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ON FUZZY TOPOLOGICAL GROUPS AND FUZZY CONTINUOUS FUNCTIONS

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Abstract

Function spaces play an important role in complex analysis, in the theory of differential equations, in functional analysis and in almost every other branch of modern mathematics. Let $FC(Y, Z)$ be the set of all fuzzy continuous functions from a fuzzy topological space Y into a fuzzy topological space Z. Our aim in this paper is to study the notion of group, fuzzy group, topological group, and fuzzy topological group on the (fuzzy) function space $FC(Y, Z)$.

Keywords: Group, Fuzzy group, Topological group, Fuzzy topological group, Function space.

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1. Introduction

The definitions and results for fuzzy topological spaces (X, τ) and fuzzy topological groups $(X, \cdot, \tau)$ which are used in this paper, have already been standardized. Some definitions and results of [1], [4], [7], [9], [10], [13], and [14], which will be needed in the sequel are recalled here.

Throughout this paper, the symbol I will denote the unit interval $[0, 1]$.

Let X be a nonempty set. A *fuzzy set* in X is a function with domain X and values in I, that is, an element of I^X.

Let $A, B \in I^X$. We define the following fuzzy sets (see [14]):

(1) $A \wedge B \in I^X$ by $(A \wedge B)(x) = \min\{A(x), B(x)\}$ for every $x \in X$ (intersection).
(2) $A \vee B \in I^X$ by $(A \vee B)(x) = \max\{A(x), B(x)\}$ for every $x \in X$ (union).

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Let X, Y be two nonempty sets, $f : X \to Y$, $A \in I^X$, and $B \in I^Y$. Then, $f(A)$ is the fuzzy set in Y defined by

$$f(A)(y) = \begin{cases} \sup\{A(x) : x \in f^{-1}(y)\}, & \text{if } f^{-1}(y) \neq \emptyset, \\ 0, & \text{if } f^{-1}(y) = \emptyset, \quad y \in Y, \end{cases}$$

and $f^{-1}(B)$ is the fuzzy set in X defined by

$$f^{-1}(B)(x) = B(f(x)), \quad x \in X.$$

A fuzzy set in X is called a *fuzzy point* if and only if it takes the value 0 for all $y \in X$ except one, say $x \in X$. If its value at x is r $(0 < r \leq 1)$ we denote the fuzzy point by p_x^r, where the point x is called its *support* and is denoted by supp p_x^r (see, for example [9], [10] and [13]).

The fuzzy point p_x^r is said to be *contained* in a fuzzy set A, or to belong to A, denoted by $p_x^r \in A$, if and only if $r \leq A(x)$. Evidently, every fuzzy set A can be expressed as the union of all the fuzzy points which belongs to A (see [9]).

A fuzzy set A in a fuzzy topological space (X, τ) is called a *neighbourhood* of a fuzzy point p_x^r if and only if there exists a $V \in \tau$ such that $p_x^r \in V \leq A$ (see [9]). A neighbourhood A is said to be open if and only if A is open.

A fuzzy point p_x^r is said to be *quasi-coincident* with A, denoted by $p_x^r qA$, if and only if $r > A^c(x)$, or equivalently $r + A(x) > 1$ (see [9]).

A fuzzy set A is said to be *quasi-coinident* with B, denoted by AqB, if and only if there exists $x \in X$ such that $A(x) > B^c(x)$, that is $A(x) + B(x) > 1$ (see [9]). If A is not quasi-coincident with B, then we write $A \not{q} B$.

A fuzzy set A in a fuzzy topological space (X, τ) is called a *Q-neighbourhood* of p_x^r if and only if there exists $B \in \tau$ such that $p_x^r qB$ and $B \leq A$. The family of all Q-neighbourhoods of p_x^r is called the system of Q-neighbourhoods of p_x^r (see [9]). A Q-neighbourhood of a fuzzy point generally does not contain the point itself.

Let f be a function from X to Y. Then (see for example [7], [9], [10] and [13]):

(1) Let p be a fuzzy point of X, A be a fuzzy set in X and B be a fuzzy set in Y. Then, we have:
  - If $f(p)\ q\ B$, then $p\ q\ f^{-1}(B)$.
  - If $p\ q\ A$, then $f(p)\ q\ f(A)$.

