Hacettepe Journal of Mathematics and Statistics
Volume 44 (2) (2015), 351 – 362

Periodic and subharmonic solutions for a 2nth-order nonlinear difference equation

Xia Liu* , Yuanbiao Zhang† and Haiping Shi‡

Abstract

By using the critical point method, some new criteria are obtained for the existence and multiplicity of periodic and subharmonic solutions to a 2nth-order nonlinear difference equation. The proof is based on the Linking Theorem in combination with variational technique. Our results generalize and improve some known ones.

2000 AMS Classification: 39A11.

Keywords: Periodic and subharmonic solutions; 2nth-order; Nonlinear difference equations; Discrete variational theory.

Received 02 /12 /2013 : Accepted 06 /02 /2014 Doi : 10.15672/HJMS.2015449097

---

*Oriental Science and Technology College, Hunan Agricultural University, Changsha 410128, China.
Science College, Hunan Agricultural University, Changsha 410128, China
Email: xia991002@163.com. Corresponding author.
†Packaging Engineering Institute, Jinan University, Zhuhai 519070, China.
‡Modern Business and Management Department, Guangdong Construction Vocational Technology Institute, Guangzhou 510440, China.
This project is supported by the Specialized Research Fund for the Doctoral Program of Higher Eduction of China (No. 20114410110002), National Natural Science Foundation of China (No. 11101098), Natural Science Foundation of Guangdong Province (No. S2013010014460), Science and Research Program of Hunan Provincial Science and Technology Department (No. 2012FJ4109) and Scientific Research Fund of Hunan Provincial Education Department (No. 12C0170).

1. Introduction

Existence of periodic solutions of higher-order differential equations has been the subject of many investigations [8,19-21,34,38,39]. By using various methods and techniques, such as fixed point theory, the Kaplan-Yorke method, critical point theory, coincidence degree theory, bifurcation theory and dynamical system theory etc., a series of existence results for periodic solutions have been obtained in the literature. Difference equations, the discrete analogs of differential equations, occur widely in numerous settings and forms, both in mathematics itself and in its applications to statistics, computing, electrical circuit analysis, dynamical systems, economics, biology and other fields. For the general background of difference equations, one can refer to monographs [1,3,4,31]. Since the last decade, there has been much progress on the qualitative properties of difference equations, which included results on stability and attractivity [22,31,33,48] and results on oscillation and other topics [1-4,7,11-15,17,18,28-30,32,44-47]. Only a few papers discuss the periodic solutions of higher-order difference equations. Therefore, it is worthwhile to explore this topic.

Let $\mathbf{N}$, $\mathbf{Z}$ and $\mathbf{R}$ denote the sets of all natural numbers, integers and real numbers respectively. For a, $b \in \mathbf{Z}$, define $\mathbf{Z}(a) = \{a, a+1, \cdots\}$, $\mathbf{Z}(a, b) = \{a, a+1, \cdots, b\}$ when $a \leq b$. $*$ denotes the transpose of a vector.

In this paper, we consider the following forward and backward difference equation

$$\Delta^n \left(r_{k-n}\Delta^n u_{k-n}\right) = (-1)^n f(k, u_{k+1}, u_k, u_{k-1}),\ n \in \mathbf{Z}(3),\ k \in \mathbf{Z}, \tag{1.1}$$

where Δ is the forward difference operator $\Delta u_k = u_{k+1} - u_k$, $\Delta^n u_k = \Delta(\Delta^{n-1} u_k)$, r_k is real valued for each $k \in \mathbf{Z}$, $f \in C(\mathbf{Z} \times \mathbf{R}^3, \mathbf{R})$, r_k and $f(k, v_1, v_2, v_3)$ are T-periodic in k for a given positive integer T.

We may think of (1.1) as a discrete analogue of the following $2n$th-order functional differential equation

$$\frac{d^n}{dt^n}\left[r(t)\frac{d^n u(t)}{dt^n}\right] = (-1)^n f(t, u(t+1), u(t), u(t-1)),\ t \in \mathbf{R}. \tag{1.2}$$

Equations similar in structure to (1.2) arise in the study of the existence of solitary waves of lattice differential equations, see Smets and Willem [42].

The widely used tools for the existence of periodic solutions of difference equations are the various fixed point theorems in cones [1,3,4,27]. It is well known that critical point theory is a powerful tool that deals with the problems of differential equations [8,10,16,25,26,43]. Only since 2003, critical point theory has been employed to establish sufficient conditions on the existence of periodic solutions of difference equations. By using the critical point theory, Guo and Yu [28-30] and Shi *et al.*[41] established sufficient conditions on the existence of periodic solutions of second-order nonlinear difference equations. Compared to first-order or second-order difference equations, the study of higher-order equations has received considerably less attention (see, for example, [1,5,6,11-15,17,18,23,31,35,37] and the references contained therein). Ahlbrandt and Peterson [5] in 1994 studied the $2n$th-order difference equation of the form,

$$\sum_{i=0}^{n} \Delta^i \left(r_i(k-i)\Delta^i u(k-i)\right) = 0 \tag{1.3}$$

in the context of the discrete calculus of variations, and Peil and Peterson [37] studied the asymptotic behavior of solutions of (1.3) with $r_i(k) \equiv 0$ for $1 \leq i \leq n-1$. In 1998, Anderson [6] considered (1.3) for $k \in \mathbf{Z}(a)$, and obtained a formulation of generalized zeros and (n, n)-disconjugacy for (1.3). Migda [35] in 2004 studied an mth-order linear difference equation. In 2007, Cai and Yu [9] have obtained some criteria for the existence of periodic solutions of a $2n$th-order difference equation

