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ON A CONNECTION BETWEEN THE THEORY OF TACHIBANA OPERATORS AND THE THEORY OF B-MANIFOLDS

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Abstract

The main purpose of the paper is to study the Tachibana operator for a pure Riemannian metric tensor field and then to apply the results obtained to the study of paraholomorphic B-manifolds.

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1. Introduction

Let M_n be a Riemannian manifold with metric g, which is not necessarily positive definite. We denote by $\mathfrak{T}^p_q(M_n)$ the set of all tensor fields of type (p,q) on M_n. Manifolds, tensor fields and connections are always assumed to be differentiable and of class C^∞.

An almost paracomplex manifold is an almost product manifold (M_n, φ), $\varphi \in I$, such that the two eigenbundles T^+M_n and T^-M_n associated with the two eigenvalues $+1$ and -1 of φ, respectively, have the same rank. Note that the dimension of an almost paracomplex manifold is necessarily even.

Considering the paracomplex structure φ, we obtain the following set of affinors on M_n : $\{I, \varphi\}$, $\varphi^2 = I$, which form a basis of a representation of an algebra of order 2 over the field of real numbers $\mathbb{R}$, which is called the algebra of paracomplex (or double) numbers and is denoted by $\mathbb{R}(j) = \{a_0 + a_1 j \mid j^2 = 1,\ a_0, a_1 \in \mathbb{R}\}$. Obviously, it is associative, commutative and unitial, i.e., it admits a principal unit 1. The canonical basis of this algebra has the form $\{1, j\}$. The structural constants of an algebra are defined by the multiplication law of the base units of this algebra: $e_i e_j = C^k_{ij} e_k$. With respect to the canonical basis of $\mathbb{R}(j)$ the components of C^k_{ij} are given by $C^1_{11} = C^2_{12} = C^2_{21} = C^1_{22} = 1$, all the others being zero.

Consider $\mathbb{R}(j)$ endowed with the usual topology of $\mathbb{R}^2$ and a domain U of $\mathbb{R}(j)$. Let

$$X = x^1 + jx^2$$

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be a variable in $\mathbb{R}(j)$, where x^i are the real coordinates of a point of a certain domain U for $i = 1, 2$. Using two real-valued functions $f^i(x^1, x^2)$, $i = 1, 2$, we introduce a paracomplex function

$$F = f^1 + jf^2$$

of the variable X. It is said to be paraholomorphic if we have

$$dF = F'(X)dX$$

for the differentials $dX = dx^1 + jdx^2$, $dF = df^1 + jdf^2$ and the derivative $F'(X)$. The paraholomorphy of the function $F = f^1 + jf^2$ in the variable $X = x^1 + jx^2$ is equivalent to the requirement that the Jacobian matrix $D = (\partial_k f^i)$ should commute with the matrix $C_2 = (C_{ij}^k) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (see [5, p. 87]). It follows that F is paraholomorphic if and only if f^1 and f^2 satisfy the para-Cauchy-Riemann equations:

$$\frac{\partial f^1}{\partial x^1} = \frac{\partial f^2}{\partial x^2}, \ \frac{\partial f^1}{\partial x^2} = \frac{\partial f^2}{\partial x^1}.$$

For almost paracomplex structures, integrability is equivalent to the vanishing of the Nijenhuis tensor

$$N_\varphi(X, Y) = [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + [X, Y].$$

On the other hand, in order that an almost paracomplex structure be integrable, it is necessary and sufficient that we can introduce on M a torsion free linear connection such that $\nabla\varphi = 0$.

A paracomplex manifold is an almost paracomplex manifold (M_{2k}, φ) such that the G-structure defined by the affinor field φ is integrable. We can give another, equivalent, definition of paracomplex manifold in terms of local homeomorphisms in the space $\mathbb{R}^k(j) = \{(X^1, \dots, X^k) \mid X^i \in \mathbb{R}(j), i = 1, \dots, k\}$ and paraholomorphic changes of charts in a way similar to [1] (for more details see [5]), i.e. a manifold M_{2k} with an integrable paracomplex structure φ is a real realization of the paraholomorphic manifold $M_k(\mathbb{R}(j))$ over the algebra $\mathbb{R}(j)$. Let t^* be a paracomplex tensor field on $M_k(\mathbb{R}(j))$. The real model of such a tensor field is a tensor field on M_{2k} of the same order, that is independent of whether its vector or covector arguments is subject to the action of the affinor structure φ. Such tensor fields are said to be pure with respect to φ. They have been studied by many authors (see, e.g., [2–5, 7]). In particular, when applied to a $(0, q)$-tensor field ω, purity means that for any $X_1, X_2, \dots, X_q \in \mathfrak{T}_0^1(M_{2k})$, the following conditions should hold:

$$\omega(\varphi X_1, X_2, \dots, X_q) = \omega(X_1, \varphi X_2, \dots, X_q) = \cdots = \omega(X_1, X_2, \dots, \varphi X_q).$$

