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ON THE CONSTRUCTION OF 2-SYMBOL ORTHOGONAL ARRAYS

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Abstract

The application of Hadamard matrices to the theory and construction of experimental designs is of considerable importance. The purpose of this study is to point out some connections between Hadamard matrices, balanced incomplete block designs(BIB designs) and orthogonal arrays.

Key Words: Orthogonal array, Hadamard matrix, balanced incomplete block design, projective geometry.

1. Introduction

Orthogonal arrays, first introduced by Rao [10, 11] have been used extensively in factorial designs. Specifically, an orthogonal array of size N, k constraints, s levels and strength t, denoted by $\mathrm{OA}(N, k, s, t)$, is a $k \times N$ matrix X of s symbols, such that all the ordered t-tuples of the symbols occur equally often as column vectors of any $t \times N$ submatrix of X. It is clear that N must be of the form λs^t, where λ is usually called the index of the orthogonal array. In applications to factorial designs, each row corresponds to a factor, the symbols are factor levels and each column represents a combination of the factor levels. Thus every $\mathrm{OA}(N, k, s, t)$ defines an N-run factorial design for k factors, each having s levels [5]. When $\lambda = 1$ we refer to such arrays as "orthogonal arrays of index unity" [3].

Most of the techniques for the construction of 2-symbol orthogonal arrays are special cases of techniques for s-symbol arrays. In this study, a structure of primary interest to us is that of a Hadamard matrix. A Hadamard matrix of order n is an $n \times n$ matrix H_n with entries 1 or -1, such that

$$H_n H_n' = nI_n$$

where I_n denotes the $n \times n$ identity matrix. There are numerous results on the construction and applications of Hadamard matrices, many of which can be found

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in Hedayat and Wallis [7]. A necessary condition for their existence is that $n = 2$ or $n \equiv 0 \pmod 4$.

If we multiply each entry in a row or column of a Hadamard matrix with -1, it remains a Hadamard matrix. Thus, by multiplying the entries in appropriate columns with -1, we may always obtain a *normalized Hadamard matrix* of the form

$$H_n = \begin{bmatrix} \underline{1} & 1 \cdots\cdots 1 \\ & A_{n-1,n-1} \end{bmatrix}$$

provided that a Hadamard matrix of order n exists [6]. Here $\underline{1}$ denotes a column of 1's and the $(n-1) \times (n-1)$ matrix $A = A_{n-1,n-1}$ is called the *core* of H_n.

2. The use of Hadamard Matrices in the construction of Orthogonal Arrays

Paley [8] was interested in orthogonal arrays with $t = 2$, $s = 2$ because of their applications to the theory of polytopes. Plotkin [9] makes the very strong conjecture that every Hadamard matrix of order $8n$ can be obtained by specializing some orthogonal design of order $8n$. He showed that the existence of a Hadamard matrix of order n implies the existence of three types of orthogonal design.

If we write $n = 4\lambda$ in the Hadamard matrix, it is easily seen that $A_{n,n-1}$ is an $\mathrm{OA}(4\lambda, 4\lambda-1, 2, 2)$, based on the symbols 1 and -1. This construction can also be reversed since $f(4\lambda, 2, 2) = k \leq 4\lambda - 1$ by Theorem 2.1 below.

2.1. Theorem: *In an* $\mathrm{OA}(\lambda 2^t, k, 2, t)$ *the inequalities*

$$N - 1 \geq \binom{k}{1}(s-1) + \cdots + \binom{k}{u}(s-1)^u \text{ if } t = 2u$$

and

$$N - 1 \geq \binom{k}{1}(s-1) + \cdots + \binom{k}{u}(s-1)^u + \binom{k-1}{u}(s-1)^{u+1} \text{ if } t = 2u+1$$

must hold [11].

Thus, we obtain the following result.

2.2. Theorem: [6] *A necessary and sufficient condition for* $f(4\lambda, 2, 2) = 4\lambda - 1$ *is the existence of a Hadamard matrix of order* 4λ.

2.3. Example. Let us take a Hadamard matrix of order 8 for $t = 2$ as follows

$$H_n = \begin{bmatrix} + & + & + & + & + & + & + & + \\ + & + & + & - & + & - & - & - \\ + & - & + & + & - & + & - & - \\ + & - & - & + & + & - & + & - \\ + & - & - & - & + & + & - & + \\ + & + & - & - & - & + & + & - \\ + & - & + & - & - & - & + & + \\ + & + & - & + & - & - & - & + \end{bmatrix}$$

and let A be the core of H_n.

$$A = \begin{bmatrix} + & + & - & + & - & - & - \\ - & + & + & - & + & - & - \\ - & - & + & + & - & + & - \\ - & - & - & + & + & - & + \\ + & - & - & - & + & + & - \\ - & + & - & - & - & + & + \\ + & - & + & - & - & - & + \end{bmatrix}$$

Here $+$ denotes 1 and $-$ denotes -1. Replacing $+$ by 1 and $-$ by 0 we obtain the orthogonal array $\mathrm{OA}(8,7,2,2)$ over the symbols 0, 1 as follows:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

An $\mathrm{OA}(4\lambda, 4\lambda - 1, 2, 2)$ as above can be used to construct an $\mathrm{OA}(8\lambda, 4\lambda, 2, 3)$. This follows from a stronger result by Seiden and Zemach [13]. For a given orthogonal array C, let $\overline{C}$ denote the array obtained from C by permuting the 0's and 1's.