(1.4) $$\Delta^n\left(r_{k-n}\Delta^n u_{k-n}\right)+f(k,u_k)=0,\ n\in \mathbf{Z}(3),\ k\in\mathbf{Z},$$

for the case where f grows superlinearly at both 0 and ∞. However, to the best of our knowledge, the results on periodic solutions of higher-order nonlinear difference equations are very scarce in the literature. Furthermore, since (1.1) contains both advance and retardation, there are very few manuscripts dealing with this subject. The main purpose of this paper is to give some sufficient conditions for the existence and multiplicity of periodic and subharmonic solutions to a $2n$th-order nonlinear difference equation. The main approach used in our paper is a variational technique and the Linking Theorem. Particularly, our results not only generalize the results in the literature [9], but also improve them. In fact, one can see the following Remarks 1.2 and 1.4 for details. The motivation for the present work stems from the recent papers in [13,24].

Let

$$\underline{r}=\min_{k\in\mathbf{Z}(1,T)}\{r_k\},\ \bar{r}=\max_{k\in\mathbf{Z}(1,T)}\{r_k\}.$$

Our main results are as follows.

Theorem 1.1. *Assume that the following hypotheses are satisfied:*
(r) $r_k>0,\forall k\in \boldsymbol{Z}$;
(F_1) *there exists a functional* $F(k,v_1,v_2)\in C^1(\boldsymbol{Z}\times\boldsymbol{R}^2,\boldsymbol{R})$ *with* $F(k,v_1,v_2)\geq 0$ *and it satisfies*

$$F(k+T,v_1,v_2)=F(k,v_1,v_2),$$
$$\frac{\partial F(k-1,v_2,v_3)}{\partial v_2}+\frac{\partial F(k,v_1,v_2)}{\partial v_2}=f(k,v_1,v_2,v_3);$$

(F_2) *there exist constants* $\delta_1>0$, $\alpha\in\left(0,\frac{1}{4}\underline{r}\lambda_{\min}^n\right)$ *such that*

$$F(k,v_1,v_2)\leq\alpha\left(v_1^2+v_2^2\right),\ \text{for } k\in\boldsymbol{Z} \text{ and } v_1^2+v_2^2\leq\delta_1^2;$$

(F_3) *there exist constants* $\rho_1>0$, $\zeta>0$, $\beta\in\left(\frac{1}{4}\bar{r}\lambda_{\max}^n,+\infty\right)$ *such that*

$$F(k,v_1,v_2)\geq\beta\left(v_1^2+v_2^2\right)-\zeta,\ \text{for } k\in\boldsymbol{Z} \text{ and } v_1^2+v_2^2\geq\rho_1^2,$$

where $\lambda_{\min}$, $\lambda_{\max}$ *are constants which can be referred to (2.7).*

Then for any given positive integer $m>0$*, (1.1) has at least three* mT*-periodic solutions.*

Remark 1.1. By (F_3) it is easy to see that there exists a constant $\zeta'>0$ such that

(F_3') $F(k,v_1,v_2)\geq\beta\left(v_1^2+v_2^2\right)-\zeta',\ \forall(k,v_1,v_2)\in\mathbf{Z}\times\mathbf{R}^2$.

As a matter of fact, let $\zeta_1=\max\left\{\left|F(k,v_1,v_2)-\beta\left(v_1^2+v_2^2\right)+\zeta\right|:k\in\mathbf{Z},v_1^2+v_2^2\leq\rho_1^2\right\}$, $\zeta'=\zeta+\zeta_1$, we can easily get the desired result.

Corollary 1.1. *Assume that* (r) *and* $(F_1)-(F_3)$ *are satisfied. Then for any given positive integer* $m>0$*, (1.1) has at least two nontrivial* mT*-periodic solutions.*

Remark 1.2. Corollary 1.1 reduces to Theorem 1.1 in [9].

Theorem 1.2. *Assume that* (r), (F_1) *and the following conditions are satisfied:*

(F_4) $\lim\limits_{\rho\to 0}\frac{F(k,v_1,v_2)}{\rho^2}=0,\ \rho=\sqrt{v_1^2+v_2^2},\ \forall(k,v_1,v_2)\in\boldsymbol{Z}\times\boldsymbol{R}^2$;

(F_5) *there exist constants* $R_1>0$ *and* $\theta>2$ *such that for* $k\in\boldsymbol{Z}$ *and* $v_1^2+v_2^2\geq R_1^2$,

$$0<\theta F(k,v_1,v_2)\leq\frac{\partial F(k,v_1,v_2)}{\partial v_1}v_1+\frac{\partial F(k,v_1,v_2)}{\partial v_2}v_2.$$

Then for any given positive integer $m > 0$, (1.1) has at least three mT-periodic solutions.

Remark 1.3. Assumption (F_5) implies that there exist constants $a_1 > 0$ and $a_2 > 0$ such that

(F_5') $F(k, v_1, v_2) \geq a_1 \left(\sqrt{v_1^2 + v_2^2}\right)^\theta - a_2, \ \forall (k, v_1, v_2) \in \mathbf{Z} \times \mathbf{R}^2.$

Corollary 1.2. *Assume that (r) and $(F_1), (F_4), (F_5)$ are satisfied. Then for any given positive integer $m > 0$, (1.1) has at least two nontrivial mT-periodic solutions.*

If $f(k, u_{k+1}, u_k, u_{k-1}) = q_k g(u_k)$, (1.1) reduces to the following $2n$th-order nonlinear equation,

(1.5) $$\Delta^n (r_{k-n} \Delta^n u_{k-n}) = (-1)^n q_k g(u_k), \ k \in \mathbf{Z},$$

where $g \in C(\mathbf{R}, \mathbf{R}), q_{k+T} = q_k > 0$, for all $k \in \mathbf{Z}$. Then, we have the following results.