We define an operator $\phi_\varphi : \mathfrak{T}_q^0(M_{2k}) \to \mathfrak{T}_{q+1}^0(M_{2k})$, applied to the pure tensor field ω by

$$(1.1) \quad \begin{aligned} (\phi_\varphi\omega)(X, Y_1, Y_2, \dots, Y_q) &= (\varphi X)(\omega(Y_1, Y_2, \dots, Y_q)) - X(\omega(\varphi Y_1, Y_2, \dots, Y_q)) + \\ &+ \omega((L_{Y_1}\varphi)X, Y_2, \dots, Y_q) + \cdots + \omega(Y_1, Y_2, \dots, (L_{Y_q}\varphi)X) \end{aligned}$$

(see [7]), where L_Y denotes Lie differentiation with respect to Y.

When φ is paracomplex structure on M_{2k}, and the tensor field $\phi_\varphi\omega$ vanishes, the paracomplex tensor field ω^* on $M_k(\mathbb{R}(j))$ is said to be paraholomorphic [2]. Thus a paraholomorphic paracomplex tensor field ω^* on $M_2(\mathbb{R}(j))$ is realized on M_{2k} in the form of a pure tensor field ω such that

$$(1.2) \quad (\phi_\varphi\omega)(X, Y_1, Y_2, \dots, Y_q) = 0$$

for any $X, Y_1, Y_2, \dots, Y_q \in \mathfrak{T}_0^1(M_n)$. Therefore such a tensor field ω on M_{2k} is also called a paraholomorphic tensor field.

2. Holomorphic B-Manifolds

A *pure metric* with respect to an almost paracomplex structure is a Riemannian metric g such that

(2.1) $\quad g(\varphi X, Y) = g(X, \varphi Y)$

for any $X, Y \in \mathfrak{T}_0^1(M_n)$. Such Riemannian metrics were studied in [6], where they were called *B-metrics*, since the metric tensor g with respect to the structure φ is a B-tensor according to the terminology accepted in [5]. If (M_{2k}, φ) is an almost paracomplex manifold with a B-metric, we say that (M_{2k}, φ, g) is an *almost B-manifold*. If φ is integrable, we say that (M_{2k}, φ, g) is an *B-manifold*.

In a B-manifold, the B-metric is called *paraholomorphic* if $(\phi_\varphi g)(X, Y, Z) = 0$. If (M_{2k}, φ, g) is a B-manifold with a paraholomorphic B-metric g, we say that (M_{2k}, φ, g) is a *paraholomorphic B-manifold*.

In some respects paraholomorphic B-manifolds may viewed as analogous to Kahler manifolds. The following theorem [3] is analogous to an almost Hermitian manifold being Kahler if and only if the almost complex structure is parallel with respect to the Levi-Civita connection.

2.1. Theorem. *An almost B-manifold is a paraholomorphic B-manifold if and only if the almost paracomplex structure is parallel with respect to the Levi-Civita connection.*

Let (M_{2k}, φ, g) be an almost B-manifold. The associated B-metric of an almost B-manifold is defined by

(2.2) $\quad G(X, Y) = (g \circ \varphi)(X, Y)$

for all vector fields X and Y on M_{2k}. We shall now apply the Tachibana operator to the pure Riemannian metric G:

$$\begin{aligned}(\phi_\varphi G)(X, Y, Z) &= (L_{\varphi X} G - L_X(G \circ \varphi))(Y, Z) + G(Y, \varphi L_X Z) - G(\varphi Y, L_X Z)\\ &= (L_{\varphi X}(g \circ \varphi) - L_X((g \circ \varphi) \circ \varphi)(Y, Z) + (g \circ \varphi)(Y, \varphi L_X Z)\\ &\qquad - (g \circ \varphi)(\varphi Y, L_X Z)\\ &= ((L_{\varphi X} g) \circ \varphi + g \circ L_{\varphi X}\varphi - L_X(g \circ \varphi) \circ \varphi - (g \circ \varphi) L_X \varphi)(Y, Z)\\ &\qquad + (g \circ \varphi)(Y, \varphi L_X Z) - (g \circ \varphi)(\varphi Y, L_X Z)\\ &= (L_{\varphi X} g - L_X(g \circ \varphi))(\varphi Y, Z) + g(\varphi Y, \varphi L_X Z)\\ &\qquad - g(\varphi(\varphi Y), L_X Z) + (g \circ L_{\varphi X}\varphi - (g \circ \varphi) L_X \varphi)(Y, Z)\\ &= (\phi_\varphi g)(X, \varphi Y, Z) + g((L_{\varphi X}\varphi)Y, Z) - g(\varphi((L_X \varphi)Y), Z)\\ &= (\phi_\varphi g)(X, \varphi Y, Z) + g([\varphi X, \varphi Y] - \varphi[\varphi X, Y], Z)\\ &\qquad - g(\varphi[X, \varphi Y] - \varphi^2[X, Y], Z)\\ &= (\phi_\varphi g)(X, \varphi Y, Z) + g([\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y]\\ &\qquad + \varphi^2[X, Y], Z)\\ &= (\phi_\varphi g)(X, \varphi Y, Z) + g(N_\varphi(X, Y), Z).\end{aligned} \tag{2.3}$$

Thus (2.3) implies the following

2.2. Theorem. *In an almost B-manifold, we have*

$$\phi_\varphi G = (\phi_\varphi g) \circ \varphi + g \circ (N_\varphi).$$

From Theorem 2.1 and Theorem 2.2 we have:

2.3. Theorem. *Almost B-manifold satisfying the conditions $\phi_\varphi G = 0$, $N_\varphi \neq 0$, i.e. the analogues of the almost Kahler manifolds, do not exist.*

2.4. Corollary. *The following conditions are equivalent:*

(a) $\phi_\varphi g = 0$.
(b) $\phi_\varphi G = 0$.

We denote by ∇_g the covariant differentiation of the Levi-Civita connection of the B-metric g. Then, we have

$$\nabla_g G = (\nabla_g g) \circ \varphi + g \circ (\nabla_g \varphi) = g \circ (\nabla_g \varphi),$$

which implies $\nabla_g G = 0$ by virtue of Theorem 2.1. Therefore we have

2.5. Theorem. *Let (M_{2k}, φ, g) be a paraholomorphic B-manifold. Then the Levi-Civita connection of the B-metric g coincides with the Levi-Civita connection of the associated B-metric G.*

3. Curvature tensors in a paraholomorphic B-manifold

Let R and S be the curvature tensors formed by g and G respectively. Then for a paraholomorphic B-manifold we have $R = S$ by means of Theorem 2.5.

Applying Ricci's identity to φ, we get

$$\varphi(R(X,Y)Z) = R(X,Y)\varphi Z \tag{3.1}$$

by virtue of $\nabla\varphi = 0$. Hence $R(X_1, X_2, X_3, X_4) = g(R(X_1, X_2)X_3, X_4)$ is pure with respect to X_3 and X_4, and also pure with respect to X_1 and X_2:

$$\begin{aligned} R(X_1, X_2, \varphi X_3, X_4) &= g(R(X_1, X_2)\varphi X_3, X_4) \\ &= g(\varphi(R(X_1, X_2)X_3), X_4) \\ &= g(R(X_1, X_2)X_3, \varphi X_4) \\ &= R(X_1, X_2, X_3, \varphi X_4). \end{aligned}$$

On the other hand, S being the curvature tensor formed by the associated B-metric G, if we put $S(X_1, X_2, X_3, X_4) = G(S(X_1, X_2)X_3, X_4)$, then we have

$$S(X_1, X_2, X_3, X_4) = S(X_3, X_4, X_1, X_2) \tag{3.2}$$

Taking account of (1.1), (2.2), (3.1) and $R = S$, we find that

$$\begin{aligned} S(X_1, X_2, X_3, X_4) &= G(S(X_1, X_2)X_3, X_4) \\ &= g(\varphi(S(X_1, X_2)X_3), X_4) \\ &= g(S(X_1, X_2)X_3, \varphi X_4) \\ &= g(R(X_1, X_2)X_3, \varphi X_4) \\ &= R(X_1, X_2, X_3, \varphi X_4) \end{aligned}$$

and

$$\begin{aligned} S(X_3, X_4, X_1, X_2) &= G(S(X_3, X_4)X_1, X_2) \\ &= g(\varphi(S(X_3, X_4)X_1), X_2) \\ &= g(S(X_3, X_4)X_1, \varphi X_2) \\ &= g(R(X_3, X_4)X_1, \varphi X_2) \\ &= R(X_3, X_4, X_1, \varphi X_2) \\ &= R(X_1, \varphi X_2, X_3, X_4) \end{aligned}$$

Thus equation (3.2) becomes

$$R(X_1, X_2, X_3, \varphi X_4) = R(X_1, \varphi X_2, X_3, X_4),$$

which shows that $R(X_1, X_2, X_3, X_4)$ is pure with respect to X_2 and X_4. Therefore $R(X_1, X_2, X_3, X_4)$ is pure.