2.4. Theorem: [6] *If C is an $\mathrm{OA}(\lambda 2^t, k, 2, t)$, where t is even, then the array B, defined by*

$$B = \begin{bmatrix} C & \overline{C} \\ 0 \cdots 0 & 1 \cdots 1 \end{bmatrix}$$

is an $\mathrm{OA}(\lambda 2^{t+1}, k+1, 2, t+1)$.

2.5. Corollary. [6] *An $\mathrm{OA}(8\lambda, 4\lambda, 2, 3)$ exists if and only if a Hadamard matrix of order 4λ exists.*

2.6. Corollary. *If t is even, then $f(\lambda 2^t, 2, t) = f(\lambda 2^{t+1}, 2, t+1) - 1$ [6].*

If C in Theorem 2.3 is just an array, not necessarily orthogonal, with the property that any $(t+1)$-tuple a appears together with $\overline{C}$ in λ columns of any $(t+1) \times \lambda 2^t$ subarray, then

$$B = \begin{bmatrix} C & \overline{C} \end{bmatrix}$$

is an $\mathrm{OA}(\lambda 2^{t+1}, k, 2, t+1)$. Such an array C is a generalization of the idea of a difference scheme, introduced by Bose and Bush [2]. The generalization is due to Seiden [12].

2.7. Theorem: [7] *The existence of a Hadamard matrix of order 4λ is equivalent to the existence of:*

(i) *an* $\mathrm{OA}(4\lambda, 4\lambda-1, 2, 2)$;

(ii) *an* $\mathrm{OA}(8\lambda, 4\lambda, 2, 3)$.

In view of Theorem 2.2 and Corollary 2.1, as well as the vast literature on Hadamard matrices, one should not expect any exciting results from such an approach. What is needed are constructions for arrays of strength 4 or more. The most fruitful concept for studying this is a geometric method, introduced by Bose and Bush [2].

2.8. Theorem: [6] *A set of k points in $\mathrm{PG}(n, 2)$ such that any t of them are linearly independent, induces an $\mathrm{OA}(\lambda 2^t, k, 2, t)$ with $\lambda = 2^{n-t+1}$. Here $\mathrm{PG}(n, s)$ is an n dimensional finite projective geometry.*

3. The relations among Orthogonal Arrays and Balanced Incomplete Block Designs

Consider two sets T and B who's elements are treatments and blocks, respectively. There are ν treatments and b blocks in a balanced incomplete block design (BIB design), that satisfy the following conditions:

1. Each block has exactly m members,
2. Each treatments occurs in exactly r blocks,
3. Every pair of treatments occurs in exactly λ blocks.

The parameters of a BIB design satisfy the following equations

1. $N = bm = r\nu$,
2. $\lambda = r(m-1)/(\nu-1)$,

where N is the total number of observations. When $\nu = b$, the BIB design is said to be a symmetric BIB design. Chakravarty and Dey [4] use some patchwork methods involving incidence matrices of a BIB design to construct orthogonal arrays of strengths 2 and 3.

4. Hadamard matrices and Balanced Incomplete Block Designs

Hadamard matrices can be transformed to produce incomplete block designs, as is seen in the following theorem.

4.1. Theorem: [6] *The existence of a Hadamard matrix of order $4w$ is equivalent to the existence of a symmetric BIB design with the parameters*

(i) $\nu = b = 4w-1, r = m = 2w-1, \lambda = w-1$;

(ii) $\nu = b = 4w-1, r = m = 2w, \lambda = w$.

4.2. Example. Let H_n be the normalized Hadamard matrix of order 8 and let A be the core of H_n. Then $N_1 = \frac{1}{2}(J+A), N_2 = \frac{1}{2}(J-A)$ are the incidence matrix of a BIB design with parameters $\nu = b = 7, r = m = 3, \lambda = 1$ and $\nu = b = 7, r = m = 4, \lambda = 2$ respectively. Here J is a matrix of 1's of appropriate order. The matrices N_1 and N_2 are obtained as follows:

$$N_1 = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \quad N_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Bayrak[1] gives in detail the relation between Hadamard matrices and balanced incomplete block designs. We can regard blocks and treatments of a BIB design as the lines and the points of $\mathrm{PG}(n, s)$. Here N_1 corresponds to $\mathrm{PG}(2, 2)$, which is a Fano plane; N_2 corresponds to a biplane, which is a symmetric design with $\lambda = 2$.

Let B be the matrix

$$B = \begin{bmatrix} \underline{J} & N_1 & N_2 & \underline{0} \end{bmatrix},$$

where $\underline{J}$ is a column of 1's and $\underline{0}$ a column of 0's. Then the matrices $\begin{bmatrix} \underline{J} & N_1 \end{bmatrix}$ and $\begin{bmatrix} N_2 & \underline{0} \end{bmatrix}$ are $\mathrm{OA}(4\lambda, 4\lambda - 1, 2, 2)$'s. Furthermore, B is an $\mathrm{OA}(\lambda 2^{t+1}, k, 2, t+1)$. According to Theorem 2.4 we obtain the following $\mathrm{OA}(16, 8, 2, 3)$:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

5. Conclusion

Incidence matrices of BIB designs and Hadamard matrices are used to construct orthogonal arrays of strengths 2 and 3.

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