(2) Let A and B be fuzzy sets in X and Y, respectively. Let p be a fuzzy point in X. Then we have:
  - $p \in f^{-1}(B)$ if $f(p) \in B$.
  - $f(p) \in f(A)$ if $p \in A$.

Let f be a function from a fuzzy topological space (X, τ_1) into a fuzzy topological space (Y, τ_2). The map f is said to be *fuzzy continuous* if for every $U \in \tau_2$, $f^{-1}(U) \in \tau_1$ (see [10]).

Let f be a function from a fuzzy topological space (X, τ_1) into a fuzzy topological space (Y, τ_2). Then the following are equivalent (see Theorem 1.1 of [9]):

(1) f is fuzzy continuous,
(2) for each fuzzy point p in X and each neighbourhood V of $f(p)$ in Y, there exists a neighbourhood of p in X such that $f(U) \leq V$.
(3) for each fuzzy point p in X and each fuzzy open Q-neighbourhood V of $f(p)$ in Y, there exists a fuzzy open Q-neighbourhood U of p in X such that $f(U) \leq V$.

Let $FC(Y, Z)$ be the set of all fuzzy continuous functions from a fuzzy topological space Y into a fuzzy topological space Z. Our aim in this paper is to study the notion of group, fuzzy group, topological group, and fuzzy topological group on the (fuzzy) function space $FC(Y, Z)$.

2. Topological and fuzzy topological groups

2.1. Notation. Let $(X, \cdot)$ be a group, $A, B \in I^X$ and $C, D \subseteq X$. We define $A \bullet B \in I^X$, $A^{-1} \in I^X$, $C \cdot D \subseteq X$ and $C^{-1} \subseteq X$ by the respective formulas:

$$(A \bullet B)(x) = \sup\{\min\{A(x_1), B(x_2)\} : x_1 \cdot x_2 = x\}$$

and

$$A^{-1}(x) = A(x^{-1}),$$

for every $x \in X$. Also

$$C \cdot D = \{c \cdot d : c \in C \text{ and } d \in D\}$$

and

$$C^{-1} = \{c^{-1} : c \in C\}.$$

2.2. Definition. (see [8]) Let X be a set of elements. A triad $(X, \cdot, \tau_X)$ is called a *fuzzy topological group* if

(i) $(X, \cdot)$ is a group.
(ii) (X, τ_X) is a fuzzy topological space.
(iii) For all $x, y \in X$ and any fuzzy open Q-neighbourhood W of the fuzzy point $p^r_{x \cdot y}$ there are fuzzy open Q-neighbourhoods U and V of p^r_x and p^r_y, respectively such that:

$$U \bullet V \leq W.$$

(iv) For all $x \in X$ and any fuzzy open Q-neighbourhood V of $p^r_{x^{-1}}$, there exists a fuzzy open Q-neighbourhood U of p^r_x such that:

$$U^{-1} \leq V.$$

2.3. Theorem. *Let (Y, τ_Y) be a fuzzy topological space, $(Z, \cdot, \tau_Z)$ a fuzzy topological group, and $f, g \in FC(Y, Z)$. Then, the maps $f * g$ and f^{-1} from the fuzzy topological space Y into the fuzzy topological space Z with the types:*

$$(f * g)(y) = f(y) \cdot g(y)$$

and

$$f^{-1}(y) = (f(y))^{-1},$$

for every $y \in Y$, are fuzzy continuous.