Theorem 1.3. *Assume that (r) and the following hypotheses are satisfied:*
(G_1) *there exists a functional $G(v) \in C^1(\boldsymbol{R}, \boldsymbol{R})$ with $G(v) \geq 0$ and it satisfies*

$$G'(v) = g(v),$$

(G_2) *there exist constants $\delta_2 > 0$, $\alpha \in \left(0, \frac{1}{2}\underline{r}\lambda_{\min}^n\right)$ such that*

$$G(v) \leq \alpha|v|^2, \text{ for } |v| \leq \delta_2;$$

(G_3) *there exist constants $\rho_2 > 0$, $\zeta > 0$, $\beta \in \left(\frac{1}{2}\bar{r}\lambda_{\max}^n, +\infty\right)$ such that*

$$G(v) \geq \beta|v|^2 - \zeta, \text{ for } |v| \geq \rho_2,$$

where $\lambda_{\min}, \ \lambda_{\max}$ are constants which can be referred to (2.7).

Then for any given positive integer $m > 0$, (1.5) has at least three mT-periodic solutions.

Corollary 1.3. *Assume that (r) and $(G_1) - (G_3)$ are satisfied. Then for any given positive integer $m > 0$, (1.5) has at least two nontrivial mT-periodic solutions.*

Remark 1.4. Corollary 1.3 reduces to Corollary 1.1 in [9].

The rest of the paper is organized as follows. First, in Section 2, we shall establish the variational framework associated with (1.1) and transfer the problem of the existence of periodic solutions of (1.1) into that of the existence of critical points of the corresponding functional. Some related fundamental results will also be recalled. Then, in Section 3, we shall complete the proof of the results by using the critical point method. Finally, in Section 4, we shall give an example to illustrate the main result.

For the basic knowledge of variational methods, the reader is referred to [27,34,36,40].

2. Variational structure and some lemmas

In order to apply the critical point theory, we shall establish the corresponding variational framework for (1.1) and give some lemmas which will be of fundamental importance in proving our main results. First, we state some basic notations.

Let S be the set of sequences $u = (\cdots, u_{-k}, \cdots, u_{-1}, u_0, u_1, \cdots, u_k, \cdots) = \{u_k\}_{k=-\infty}^{+\infty}$, that is

$$S = \{\{u_k\} | u_k \in \mathbf{R}, \ k \in \mathbf{Z}\}.$$

For any $u, v \in S$, $a, b \in \mathbf{R}$, $au + bv$ is defined by

$$au + bv = \{au_k + bv_k\}_{k=-\infty}^{+\infty}.$$

Then S is a vector space.

For any given positive integers m and T, E_{mT} is defined as a subspace of S by

$$E_{mT} = \{u \in S | u_{k+mT} = u_k, \ \forall k \in \mathbf{Z}\}.$$

Clearly, E_{mT} is isomorphic to $\mathbf{R}^{mT}$. E_{mT} can be equipped with the inner product

$$\langle u, v\rangle = \sum_{j=1}^{mT} u_j v_j, \ \forall u, v \in E_{mT}, \tag{2.1}$$

by which the norm $\|\cdot\|$ can be induced by

$$\|u\| = \left(\sum_{j=1}^{mT} u_j^2\right)^{\frac{1}{2}}, \ \forall u \in E_{mT}. \tag{2.2}$$

It is obvious that E_{mT} with the inner product (2.1) is a finite dimensional Hilbert space and linearly homeomorphic to $\mathbf{R}^{mT}$.

On the other hand, we define the norm $\|\cdot\|_s$ on E_{mT} as follows:

$$\|u\|_s = \left(\sum_{j=1}^{mT} |u_j|^s\right)^{\frac{1}{s}}, \tag{2.3}$$

for all $u \in E_{mT}$ and $s > 1$.

Since $\|u\|_s$ and $\|u\|_2$ are equivalent, there exist constants $c_1, \ c_2$ such that $c_2 \geq c_1 > 0$, and

$$c_1\|u\|_2 \leq \|u\|_s \leq c_2\|u\|_2, \ \forall u \in E_{mT}. \tag{2.4}$$

Clearly, $\|u\| = \|u\|_2$. For all $u \in E_{mT}$, define the functional J on E_{mT} as follows:

$$J(u) = \frac{1}{2}\sum_{k=1}^{mT} r_{k-1}\left(\Delta^n u_{k-1}\right)^2 - \sum_{k=1}^{mT} F(k, u_{k+1}, u_k), \tag{2.5}$$

where

$$\frac{\partial F(k-1, v_2, v_3)}{\partial v_2} + \frac{\partial F(k, v_1, v_2)}{\partial v_2} = f(k, v_1, v_2, v_3).$$