Thus we get

3.1. Theorem. *In a paraholomorphic B-manifold, the Riemannian curvature tensor of the B-metric is pure.*

Since the Riemannian curvature tensor R is pure, we can apply the ϕ-operator to R. By similar devices (see the proof of Theorem 1 in [3]), we can prove that

$$(\phi_\varphi R)(X, Y_1, Y_2, Y_3, Y_4) = (\nabla_{\varphi X} R)(Y_1, Y_2, Y_3, Y_4) - (\nabla_X R)(\varphi Y_1, Y_2, Y_3, Y_4). \tag{3.3}$$

Using (3.1), and applying Bianchi's 2nd identity to (3.3), we get

$$\begin{aligned}(\phi_\varphi R)(X, Y_1, Y_2, Y_3, Y_4) &= g((\nabla_{\varphi X} R)(Y_1, Y_2, Y_3) - (\nabla_X R)(\varphi Y_1, Y_2, Y_3), Y_4) \\ &= g((\nabla_{\varphi X} R)(Y_1, Y_2, Y_3) - \varphi((\nabla_X R)(Y_1, Y_2, Y_3)), Y_4) \\ &= g(-(\nabla_{Y_1} R)(Y_2, \varphi X, Y_3) - (\nabla_{Y_1} R)(\varphi X, Y_1, Y_3) \\ &\qquad - \varphi((\nabla_X R)(Y_1, Y_2, Y_3)), Y_4).\end{aligned} \tag{3.4}$$

On the other hand, using $\nabla\varphi = 0$, we find:

$$\begin{aligned}(\nabla_{Y_2} R)(\varphi X, Y_1, Y_3) &= \nabla_{Y_2}(R(\varphi X, Y_1, Y_3)) - R(\nabla_{Y_2}(\varphi X), Y_1, Y_3) \\ &\qquad - R(\varphi X, \nabla_{Y_2} Y_1, Y_3) - R(\varphi X, Y_1, \nabla_{Y_2} Y_3) \\ &= (\nabla_{Y_2}\varphi)(R(X, Y_1, Y_3)) + \varphi(\nabla_{Y_2} R(X, Y_1, Y_3)) \\ &\qquad - R((\nabla_{Y_2}\varphi)X + \varphi(\nabla_{Y_2} X), Y_1, Y_3) \\ &\qquad - R(\varphi X, \nabla_{Y_2} Y_1, Y_3) - R(\varphi X, Y_1, \nabla_{Y_2} Y_3) \\ &= \varphi(\nabla_{Y_2} R(X, Y_1, Y_3)) - \varphi(R(\nabla_{Y_2} X, Y_1, Y_3)) \\ &\qquad - \varphi(R(X, \nabla_{Y_2} Y_1, Y_3)) - \varphi(R(X, Y_1, \nabla_{Y_2} Y_3)) \\ &= \varphi((\nabla_{Y_2} R)(X, Y_1, Y_3)).\end{aligned} \tag{3.5}$$

Similarly

$$(\nabla_{Y_1} R)(Y_2, \varphi X, Y_3) = \varphi((\nabla_{Y_1} R)(Y_2, X, Y_3)). \tag{3.6}$$

Substituting (3.5) and (3.6) in (3.4), and using again Bianchi's 2nd identity, we obtain

$$\begin{aligned}(\phi_\varphi R)(X, Y_1, Y_2, Y_3, Y_4) &= g(-\varphi((\nabla_{Y_1} R)(Y_2, X, Y_3)) - \varphi((\nabla_{Y_2} R)(X, Y_1, Y_3)) \\ &\qquad - \varphi((\nabla_X R)(Y_1, Y_2, Y_3)), Y_4) \\ &= -g(\varphi(\sigma\{(\nabla_X R)(Y_1, Y_2\}, Y_3)), Y_4) \\ &= 0,\end{aligned}$$

where σ denotes the cyclic sum with respect to X, Y_1 and Y_2. Therefore we have:

3.2. Theorem. *In a paraholomorphic B-manifold, the Riemannian curvature tensor field is a paraholomorphic tensor field*

3.3. Example. We suppose that the manifold M_{2n} is the tangent bundle $\pi : T(V_n) \to V_n$ of a Riemannian manifold V_n. If x^i are the local coordinates on V_n, then the x^i, together with the fibre coordinates $x^{\bar{\imath}} = y^i$, $\bar{\imath} = n+1, \dots 2n$, form local coordinates on $T(V_n)$.