Proof. Let p^r_y, where $r \in (0, 1]$ and $y \in Y$, be a fuzzy point of Y and W a fuzzy open Q-neighbourhood of $(f * g)(p^r_y) = p^r_{(f*g)(y)} = p^r_{f(y) \cdot g(y)}$ in Z. Since $(Z, \cdot, \tau_Z)$ is a fuzzy topological group there exist Q-neighbourhoods U and V of $p^r_{f(y)}$ and $p^r_{g(y)}$, respectively such that

$$U \bullet V \leq W. \tag{2.1}$$

Now, since the maps f and g are fuzzy continuous, there exist fuzzy open Q-neighbourhoods U_1 and V_1 of the fuzzy point p^r_y in Y such that

$$f(U_1) \leq U \text{ and } g(V_1) \leq V.$$

Clearly, the fuzzy set $U_1 \wedge V_1 \in I^Y$ is a fuzzy open Q-neighbourhood of p^r_y in Y. We prove that

$$(f * g)(U_1 \wedge V_1) \leq W.$$

Indeed, let $p_{y_1}^{r_1} \in U_1 \wedge V_1$. We prove that

$$(f * g)(p_{y_1}^{r_1}) = p_{f(y_1)\cdot g(y_1)}^{r_1} = p_{(f*g)(y_1)}^{r_1} \in W.$$

We have $p_{y_1}^{r_1} \in U_1$ and $p_{y_1}^{r_1} \in V_1$. Therefore, $f(p_{y_1}^{r_1}) = p_{f(y_1)}^{r_1} \in U$ and $g(p_{y_1}^{r_1}) = p_{g(y_1)}^{r_1} \in V$. Hence $r_1 \leq U(f(y_1))$ and $r_1 \leq V(g(y_1))$.

Also, clearly

$$(U \bullet V)(f(y_1) \cdot g(y_1)) = \sup\{\min\{U(z_1), V(z_2)\} : z_1 \cdot z_2 = f(y_1) \cdot g(y_1)\} \geq r_1.$$

Thus, by relation (2.1) we have:

$$r_1 \leq (U \bullet V)(f(y_1) \cdot g(y_1)) \leq W(f(y_1) \cdot g(y_1)),$$

and therefore,

$$p_{f(y_1)\cdot g(y_1)}^{r_1} = p_{(f*g)(y_1)}^{r_1} \in W.$$

So, $(f * g)(U_1 \wedge V_1) \leq W$ and, therefore, the map $f * g$ is fuzzy continuous.

Finally, we prove that the map f^{-1} is fuzzy continuous. Let p_y^r be a fuzzy point of Y and W a fuzzy open Q-neighbourhood of $f^{-1}(p_y^r) = p_{f^{-1}(y)}^r = p_{(f(y))^{-1}}^r$ in Z. Since $(Z, \cdot, \tau_Z)$ is a fuzzy topological group there exists a fuzzy open Q-neighbourhood U of $p_{f(y)}^r$ in Z such that

$$U^{-1} \leq W. \tag{2.2}$$

Now, since the map f is fuzzy continuous at the fuzzy point p_y^r in Y, there exists a fuzzy open Q-neighbourhood U_1 of p_y^r such that

$$f(U_1) \leq U.$$

For the fuzzy open Q-neighbourhood U_1 of p_y^r in Y we have:

$$f^{-1}(U_1) \leq W.$$

Indeed, let $p_{y_1}^{r_1} \in U_1$. It is sufficient to prove that $f^{-1}(p_{y_1}^{r_1}) \in W$, that is $r_1 \leq W(f^{-1}(y_1)) = W((f(y_1))^{-1})$.

We have $f(p_{y_1}^{r_1}) = p_{f(y_1)}^{r_1} \in U$. Thus

$$r_1 \leq U(f(y_1)) = U^{-1}((f(y_1))^{-1}) \leq W((f(y_1))^{-1}) \text{ (see relation (2.2))},$$

and, therefore, the map f^{-1} is fuzzy continuous. □

2.4. Definition. (see, for example, [8]) Let X be a set and $r \in [0, 1]$. By $r^* \in I^X$ we denote the fuzzy set of X for which $r^*(x) = r$, for every $x \in X$. Also, a fuzzy topological space (X, τ) is called *fully stratified* if $r^* \in \tau$, for every $r \in [0, 1]$.

2.5. Theorem. *Let (Y, τ_Y) be a fully stratified fuzzy topological space, $(Z, \cdot, \tau_Z)$ a fuzzy topological group, and e the identity element of the group $(Z, \cdot)$. Then, the map e' from the fuzzy topological space Y into the fuzzy topological space Z with the type:*

$$e'(y) = e,$$

for every $y \in Y$, is fuzzy continuous.

