2,0)	A	B	AB	AB^2
$[1,0,1]$	(0,1,0) (1,0,2) (1,1,2) (1,2,2)	B	AC^2	ABC^2	AB^2C^2
$[1,0,2]$	(0,1,0) (1,0,1) (1,1,1) (1,2,1)	B	AC	ABC	AB^2C

Lines	Points on Lines	Factor and Factor Combinations			
$[1,0,0]$	(0,1,0) (0,0,1) (0,1,1) (0,2,1)	B	C	BC	BC^2
$[0,1,0]$	(1,0,0) (0,0,1) (1,0,1) (1,0,2)	A	C	AC	AC^2
$[1,1,0]$	(0,0,1) (1,2,1) (2,1,1) (1,2,0)	C	AB^2C	AB^2C^2	AB^2
$[1,2,0]$	(1,1,0) (0,0,1) (1,1,1) (1,1,2)	AB	C	ABC	ABC^2

Lines	Points on Lines	Factor and Factor Combinations			
$[1,0,0]$	(0,1,0) (0,0,1) (0,1,1) (0,2,1)	B	C	BC	BC^2
$[0,2,1]$	(1,0,0) (0,1,1) (1,1,1) (1,2,2)	A	BC	ABC	AB^2C^2
$[1,1,2]$	(1,2,0) (1,0,1) (0,1,1) (1,1,2)	AB^2	AC	BC	ABC^2
$[1,2,2]$	(1,1,0) (1,0,2) (0,1,1) (1,2,1)	AB	AC^2	BC	AB^2C

Lines	Points on Lines	Factor and Factor Combinations			
$[1,0,0]$	(0,1,0) (0,0,1) (0,1,1) (0,2,1)	B	C	BC	BC^2
$[1,1,1]$	(1,1,1) (1,2,0) (1,0,2) (0,2,1)	ABC	AB^2	AC^2	BC^2
$[0,1,1]$	(1,0,0) (0,2,1) (1,2,1) (1,1,2)	A	BC^2	AB^2C	ABC^2
$[2,1,1]$	(1,1,0) (1,0,1) (2,1,1) (0,2,1)	AB	AC	AB^2C^2	BC^2

Since the line $[1,0,0]$ and points on that line are removed, we have 12 lines and 9 points. Using the above notation these 12 lines and points can be shown as follows:

Table 4.

Lines	Points on Lines	Factors and Combinations			Deleted Line : Points
$[0,0,1]$	(1,0,0) (1,1,0) (1,2,0)	A	AB	AB^2	$[1,0,0]$:
$[1,0,1]$	(1,0,2) (1,1,2) (1,2,2)	AC^2	ABC^2	AB^2C^2	(0,1,0) (0,0,1)
$[1,0,2]$	(1,0,1) (1,1,1) (1,2,1)	AC	ABC	AB^2C	(0,1,1) (0,2,1)

Lines	Points on Lines	Factors and Combinations			Deleted Line : Points
$[0,1,0]$	(1,0,0) (1,0,1) (1,0,2)	A	AC	AC^2	$[1,0,0]$:
$[1,1,0]$	(1,2,1) (2,1,1) (1,2,0)	AB^2C	AB^2C^2	AB^2	(0,1,0) (0,0,1)
$[1,2,0]$	(1,1,0) (1,1,1) (1,1,2)	AB	ABC	ABC^2	(0,1,1) (0,2,1)

Lines	Points on Lines	Factors and Combinations			Deleted Line : Points
$[0,1,2]$	(1,0,0) (1,1,1) (1,2,2)	A	ABC	AB^2C^2	$[1,0,0]$:
$[1,1,2]$	(1,2,0) (1,0,1) (1,1,2)	AB^2	AC	ABC^2	(0,1,0) (0,0,1)
$[1,2,2]$	(1,1,0) (1,0,2) (1,2,1)	AB	AC^2	AB^2C	(0,1,1) (0,2,1)

Lines	Points on Lines	Factors and Combinations			Deleted Line : Points
$[1,1,1]$	(1,1,1) (1,2,0) (1,0,2)	ABC	AB^2	AB^2C^2	$[1,0,0]$:
$[0,1,1]$	(1,0,0) (1,2,1) (1,1,2)	AB^2	AC	ABC^2	(0,1,0) (0,0,1)
$[2,1,1]$	(1,1,0) (1,0,1) (2,1,1)	AB	AC^2	AB^2C	(0,1,1) (0,2,1)

The vertices corresponding to the pencil (4) are (0,1,0), (0,0,1), (0,1,1) and (0,2,1), and we call them the factors B, C, BC and BC^2, respectively. The nine finite points are (1,0,0), (1,0,1), (1,0,2), (1,1,0), (1,1,1), (1,1,2), (1,2,0), (1,2,1) and (1,2,2), numbered in the serial order $1,2,\ldots,9$. Let us form an array of four rows and nine columns, the rows corresponding to the four factors, and the columns corresponding to the nine finite points, and fill the $(i,j)^{th}$ entry by the flat number of the i^{th} pencil passing through the j^{th} finite point ($i = B, C, BC, BC^2$; $j = 1,2,\ldots,9$). For example, the second point lies on flat number 1, passing through the vertex B, and hence the entry in the second column under row B is 0. Deleted points corresponding to vertices (factor and factor combinations) are multiplied by the finite points, and the $(i,j)^{th}$ entry filled by the number obtained. For instance, if the second point and vertex B are multiplied, the $(1,2)^{th}$ entry will be 0, and so on. The completed arrangement, constituting the orthogonal array $\mathrm{OA}(9,4,3,2)$ of unit index, will then be as follows:

Table 5. Orthogonal Array $\mathbf{OA(9,4,3,2)}$ of unit index

	Finite Points								
Factor	1	2	3	4	5	6	7	8	9
B	0	0	0	1	1	1	2	2	2
C	0	1	2	0	1	2	0	1	2
BC	0	1	2	1	2	0	2	0	1
BC^2	0	1	2	2	0	1	1	2	0

2.3. Definition. A k-net, N, is a system of undefined points and lines, together with an incidence relation, subject to the following axioms:

(i) N has at least one point.

(ii) The lines of N are partitioned into k disjoint, nonempty, "parallel classes" such that

  (a) each point of N is incident with exactly one line of each class;

  (b) for any two lines belonging to distinct classes, there corresponds exactly one point of N which is incident with both lines.

If some line of N contains exactly n distinct points, the following statements are true:

(I) Each line of N contains exactly n distinct points, where $n \geq 1$.

(II) Each point of N lies in exactly k distinct classes, where $k > 1$.

(III) N has exactly kn distinct lines. These fall into k parallel classes of n lines each. Distinct lines of the same parallel class have no common points. Two lines of different classes have exactly one common point.

(IV) N has exactly n^2 distinct points [4].

According to Definition 2.3, Table 3 defines a net. A finite net, N, of order n, deficiency zero, is precisely an affine plane of order n. Thus the design above is an affine plane.

3. Conclusion

The construction and analysis of orthogonal arrays needs special care. There are several methods which can be used to construct orthogonal arrays. We have tried to give relations between orthogonal arrays and properties of projective and related geometries. We have offered some geometrical examples, some of which can be considered original.

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