Clearly, $J \in C^1(E_{mT}, \mathbf{R})$ and for any $u = \{u_k\}_{k\in\mathbf{Z}} \in E_{mT}$, by using $u_0 = u_{mT}, \ u_1 = u_{mT+1}$, we can compute the partial derivative as

$$\frac{\partial J}{\partial u_k} = (-1)^n \Delta^n \left(r_{k-n}\Delta^n u_{k-n}\right) - f(k, u_{k+1}, u_k, u_{k-1}).$$

Thus, u is a critical point of J on E_{mT} if and only if

$$\Delta^n \left(r_{k-n}\Delta^n u_{k-n}\right) = (-1)^n f(k, u_{k+1}, u_k, u_{k-1}), \ \forall k \in \mathbf{Z}(1, mT).$$

Due to the periodicity of $u = \{u_k\}_{k\in\mathbf{Z}} \in E_{mT}$ and $f(k, v_1, v_2, v_3)$ in the first variable k, we reduce the existence of periodic solutions of (1.1) to the existence of critical points of J on E_{mT}. That is, the functional J is just the variational framework of (1.1).

Let P be the $mT \times mT$ matrix defined by

$$P = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ -1 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}.$$

By matrix theory, we see that the eigenvalues of P are

$$\lambda_j = 2\left(1 - \cos\frac{2j}{mT}\pi\right), j = 0, 1, 2, \cdots, mT-1. \tag{2.6}$$

Thus, $\lambda_0 = 0, \lambda_1 > 0, \lambda_2 > 0, \cdots, \lambda_{mT-1} > 0$. Therefore,

$$\left.\begin{aligned} &\lambda_{\min} = \min\{\lambda_1, \lambda_2, \cdots, \lambda_{mT-1}\} = 2\left(1 - \cos\tfrac{2}{mT}\pi\right), \\ &\lambda_{\max} = \max\{\lambda_1, \lambda_2, \cdots, \lambda_{mT-1}\} = \begin{cases} 4, & \text{when mT is even,} \\ 2\left(1 + \cos\tfrac{1}{mT}\pi\right), & \text{when mT is odd.} \end{cases} \end{aligned}\right\} \tag{2.7}$$

Let

$$W = \ker P = \{u \in E_{mT} | Pu = 0 \in \mathbf{R}^{mT}\}.$$

Then

$$W = \{u \in E_{mT} | u = \{c\}, \ c \in \mathbf{R}\}.$$

Let V be the direct orthogonal complement of E_{mT} to W, i.e., $E_{mT} = V \oplus W$. For convenience, we identify $u \in E_{mT}$ with $u = (u_1, u_2, \cdots, u_{mT})^*$.

Let E be a real Banach space, $J \in C^1(E, \mathbf{R})$, i.e., J is a continuously Fréchet-differentiable functional defined on E. J is said to satisfy the Palais-Smale condition (P.S. condition for short) if any sequence $\left\{u^{(i)}\right\} \subset E$ for which $\left\{J\left(u^{(i)}\right)\right\}$ is bounded and $J'\left(u^{(i)}\right) \to 0 (i \to \infty)$ possesses a convergent subsequence in E.

Let B_ρ denote the open ball in E about 0 of radius ρ and let ∂B_ρ denote its boundary.

Lemma 2.1 (Linking Theorem [40]). *Let E be a real Banach space, $E = E_1 \oplus E_2$, where E_1 is finite dimensional. Suppose that $J \in C^1(E, \boldsymbol{R})$ satisfies the P.S. condition and*
(J_1) there exist constants $a > 0$ and $\rho > 0$ such that $J|_{\partial B_\rho \cap E_2} \geq a$;
(J_2) there exists an $e \in \partial B_1 \cap E_2$ and a constant $R_0 \geq \rho$ such that $J|_{\partial Q} \leq 0$, where $Q = (\bar{B}_{R_0} \cap E_1) \oplus \{se | 0 < s < R_0\}$.
Then J possesses a critical value $c \geq a$, where

$$c = \inf_{h \in \Gamma} \sup_{u \in Q} J(h(u)),$$

and $\Gamma = \{h \in C(\bar{Q}, E) \mid h|_{\partial Q} = id\}$, where id denotes the identity operator.

Lemma 2.2. *Assume that (r), (F_1) and (F_3) are satisfied. Then the functional J is bounded from above in E_{mT}.*

Proof. By (F_3') and (2.4), for any $u \in E_{mT}$,

$$\begin{aligned} J(u) &= \frac{1}{2}\sum_{k=1}^{mT} r_{k-1}\left(\Delta^n u_{k-1}, \Delta^n u_{k-1}\right) - \sum_{k=1}^{mT} F(k, u_{k+1}, u_k) \\ &= \frac{1}{2}\sum_{k=1}^{mT} r_k\left(\Delta^n u_k, \Delta^n u_k\right) - \sum_{k=1}^{mT} F(k, u_{k+1}, u_k) \\ &\leq \frac{\bar{r}}{2} x^* P x - \sum_{k=1}^{mT}\left[\beta(u_{k+1}^2 + u_k^2) - \zeta'\right] \\ &\leq \frac{\bar{r}}{2}\lambda_{\max}\|x\|_2^2 - 2\beta\|u\|_2^2 + mT\zeta', \end{aligned}$$

where $x = (\Delta^{n-1}u_1, \Delta^{n-1}u_2, \cdots, \Delta^{n-1}u_{mT})^*$. Since

$$\|x\|_2^2 = \sum_{k=1}^{mT}\left(\Delta^{n-2}u_{k+1} - \Delta^{n-2}u_k\right)^2 \leq \lambda_{\max}\sum_{k=1}^{mT}\left(\Delta^{n-2}u_k\right)^2 \leq \lambda_{\max}^{n-1}\|u\|_2^2,$$

we have

$$J(u) \le \left(\frac{\bar{r}}{2}\lambda_{\max}^n - 2\beta\right)\|u\|_2^2 + mT\zeta' \le mT\zeta'.$$

The proof of Lemma 2.2 is complete. □

Remark 2.1. The case $mT = 1$ is trivial. For the case $mT = 2$, P has a different form, namely,

$$P = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

However, in this special case, the argument need not to be changed and we omit it.