Proof. Let $U \in \tau_Z$. It suffices to prove that

$$(e')^{-1}(U) \in \tau_Y.$$

We have

$$((e')^{-1}(U))(y) = U(e'(y)) = U(e),$$

for every $y \in Y$. This means that the fuzzy set $(e')^{-1}(U)$ is constant. Therefore, since the fuzzy space (Y, τ_Y) is fully stratified, we have $(e')^{-1}(U) \in \tau_Y$.

Thus, the map e' is fuzzy continuous. □

2.6. Theorem. *Let (Y, τ_Y) be a fully stratified fuzzy topological space and $(Z, \cdot, \tau_Z)$ a fuzzy topological group. Then, the pair $(FC(Y,Z), *)$ is a group.*

Proof. Let $f, g, h \in FC(Y,Z)$. Then, we have:

$$\begin{aligned}((f * g) * h)(y) &= (f * g)(y) \cdot h(y) \\ &= (f(y) \cdot g(y)) \cdot h(y) \\ &= f(y) \cdot (g(y) \cdot h(y)) \\ &= f(y) \cdot (g * h)(y) \\ &= (f * (g * h))(y)\end{aligned}$$

for every $y \in Y$, and therefore $(f * g) * h = f * (g * h)$.

Also, let $f \in FC(Y,Z)$. Then, for the map $e' \in FC(Y,Z)$ we have:

$$f * e' = e' * f = f.$$

Thus, the map $e' \in FC(Y,Z)$ is the identity element.

Finally, for every $f \in FC(Y,Z)$ there exists the map $f^{-1} \in FC(Y,Z)$ such that

$$f * f^{-1} = f^{-1} * f = e'.$$

Thus, the map $f^{-1} \in FC(Y,Z)$ is the inverse element of f.

By the above the pair $(FC(Y,Z), *)$ is a group. □

2.7. Theorem. *Let (Y, τ_Y) be a fully stratified fuzzy topological space, and $(Z, \cdot, \tau_Z)$ a fuzzy topological group. If $(Z, \cdot)$ is an abelian group, then the pair $(FC(Y,Z), *)$ is an abelian group.*

Proof. By Theorem 2.6 the pair $(FC(Y,Z), *)$ is a group. Also, for every $f, g \in FC(Y,Z)$ we have:

$$(f * g)(y) = f(y) \cdot g(y) = g(y) \cdot f(y) = (g * f)(y),$$

for every $y \in Y$. Thus $f * g = g * f$, for every $f, g \in FC(Y,Z)$, and, therefore, the pair $(FC(Y,Z), *)$ is an abelian group. □

2.8. Definition. (see [12]) Let $(X, \cdot)$ be a group and $G \in I^X$. Then G is a *fuzzy group* in X if the following conditions are satisfied:

(i) $G(x \cdot y) \geq \min\{G(x), G(y)\}$, for all $x, y \in X$,
(ii) $G(x^{-1}) \geq G(x)$, for all $x \in X$.

2.9. Theorem. *Let (Y, τ_Y) be a fully stratified fuzzy topological space, $(Z, \cdot, \tau_Z)$ a fuzzy topological group, and $Z_1 \in I^Z$ a fuzzy group. Then, the fuzzy set $G \in I^{FC(Y,Z)}$ for which*

$$G(f) = \inf\{Z_1(f(y)) : y \in Y\}, \ f \in FC(Y,Z)$$

is a fuzzy group.

Proof. By Theorem 2.6, the pair $(FC(Y,Z), *)$ is a group.