A tensor field of type $(0, q)$ on $T(V_n)$ is completely determined by its action on all vector fields $\widetilde{X}_i$, $i = 1, 2, \dots, q$, which are of the form ${}^V\!X$ (vertical lift) or ${}^H\!X$ (horizontal lift) [8, p.101]:

$${}^V\!X = X^i \frac{\partial}{\partial x^{\bar{\imath}}}, \quad {}^H\!X = X^i \frac{\partial}{\partial x^i} - y^s \Gamma^i_{sh} X^h \frac{\partial}{\partial x^{\bar{\imath}}}.$$

Therefore, we define the Sasakian metric ${}^{s}g$ on $T(V_n)$ by

$$(3.7)\qquad \begin{cases} {}^{s}g({}^{H}X, {}^{H}Y) = {}^{V}(g(X,Y)), \\ {}^{s}g({}^{V}X, {}^{V}Y) = {}^{V}(g(X,Y)), \\ {}^{s}g({}^{V}X, {}^{H}Y) = 0, \end{cases}$$

for any $X, Y \in \Im_0^1(V_n)$. The metric ${}^{s}g$ has local components

$${}^{s}g = \begin{pmatrix} g_{ji} + g_{ts} y^k y^l \Gamma_{kj}^t \Gamma_{li}^s & y^k \Gamma_{kj}^s g_{si} \\ y^k \Gamma_{ki}^s g_{js} & g_{ji} \end{pmatrix}$$

with respect to the induced coordinates $(x^i, x^{\bar{\imath}})$ in $T(V_n)$, where Γ_{ij}^k are the components of the Levi-Civita connection ∇_g in V_n.

The diagonal lift ${}^{D}\varphi$ in $T(V_n)$ is defined by

$$(3.8)\qquad \begin{cases} {}^{D}\varphi {}^{H}X = {}^{H}(\varphi X), \\ {}^{D}\varphi {}^{V}X = -{}^{V}(\varphi X), \end{cases}$$

for any $X \in \Im_0^1(V_n)$ and $\varphi \in \Im_1^1(M_n)$. The diagonal lift ${}^{D}I$ of the identity tensor field $I \in \Im_1^1(M_n)$ has the components

$${}^{D}I = \begin{pmatrix} \delta_i^j & 0 \\ -2y^t \Gamma_{ti}^j & -\delta_i^j \end{pmatrix}$$

with respect to the induced coordinates and satisfies $({}^{D}I)^2 = I_{T(V_n)}$. Thus, ${}^{D}I$ is an almost paracomplex structure determining the horizontal distribution and the distribution consisting of the tangent planes to fibres.

We put

$$A(\widetilde{X}, \widetilde{Y}) = {}^{s}g({}^{D}I\widetilde{X}, \widetilde{Y}) - {}^{s}g(\widetilde{X}, {}^{D}I\widetilde{Y}).$$

If $A(\widetilde{X}, \widetilde{Y}) = 0$ for all vector fields $\widetilde{X}$ and $\widetilde{Y}$ which are of the form ${}^{V}X$, ${}^{V}Y$ or ${}^{H}X$, ${}^{H}Y$ then $A = 0$. We have by virtue of ${}^{D}I^{V}X = -{}^{V}X$, ${}^{D}I^{H}X = {}^{H}X$, (4.1) and (4.2),

$$A({}^{V}X, {}^{V}Y) = {}^{s}g(-{}^{V}X, {}^{V}Y) - {}^{s}g({}^{V}X, -{}^{V}Y) = 0,$$
$$A({}^{V}X, {}^{H}Y) = {}^{s}g(-{}^{V}X, {}^{H}Y) - {}^{s}g({}^{V}X, {}^{H}Y) = 0,$$
$$A({}^{H}X, {}^{V}Y) = {}^{s}g({}^{H}X, {}^{V}Y) - {}^{s}g({}^{H}X, -{}^{V}Y) = 0,$$
$$A({}^{H}X, {}^{H}Y) = {}^{s}g({}^{H}X, {}^{H}Y) - {}^{s}g({}^{H}X, {}^{H}Y) = 0,$$

i.e. ${}^{s}g$ is B-metric with respect to ${}^{D}I$. Hence we have:

3.4. Theorem. *$(T(V_n), {}^{D}I, {}^{s}g)$ is an almost B-manifold.*

Using the properties of ${}^{V}X$, ${}^{H}X$ and $\gamma R(X,Y) = y^s R_{ijs}^k X^i Y^j \frac{\partial}{\partial x^k}$, we have

$$(\phi_{{}^{D}I}\, {}^{s}g)({}^{V}X, {}^{H}Y, {}^{H}Z) = -2({}^{s}g\, {}^{V}(\nabla_Y X), {}^{H}Z + {}^{s}g({}^{H}Y, {}^{V}(\nabla_Z X))) = 0,$$
$$(\phi_{{}^{D}I}\, {}^{s}g)({}^{V}X, {}^{H}Y, {}^{V}Z) = -2\, {}^{s}g({}^{H}Y, [{}^{V}Z, {}^{V}X]) = 0,$$
$$(\phi_{{}^{D}I}\, {}^{s}g)({}^{V}X, {}^{V}Y, {}^{H}Z) = -2\, {}^{s}g([{}^{V}Y, {}^{V}X], {}^{H}Z) = 0,$$
$$(\phi_{{}^{D}I}\, {}^{s}g)({}^{V}X, {}^{V}Y, {}^{V}Z) = 0,$$
$$(\phi_{{}^{D}I}\, {}^{s}g)({}^{H}X, {}^{H}Y, {}^{H}Z) = 0,$$
$$(\phi_{{}^{D}I}\, {}^{s}g)({}^{H}X, {}^{V}Y, {}^{V}Z) = 2\, {}^{V}((\nabla_X g)(Y,Z)) = 0,$$
$$(\phi_{{}^{D}I}\, {}^{s}g)({}^{H}X, {}^{H}Y, {}^{V}Z) = -2\, {}^{s}g(\gamma R(Y,X), {}^{V}Z),$$
$$(\phi_{{}^{D}I}\, {}^{s}g)({}^{H}X, {}^{V}Y, {}^{H}Z) = -2\, {}^{s}g({}^{V}Y, \gamma R(Z,X)).$$

Therefore we have:

3.5. Theorem. *The almost B-manifold* $(T(V_n), {}^D I, {}^s g)$ *is paraholomorphic if and only if* V_n *is locally Euclidean.*

3.6. Example. Now let M_n be the locally product Riemannian manifold with integrable almost product structure $\varphi = \begin{pmatrix} \delta^i_j & 0 \\ 0 & -\delta^{\bar\imath}_{\bar\jmath} \end{pmatrix}$, $i, j = 1, \ldots, k$, $\bar\imath, \bar\jmath = k+1, \ldots, n$, and let $n = 2k$. Then the paracomplex manifold M_{2k} admits the structure of a B-manifold:

$$g = \begin{pmatrix} g_{ij} & 0 \\ 0 & g_{\bar\imath\bar\jmath} \end{pmatrix}, \ g_{ij} = g_{ij}(x^t, x^{\bar t}), \ g_{\bar\imath\bar\jmath}(x^t, x^{\bar t}).$$

Suppose that the metric of the locally product Riemannian manifold M_{2k} has the form

$$ds^2 = g_{ij}(x^t)dx^i dx^j + g_{\bar\imath\bar\jmath}(x^{\bar t})dx^{\bar\imath}dx^{\bar\jmath}, \ i, j, t = 1, \ldots, k, \ \bar\imath, \bar\jmath, \bar t = k+1, \ldots, 2k,$$

that is the $g_{ij}(x)$ are functions of x^t only, $g_{i\bar\jmath} = 0$ and the $g_{\bar\imath\bar\jmath}(x)$ are functions of $x^{\bar t}$ only, then we call the manifold a *locally decomposable* Riemannian manifold. A necessary and sufficient condition for a locally product Riemannian manifold to be a locally decomposable Riemannian manifold is that $\nabla_g \varphi = 0$ [9, p. 420]. Then from Theorem 3.2 we have

3.7. Theorem. *A locally decomposable Riemannian manifold* M_{2k} *is a paraholomorphic B-manifold.*

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