Lemma 2.3. *Assume that* (r), (F_1) *and* (F_3) *are satisfied. Then the functional* J *satisfies the P.S. condition.*

Proof. Let $\left\{J\left(u^{(i)}\right)\right\}$ be a bounded sequence from the lower bound, i.e., there exists a positive constant M_1 such that

$$-M_1 \le J\left(u^{(i)}\right), \ \forall i \in \mathbf{N}.$$

By the proof of Lemma 2.2, it is easy to see that

$$-M_1 \le J\left(u^{(i)}\right) \le \left(\frac{\bar{r}}{2}\lambda_{\max}^n - 2\beta\right)\left\|u^{(i)}\right\|_2^2 + mT\zeta', \ \forall i \in \mathbf{N}.$$

Therefore,

$$\left(2\beta - \frac{\bar{r}}{2}\lambda_{\max}^n\right)\left\|u^{(i)}\right\|_2^2 \le M_1 + mT\zeta'.$$

Since $\beta > \frac{1}{4}\bar{r}\lambda_{\max}^n$, it is not difficult to know that $\left\{u^{(i)}\right\}$ is a bounded sequence in E_{mT}. As a consequence, $\left\{u^{(i)}\right\}$ possesses a convergence subsequence in E_{mT}. Thus the P.S. condition is verified. □

3. Proof of the main results

In this Section, we shall prove our main results by using the critical point theory.

3.1. Proof of Theorem 1.1

Assumptions (F_1) and (F_2) imply that $F(k, 0) = 0$ and $f(k, 0) = 0$ for $k \in \mathbf{Z}$. Then $u = 0$ is a trivial mT-periodic solution of (1.1).

By Lemma 2.2, J is bounded from the upper on E_{mT}. We define $c_0 = \sup_{u \in E_{mT}} J(u)$. The proof of Lemma 2.2 implies $\lim_{\|u\|_2 \to +\infty} J(u) = -\infty$. This means that $-J(u)$ is coercive. By the continuity of $J(u)$, there exists $\bar{u} \in E_{mT}$ such that $J(\bar{u}) = c_0$. Clearly, $\bar{u}$ is a critical point of J.

We claim that $c_0 > 0$. Indeed, by (F_2), for any $u \in V$, $\|u\|_2 \le \delta_1$, we have

$$\begin{aligned}
J(u) &= \frac{1}{2}\sum_{k=1}^{mT} r_{k-1}\left(\Delta^n u_{k-1}, \Delta^n u_{k-1}\right) - \sum_{k=1}^{mT} F(k, u_{k+1}, u_k) \\
&= \frac{1}{2}\sum_{k=1}^{mT} r_k\left(\Delta^n u_k, \Delta^n u_k\right) - \sum_{k=1}^{mT} F(k, u_{k+1}, u_k) \\
&\ge \frac{1}{2}\underline{r}x^* Px - \alpha\sum_{k=1}^{mT}(u_{k+1}^2 + u_k^2)
\end{aligned}$$

$$\geq \frac{1}{2}\underline{r}\lambda_{\min}\|x\|_2^2 - 2\alpha\|u\|_2^2,$$

where $x = (\Delta^{n-1}u_1, \Delta^{n-1}u_2, \cdots, \Delta^{n-1}u_{mT})^*$. Since

$$\|x\|_2^2 = \sum_{k=1}^{mT}\left(\Delta^{n-2}u_{k+1} - \Delta^{n-2}u_k\right)^2 \geq \lambda_{\min}\sum_{k=1}^{mT}\left(\Delta^{n-2}u_k\right)^2 \geq \lambda_{\min}^{n-1}\|u\|_2^2,$$

we have

$$J(u) \geq \left(\frac{1}{2}\underline{r}\lambda_{\min}^n - 2\alpha\right)\|u\|_2^2.$$

Take $\sigma = \left(\frac{1}{2}\underline{r}\lambda_{\min}^n - 2\alpha\right)\delta_1^2$. Then

$$J(u) \geq \sigma, \ \forall u \in V \cap \partial B_{\delta_1}.$$

Therefore, $c_0 = \sup_{u\in E_{mT}} J(u) \geq \sigma > 0$. At the same time, we have also proved that there exist constants $\sigma > 0$ and $\delta_1 > 0$ such that $J|_{\partial B_{\delta_1}\cap V} \geq \sigma$. That is to say, J satisfies the condition (J_1) of the Linking Theorem.

Noting that $\sum_{k=1}^{mT} r_{k-1}\left(\Delta^n u_{k-1}\right)^2 = 0$, for all $u \in W$, we have

$$J(u) = \frac{1}{2}\sum_{k=1}^{mT} r_{k-1}\left(\Delta^n u_{k-1}\right)^2 - \sum_{k=1}^{mT} F(k, u_{k+1}, u_k) = -\sum_{k=1}^{mT} F(k, u_{k+1}, u_k) \leq 0.$$

Thus, the critical point $\bar{u}$ of J corresponding to the critical value c_0 is a nontrivial mT-periodic solution of (1.1).