Now, let $f, g \in FC(Y, Z)$. Then, we have:

$$\begin{aligned} G(f * g) &= \inf\{Z_1((f * g)(y)) : y \in Y\} \\ &= \inf\{Z_1(f(y) \cdot g(y)) : y \in Y\} \\ &\geq \inf\{\min\{Z_1(f(y), Z_1(g(y))\} : y \in Y\} \\ &\geq \min\{\inf\{Z_1(f(y)) : y \in Y\},\ \inf\{Z_1(g(y)) : y \in Y\}\} \\ &= \min\{G(f),\ G(g)\} \end{aligned}$$

Also, for the map $f^{-1} \in FC(Y, Z)$ we have:

$$\begin{aligned} G(f^{-1}) &= \inf\{Z_1(f^{-1}(y)) : y \in Y\} \\ &= \inf\{Z_1((f(y))^{-1}) : y \in Y\} \\ &\geq \inf\{Z_1(f(y)) : y \in Y\} \\ &= G(f). \end{aligned}$$

Thus, $G \in I^{FC(Y,Z)}$ is a fuzzy group. □

2.10. Definition. (see [3]) Let $U \in I^Z$ be a fuzzy open set of Z and p_y^r, where $r \in (0, 1]$ and $y \in Y$, a fuzzy point of Y. Then by $[p_y^r; U]$ we denote the following subset of $FC(Y, Z)$

$$[p_y^r; U] = \{f \in FC(Y, Z) : f(p_y^r) < U\}.$$

The *F-point open topology* $\tau_{\text{F-p-o}}$ on $FC(Y, Z)$ is the topology which has as a subbase the family $\mathcal{B} = \{[p_y^r; U] : U \in I^Z$ is a fuzzy open set of Z and p_y^r a fuzzy point of $Y\} \cup \{FC(Y, Z)\}$.

2.11. Theorem. *Let (Y, τ_Y) be a fully stratified fuzzy topological space, and $(Z, \cdot, \tau_Z)$ a fuzzy topological group. Then, the triad $(FC(Y, Z), *, \tau_{F\text{-}p\text{-}o})$ is a topological group.*

Proof. Clearly, we have:

(i) The pair $(FC(Y, Z), *)$ is a group (see Theorem 2.6), and
(ii) The pair $(FC(Y, Z), \tau_{\text{F-p-o}})$ is a topological space.

Now, let $f, g \in FC(Y, Z)$ and let $[p_y^r; W]$ be a subbasic neighbourhood of $f * g$ in $\tau_{\text{F-p-o}}$. We prove that there exist subbasic neighbourhoods $[p_{y_1}^{r_1}; U]$ and $[p_{y_1}^{r_1}; V]$ of f and g, respectively, such that

$$[p_{y_1}^{r_1}; U] * [p_{y_2}^{r_2}; V] \subseteq [p_y^r; W].$$

Since $f * g \in [p_y^r; W]$ we have $(f * g)(p_y^r) < W$ and, therefore,

$$r < W((f * g)(y)),$$

that is

$$r < W(f(y) \cdot g(y)).$$

Let us suppose that $r = 1 - m$. Then the fuzzy set W is a fuzzy open Q-neighbourhood of $p_{f(y)\cdot g(y)}^m$. Indeed, we have

$$r < W(f(y) \cdot g(y)),$$

that is

$$1 - m < W(f(y) \cdot g(y)).$$

Thus $1 < m + W(f(y) \cdot g(y))$ and therefore $p_{f(y)\cdot g(y)}^m \, q \, W$.

Now, since $(Z, \cdot, \tau_Z)$ is a fuzzy topological group, there exist fuzzy open Q-neighbourhoods U and V of $p^m_{f(y)}$ and $p^m_{g(y)}$, respectively such that:

$$U \bullet V \leq W.$$

We consider the subsets:

$$[p^{1-m}_y; U] \text{ and } [p^{1-m}_y; V]$$

of $FC(Y, Z)$. Clearly, the above subsets are subbasic neighbourhoods of f and g, respectively. We prove that:

$$[p^{1-m}_y; U] * [p^{1-m}_y; V] \subseteq [p^r_y; W].$$

Let $f_1 * g_1 \in [p^{1-m}_y; U] * [p^{1-m}_y; V]$. Then, we have:

$$1 - m < U(f_1(y)) \text{ and } 1 - m < V(g_1(y)).$$

Thus,

$$\begin{aligned} W((f_1 * g_1)(y)) &= W(f_1(y) \cdot g_1(y)) \\ &\geq (U \bullet V)(f_1(y) \cdot g_1(y)) \\ &= \sup\{\min\{U(z_1), V(z_2)\} : z_1 \cdot z_2 = f_1(y) \cdot g_1(y)\} \\ &\geq \min\{U(f_1(y)), V(g_1(y))\} \\ &> 1 - m. \end{aligned}$$

Hence $(f_1 * g_1)(p^r_y) = p^r_{(f_1*g_1)(y)} < W$ and therefore $f_1 * g_1 \in [p^r_y : W]$.