In order to obtain another nontrivial mT-periodic solution of (1.1) different from $\bar{u}$, we need to use the conclusion of Lemma 2.1. We have known that J satisfies the P.S. condition on E_{mT}. In the following, we shall verify the condition (J_2).

Take $e \in \partial B_1 \cap V$, for any $z \in W$ and $s \in \mathbf{R}$, let $u = se + z$. Then

$$\begin{aligned}
J(u) &= \frac{1}{2}\sum_{k=1}^{mT} r_k\left(\Delta^n u_k, \Delta^n u_k\right) - \sum_{k=1}^{mT} F(k, u_{k+1}, u_k) \\
&\leq \frac{\bar{r}}{2}s^2\sum_{k=1}^{mT}\left(\Delta^n e_k, \Delta^n e_k\right) - \sum_{k=1}^{mT} F(k, se_{k+1} + z_{k+1}, se_k + z_k) \\
&\leq \frac{\bar{r}}{2}s^2 y^* P y - \sum_{k=1}^{mT}\left\{\beta\left[(se_{k+1} + z_{k+1})^2 + (se_k + z_k)^2\right] - \zeta'\right\} \\
&\leq \frac{\bar{r}}{2}s^2\lambda_{\max}\|y\|_2^2 - 2\beta\sum_{k=1}^{mT}(se_k + z_k)^2 + mT\zeta' \\
&= \frac{\bar{r}}{2}s^2\lambda_{\max}\|y\|_2^2 - 2\beta s^2 - 2\beta\|z\|_2^2 + mT\zeta',
\end{aligned}$$

where $y = (\Delta^{n-1}e_1, \Delta^{n-1}e_2, \cdots, \Delta^{n-1}e_{mT})^*$. Since

$$\|y\|_2^2 = \sum_{k=1}^{mT}\left(\Delta^{n-2}e_{k+1} - \Delta^{n-2}e_k\right)^2 \leq \lambda_{\max}\sum_{k=1}^{mT}\left(\Delta^{n-2}e_k\right)^2 \leq \lambda_{\max}^{n-1},$$

we have

$$J(u) \leq \left(\frac{\bar{r}}{2}\lambda_{\max}^n - 2\beta\right)s^2 - 2\beta\|z\|_2^2 + mT\zeta' \leq -2\beta\|z\|_2^2 + mT\zeta'.$$

Thus, there exists a positive constant $R_2 > \delta_1$ such that for any $u \in \partial Q$, $J(u) \le 0$, where $Q = (\bar{B}_{R_2} \cap W) \oplus \{se | 0 < s < R_2\}$. By the Linking Theorem, J possesses a critical value $c \ge \sigma > 0$, where

$$c = \inf_{h \in \Gamma} \sup_{u \in Q} J(h(u)),$$

and $\Gamma = \{h \in C(\bar{Q}, E_{mT}) \mid h|_{\partial Q} = id\}$.

Let $\tilde{u} \in E_{mT}$ be a critical point associated to the critical value c of J, i.e., $J(\tilde{u}) = c$. If $\tilde{u} \ne \bar{u}$, then the conclusion of Theorem 1.1 holds. Otherwise, $\tilde{u} = \bar{u}$. Then $c_0 = J(\bar{u}) = J(\tilde{u}) = c$, that is $\sup\limits_{u \in E_{mT}} J(u) = \inf\limits_{h \in \Gamma} \sup\limits_{u \in Q} J(h(u))$. Choosing $h = id$, we have $\sup\limits_{u \in Q} J(u) = c_0$. Since the choice of $e \in \partial B_1 \cap V$ is arbitrary, we can take $-e \in \partial B_1 \cap V$. Similarly, there exists a positive number $R_3 > \delta_1$, for any $u \in \partial Q_1$, $J(u) \le 0$, where $Q_1 = (\bar{B}_{R_3} \cap W) \oplus \{-se | 0 < s < R_3\}$.

Again, by the Linking Theorem, J possesses a critical value $c' \ge \sigma > 0$, where

$$c' = \inf_{h \in \Gamma_1} \sup_{u \in Q_1} J(h(u)),$$

and $\Gamma_1 = \{h \in C(\bar{Q}_1, E_{mT}) \mid h|_{\partial Q_1} = id\}$.

If $c' \ne c_0$, then the proof is finished. If $c' = c_0$, then $\sup\limits_{u \in Q_1} J(u) = c_0$. Due to the fact $J|_{\partial Q} \le 0$ and $J|_{\partial Q_1} \le 0$, J attains its maximum at some points in the interior of sets Q and Q_1. However, $Q \cap Q_1 \subset W$ and $J(u) \le 0$ for any $u \in W$. Therefore, there must be a point $u' \in E_{mT},\ u' \ne \tilde{u}$ and $J(u') = c' = c_0$. The proof of Theorem 1.1 is complete. □

Remark 3.1. Similarly to above argument, we can also prove Theorems 1.2 and 1.3. For simplicity, we omit their proofs.

Remark 3.2. Due to Theorems 1.1, 1.2 and 1.3, the conclusion of Corollaries 1.1, 1.2 and 1.3 is obviously true.

4. Example

As an application of Theorem 1.1, we give an example to illustrate our main result.

Example 4.1. For all $n \in \mathbf{Z}(3),\ k \in \mathbf{Z}$, assume that

$$\Delta^n (r_{k-n} \Delta^n u_{k-n}) =$$

(4.1)

$$(-1)^n \mu u_k \left[\left(8 + \sin^2\left(\frac{\pi k}{T}\right)\right) \left(u_{k+1}^2 + u_k^2\right)^{\frac{\mu}{2}-1} + \left(8 + \sin^2\left(\frac{\pi(k-1)}{T}\right)\right) \left(u_k^2 + u_{k-1}^2\right)^{\frac{\mu}{2}-1} \right],$$

where r_k is real valued for each $k \in \mathbf{Z}$ and $r_{k+T} = r_k > 0$, $\mu > 2$, T is a given positive integer.

We have

$$f(k, v_1, v_2, v_3) =$$

$$\mu v_2 \left[\left(8 + \sin^2\left(\frac{\pi k}{T}\right)\right) \left(v_1^2 + v_2^2\right)^{\frac{\mu}{2}-1} + \left(8 + \sin^2\left(\frac{\pi(k-1)}{T}\right)\right) \left(v_2^2 + v_3^2\right)^{\frac{\mu}{2}-1} \right]$$

and

$$F(k, v_1, v_2) = \left[8 + \sin^2\left(\frac{\pi k}{T}\right)\right] \left(v_1^2 + v_2^2\right)^{\frac{\mu}{2}}.$$

Then

$$\frac{\partial F(k-1,v_2,v_3)}{\partial v_2}+\frac{\partial F(k,v_1,v_2)}{\partial v_2}$$

$$=\mu v_2\left[\left(8+\sin^2\left(\frac{\pi k}{T}\right)\right)\left(v_1^2+v_2^2\right)^{\frac{\mu}{2}-1}+\left(8+\sin^2\left(\frac{\pi(k-1)}{T}\right)\right)\left(v_2^2+v_3^2\right)^{\frac{\mu}{2}-1}\right].$$

It is easy to verify all the assumptions of Theorem 1.1 are satisfied. Consequently, for any given positive integer $m>0$, (4.1) has at least three mT-periodic solutions.

References

[1] R.P. Agarwal. *Difference Equations and Inequalities: Theory, Methods and Applications.* Marcel Dekker, New York (1992).

[2] R.P. Agarwal, K. Perera and D. O'regan. *Multiple positive solutions of singular and nonsingular discrete problems via variational methods.* Nonlinear Anal., **58**(1-2) (2004) 69-73.

[3] R.P. Agarwal and P.J.Y. Wong. *Advanced Topics in Difference Equations.* Kluwer Academic Publishers, Dordrecht (1997).

[4] C.D. Ahlbrandt and A.C. Peterson. *Discrete Hamiltonian Systems: Difference Equations, Continued Fraction and Riccati Equations.* Kluwer Academic Publishers, Dordrecht (1996).

[5] C.D. Ahlbrandt and A.C. Peterson. *The (n,n)-disconjugacy of a 2nth-order linear difference equation.* Comput. Math. Appl., **28**(1-3) (1994) 1-9.

[6] D. Anderson. *A 2nth-order linear difference equation.* Comput. Math. Appl., **2**(4) (1998) 521-529.

[7] R.I. Avery and A.C. Pererson. *Three positive fixed points of nonlinear operators on ordered Banach space.* Comput. Math. Appl., **42**(3-5) (2001) 313-322.

[8] V. Benci and P.H. Rabinowitz. *Critical point theorems for indefinite functionals.* Invent Math., **52**(3) (1979) 241-273.

[9] X.C. Cai and J.S.Yu. *Existence of periodic solutions for a 2nth-order nonlinear difference equation.* J. Math. Anal. Appl., **329**(2) (2007) 870-878.

[10] K.C. Chang. *Infinite Dimensional Morse Theory and Multiple Solution Problems.* Birkhäuser, Boston (1993).

[11] P. Chen. *Existence of homoclinic orbits in discrete Hamilton systems without Palais-Smale condition.* J. Difference Equ. Appl., **19**(11) (2013) 1781-1794.

[12] P. Chen and H. Fang. *Existence of periodic and subharmonic solutions for second-order p-Laplacian difference equations.* Adv. Difference Equ., **2007** (2007) 1-9.

[13] P. Chen and X.H. Tang. *Existence and multiplicity of homoclinic orbits for 2nth-order nonlinear difference equations containing both many advances and retardations.* J. Math. Anal. Appl., **381**(2) (2011) 485-505.

[14] P. Chen and X.H. Tang. *Existence of infinitely many homoclinic orbits for fourth-order difference systems containing both advance and retardation.* Appl. Math. Comput., **217**(9) (2011) 4408-4415.

[15] P. Chen and X.H.Tang. *Existence of homoclinic solutions for the second-order discrete p-Laplacian systems.* Taiwanese J. Math., **15**(5) (2011) 2123-2143.

[16] P. Chen and X.H. Tang. *New existence and multiplicity of solutions for some Dirichlet problems with impulsive effects.* Math. Comput. Modelling, **55**(3-4) (2012) 723-739.

[17] P. Chen and X.H.Tang. *Infinitely many homoclinic solutions for the second-order discrete p-Laplacian systems.* Bull. Belg. Math. Soc., **20**(2) (2013) 193-212.

[18] P. Chen and X.H. Tang. *Existence of homoclinic solutions for some second-order discrete Hamiltonian systems.* J. Difference Equ. Appl., **19**(4) (2013) 633-648.

[19] F.H. Clarke. *Periodic solutions to Hamiltonian inclusions.* J. Differential Equations, **40**(1) (1981) 1-6.