Finally, let $f \in [p^r_y; W]$. We prove that there exists a subbasic neighbourhood $[p^{r_1}_{y_1}; U]$ of f^{-1} such that:

$$[p^{r_1}_{y_1}; U]^{-1} \subseteq [p^r_y; W].$$

Since, $f \in [p^r_y; W]$ we have $r < W(f(y))$. Let $r = 1 - m$. Then the fuzzy set W is a fuzzy open Q-neighbourhood of $p^m_{f(y)}$ in Z. Since $(Z, \cdot, \tau_Z)$ is a fuzzy topological group there exists a fuzzy open Q-neigbourhood U of $p^m_{(f(y))^{-1}}$ such that:

$$U^{-1} \leq W.$$

Clearly, the subset $[p^{1-m}_y; U]$ is a neighbourhood of f^{-1}. We prove that

$$[p^{1-m}_y; U]^{-1} \subseteq W.$$

Let $f_1 \in [p^{1-m}_y; U]^{-1}$. Then $f_1^{-1} \in [p^{1-m}_y; U]$ and therefore $1 - m < U((f_1(y))^{-1})$. Thus $W(f_1(y)) \geq U^{-1}((f_1(y)) \geq U((f_1(y))^{-1}) > 1 - m$ and therefore:

$$[p^{1-m}_y; U]^{-1} \subseteq W,$$

as required. □

2.12. Definition. (see [2], and for a similar definition see [4], [5], [6], and [11]) Let Y, Z be two fixed fuzzy topological spaces, $U \in I^Z$ a fuzzy open set of Z, and $y \in Y$. Then, by $(y; U) \in I^{FC(Y,Z)}$ we denote the fuzzy set for which

$$(y; U)(f) = U(f(y)),$$

for every $f \in FC(Y, Z)$.

The *fuzzy point-open topology* τ_{fp} on $FC(Y, Z)$ is generated by fuzzy sets of the form:

$$(y; U),$$

where $y \in Y$ and $U \in I^Z$ is a fuzzy open set of Z.

2.13. Theorem. *Let (Y, τ_Y) be a fully stratified fuzzy topological space, and $(Z, \cdot, \tau_Z)$ a fuzzy topological group. Then, the triad $(FC(Y, Z), *, \tau_{fp})$ is a fuzzy topological group.*

Proof. Clearly, we have:

(i) The pair $(FC(Y,Z), *)$ is a group (see Theorem 2.6).
(ii) The pair $(FC(Y,Z), \tau_{\mathrm{fp}})$ is a fuzzy topological space.

Now, let $f, g \in FC(Y,Z)$ and let $(y;W)$ be a fuzzy subbasic Q-neighbourhood of p^r_{f*g} in τ_{fp}. We prove that there exist fuzzy subbasic Q-neighbourhoods $(y_1;U)$ and $(y_2;V)$ of p^r_f and p^r_g, respectively such that

$$(y_1;U) \bullet (y_2;V) \leq (y;W).$$

Since $p^r_{f*g}\, q\, (y;W)$ we have:

$$r + W((f*g)(y)) > 1,$$

that is

$$r + W(f(y) \cdot g(y)) > 1$$

and so

$$p^r_{f(y)\cdot g(y)}\ q\ W.$$

Thus, the fuzzy set W is a fuzzy open Q-neighbourhood of $p^r_{f(y)\cdot g(y)}$. Now, since $(Z, \cdot, \tau_Z)$ is a fuzzy topological group there exist fuzzy open Q-neighbourhoods U and V of $p^r_{f(y)}$ and $p^r_{g(y)}$ such that