[20] F.H. Clarke. *Periodic solutions of Hamilton's equations and local minima of the dual action.* Trans. Amer. Math. Soc., **287**(1) (1985) 239-251.

[21] G. Cordaro. *Existence and location of periodic solution to convex and non coercive Hamiltonian systems.* Discrete Contin. Dyn. Syst., **12**(5) (2005) 983-996.

[22] L.H. Erbe, H. Xia and J.S. Yu. *Global stability of a linear nonautonomous delay difference equations.* J. Difference Equ. Appl., **1**(2) (1995) 151-161.

[23] H. Fang and D.P. Zhao. *Existence of nontrivial homoclinic orbits for fourth-order difference equations.* Appl. Math. Comput., **214**(1) (2009) 163-170.

[24] C.J. Guo, D. O'Regan and R.P. Agarwal. *Existence of multiple periodic solutions for a class of first-order neutral differential equations.* Appl. Anal. Discrete Math., **5**(1) (2011) 147-158.

[25] C.J. Guo, D. O'Regan, Y.T. Xu and R.P. Agarwal. *Existence and multiplicity of homoclinic orbits of a second-order differential difference equation via variational methods.* Appl. Math. Inform. Mech., **4**(1) (2012) 1-15.

[26] C.J. Guo and Y.T. Xu. *Existence of periodic solutions for a class of second order differential equation with deviating argument.* J. Appl. Math. Comput., **28**(1-2) (2008) 425-433.

[27] D.J. Guo. *Nonlinear Functional Analysis.* Shandong Scientific Press, Jinan (1985).

[28] Z.M. Guo and J.S. Yu. *Applications of critical point theory to difference equations.* Fields Inst. Commun., **42** (2004) 187-200.

[29] Z.M. Guo and J.S. Yu. *Existence of periodic and subharmonic solutions for second-order superlinear difference equations.* Sci. China Math, **46**(4) (2003) 506-515.

[30] Z.M. Guo and J.S. Yu. *The existence of periodic and subharmonic solutions of subquadratic second order difference equations.* J. London Math. Soc., **68**(2) (2003) 419-430.

[31] V.L. Kocic and G. Ladas. *Global Behavior of Nonlinear Difference Equations of Higher Order with Applications.* Kluwer Academic Publishers, Dordrecht (1993).

[32] Y.J. Liu and W.G. Ge. *Twin positive solutions of boundary value problems for finite difference equations with p-Laplacian operator.* J. Math. Anal. Appl., **278**(2) (2003) 551-561.

[33] H. Matsunaga, T. Hara and S. Sakata. *Global attractivity for a nonlinear difference equation with variable delay.* Computers Math. Appl., **41**(5-6) (2001) 543-551.

[34] J. Mawhin and M. Willem. *Critical Point Theory and Hamiltonian Systems.* Springer, New York (1989).

[35] M. Migda. *Existence of nonoscillatory solutions of some higher order difference equations.* Appl. Math. E-notes, **4**(2) (2004) 33-39.

[36] A. Pankov and N. Zakhrchenko. *On some discrete variational problems.* Acta Appl. Math., **65**(1-3) (2001) 295-303.

[37] T. Peil and A. Peterson. *Asymptotic behavior of solutions of a two-term difference equation.* Rocky Mountain J. Math., **24**(1) (1994) 233-251.

[38] P.H. Rabinowitz. *Periodic solutions of Hamiltonian systems.* Comm. Pure Appl. Math., **31**(2) (1978) 157-184.

[39] P.H. Rabinowitz. *On subharmonic solutions of Hamiltonian systems.* Comm. Pure Appl. Math., **33**(5) (1980) 609-633.

[40] P.H. Rabinowitz. *Minimax Methods in Critical Point Theory with Applications to Differential Equations.* Amer. Math. Soc., Providence, RI, New York (1986).

[41] H.P. Shi, W.P. Ling, Y.H. Long and H.Q. Zhang. *Periodic and subharmonic solutions for second order nonlinear functional difference equations.* Commun. Math. Anal., **5**(2) (2008) 50-59.

[42] D. Smets and M. Willem. *Solitary waves with prescribed speed on infinite lattices.* J. Funct. Anal., **149**(1) (1997) 266-275.

[43] Y.T. Xu and Z.M. Guo. *Applications of a Z_p index theory to periodic solutions for a class of functional differential equations.* J. Math. Anal. Appl., **257**(1) (2001) 189-205.

[44] J.S. Yu and Z.M. Guo. *On boundary value problems for a discrete generalized Emden-Fowler equation.* J. Differential Equations, **231**(1) (2006) 18-31.

[45] J.S. Yu, Y.H. Long and Z.M. Guo. *Subharmonic solutions with prescribed minimal period of a discrete forced pendulum equation.* J. Dynam. Differential Equations, **16**(2) (2004) 575-586.

[46] Z. Zhou, J.S. Yu and Y. M. Chen. *Homoclinic solutions in periodic difference equations with saturable nonlinearity.* Sci. China Math, **54**(1) (2011) 83-93.

[47] Z. Zhou, J.S. Yu and Z.M. Guo. *Periodic solutions of higher-dimensional discrete systems.* Proc. Roy. Soc. Edinburgh (Section A), **134**(5) (2004) 1013-1022.

[48] Z. Zhou and Q. Zhang. *Uniform stability of nonlinear difference systems.* J. Math. Anal. Appl., **225**(2) (1998) 486-500.