$$U \bullet V \leq W.$$

We consider the fuzzy sets $(y;U)$ and $(y;V)$. Clearly, the fuzzy sets $(y;U)$ and $(y;V)$ are Q-neighbourhoods of p^r_f and p^r_g, respectively. We prove that:

$$(y;U) \bullet (y;V) \leq (y;W).$$

Indeed, let $f \in FC(Y,Z)$. Then, we have:

$$\begin{aligned} ((y;U) \bullet (y;V))(f) &= \sup\{\min\{(y;U)(f_1), (y;V)(f_2)\} : f_1 * f_2 = f\} \\ &= \sup\{\min\{U(f_1(y)), V(f_2(y))\} : f_1 * f_2 = f\} \\ &\leq \sup\{\min\{U(z_1), V(z_2)\} : z_1 \cdot z_2 = f(y)\} \\ &= (U \bullet V)(f(y)) \leq W(f(y)) = (y;W)(f). \end{aligned}$$

Also, it is not difficult to prove that for every $f \in FC(Y,Z)$ and for every fuzzy open Q-neighbourhood $(y;V)$ of $p^r_{f^{-1}}$, there exists a fuzzy open Q-neighbourhood $(y_1;U)$ of p^r_f such that

$$(y_1;U)^{-1} \leq (y;V).$$

Thus, the triad $(FC(Y,Z), *, \tau_{\mathrm{fp}})$ is a fuzzy topological group. □

2.14. Theorem. *Let (Y, τ_Y) be a fully stratified fuzzy topological space, $(Z, \cdot, \tau_Z)$ a fuzzy topological group, and (Z, τ_Z) a fully stratified space. Then, the triad $(FC(Y,Z), *, \tau_{fp})$ is a fuzzy topological group and the pair $(FC(Y,Z), \tau_{fp})$ is a fully stratified space.*

Proof. By Theorem 2.13 the triad $(FC(Y,Z), *, \tau_{\mathrm{fp}})$ is a fuzzy topological group.

Now, let $r \in [0,1]$ and denote by r^* the constant fuzzy subset of Z with value r. If y is a fixed but arbitrary element of Y then for any $f \in FC(Y,Z)$ we have

$$(y;r^*)(f) = r^*(f(y)) = r,$$

and so $(y;f^*)$ is the constant fuzzy subset of $FC(Y,Z)$ with value r. Since $(y;r^*) \in \tau_{\mathrm{fp}}$ it follows that $(FC(Y,Z), \tau_{\mathrm{fp}})$ is fully stratified. □

2.15. Theorem. *Let (Y, τ_Y) be a fully stratified fuzzy topological space, $(Z, \cdot, \tau_Z)$ a fuzzy topological group, and (Z, τ_Z) a fully stratified space. Then the mappings $F(f) = f * a$, $G(f) = a * f$, and $H(f) = f^{-1}$ are all homeomorphic mappings of $FC(Y, Z)$ onto itself, where $a \in FC(Y, Z)$ is a definite point.*

Proof. By Theorem 2.14 the pair $(FC(Y, Z), \tau_{\rm fp})$ is a fully stratified space. Thus, by Proposition 2.4 of [8] the maps F, G, and H are all homeomorphic mappings of $FC(Y, Z)$ onto itself. □

2.16. Theorem. *Let (Y, τ_Y) be a fully stratified fuzzy topological space, $(Z, \cdot, \tau_Z)$ a fuzzy topological group, and (Z, τ_Z) a fully stratified space. Then for every fuzzy points p^r_f and p^r_g of $FC(Y, Z)$, there exists a homeomorphic mapping F of $FC(Y, Z)$ onto itself such that $F(p^r_f) = p^r_g$. This property is called the homogeneity of the fuzzy topological group $(FC(Y, Z), *, \tau_{fp})$.*

Proof. By Theorem 2.14 the pair $(FC(Y, Z), \tau_{\rm fp})$ is a fully stratified space. Thus, by Propositions 2.4 and 2.6 of [8], for all fuzzy points p^r_f and p^r_g of $FC(Y, Z)$, there exists a homeomorphic mapping F of $FC(Y, Z)$ onto itself such that $F(p^r_f) = p^r_g$. □

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