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CONVERGENCE OF REGULAR DIFILTERS AND THE COMPLETENESS OF DI-UNIFORMITIES

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Abstract

The authors propose a theory of convergence for regular difilters on a ditopological texture space and go on to discuss completeness and total boundedness of di-uniformities.

Keywords: Difilters, Regular difilters, Convergence of difilters, Clustering of difilters, Dicompactness, Di-uniformity, Cauchy difilter, Completeness, Total boundedness.

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1. Introduction

Let S be a non-empty set. We recall [3] that a *texturing* on S is a point separating, complete, completely distributive lattice $\mathcal{S}$ of subsets of S with respect to inclusion, which contains S, $\emptyset$, and for which meet $\bigwedge$ coincides with intersection $\bigcap$ and finite joins $\vee$ coincide with unions $\cup$. Textures first arose in connection with the representation of Hutton algebras and lattices of $\mathbb{L}$-fuzzy sets in a point-based setting [3, 5], and have subsequently proved to be a fruitful setting for the investigation of complement-free concepts in mathematics. The sets

$$P_s = \bigcap\{A \in \mathcal{S} \mid s \in A\},\ Q_s = \bigvee\{P_u \mid u \in S,\ s \notin P_u\},\ s \in S,$$

are important in the study of textures, and the following facts concerning these so called p–sets and q–sets will be used extensively below.

1.1. Lemma. [6, Theorem 1.2]

(1) $s \notin A \implies A \subseteq Q_s \implies s \notin A^\flat$ *for all* $s \in S$, $A \in \mathcal{S}$.
(2) $A^\flat = \{s \mid A \not\subseteq Q_s\}$ *for all* $A \in \mathcal{S}$.
(3) *For* $A_i \in \mathcal{S}$, $i \in I$ *we have* $(\bigvee_{i\in I} A_i)^\flat = \bigcup_{i\in I} A_I^\flat$.

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(4) *A is the smallest element of $\mathcal{S}$ containing $A^\flat$ for all $A \in \mathcal{S}$.*
(5) *For $A, B \in \mathcal{S}$, if $A \not\subseteq B$ then there exists $s \in S$ with $A \not\subseteq Q_s$ and $P_s \not\subseteq B$.*
(6) *$A = \bigcap\{Q_s \mid P_s \not\subseteq A\}$ for all $A \in \mathcal{S}$.*
(7) *$A = \bigvee\{P_s \mid A \not\subseteq Q_s\}$ for all $A \in \mathcal{S}$.*

Here $A^\flat$ is defined by

$$A^\flat = \bigcap\Big\{\bigcup\{A_i \mid i \in I\} \mid \{A_i \mid i \in I\} \subseteq \mathcal{S},\ A = \bigvee\{A_i \mid i \in I\}\Big\}$$

and known as the *core* of $A \in \mathcal{S}$. The above lemma exposes an important formal duality in $(S, \mathcal{S})$, namely that between $\bigcap$ and $\bigvee$, Q_s and P_s, and $P_s \not\subseteq A$ and $A \not\subseteq Q_s$. Indeed, it is to emphasize this duality that we normally write $P_s \not\subseteq A$ in preference to $s \notin A$.

The simplest example of a texture is the *discrete texture* $(X, \mathcal{P}(X))$ *on* X, for which $P_x = \{x\}$ and $Q_x = X \setminus \{x\}$, $x \in X$. Two other important textures which we will consider in this paper are given below.

1.2. Examples. (1) Let $L = (0, 1]$ and $\mathcal{L} = \{(0, r] \mid 0 \le r \le 1\}$, where $(0, 0]$ is interpreted as $\emptyset$. Then $(L, \mathcal{L})$ is a texture with $P_r = Q_r = (0, r]$ for all $r \in L$. This texture is of particular interest since it is the Hutton texture corresponding to the Hutton algebra $\mathbb{I} = [0, 1]$, which is closely involved in the theory of classical fuzzy sets [5, 6].

(2) The *unit interval texture* $(\mathbb{I}, \mathcal{I})$ has $\mathbb{I} = [0, 1]$ and the texturing $\mathcal{I} = \{[0, r) \mid 0 \le r \le 1\} \cup \{[0, r] \mid 0 \le r \le 1\}$. For this texture $P_r = [0, r]$ and $Q_r = [0, r)$, $r \in \mathbb{I}$.

We recall that a texture $(S, \mathcal{S})$ is called *plain* if arbitrary joins coincide with unions; equivalently if $\mathcal{S}$ is closed under arbitrary unions or if $P_s \not\subseteq Q_s$ for all $s \in S$. For the above examples, $(X, \mathcal{P}(X))$ and $(\mathbb{I}, \mathcal{I})$ are plain but $(L, \mathcal{L})$ is not.

In general a texturing $\mathcal{S}$ need not be closed under set complement, so in the context of a texture $(S, \mathcal{S})$ the notion of topology is replaced by that of dichotomous topology. A *dichotomous topology*, or *ditopology* for short, on a texture space $(S, \mathcal{S})$ is a pair (τ, κ) of subsets of $\mathcal{S}$, where the set of *open sets* τ satisfies

(1) $S, \emptyset \in \tau$,
(2) $G_1, G_2 \in \tau \implies G_1 \cap G_2 \in \tau$ and
(3) $G_i \in \tau$, $i \in I \implies \bigvee_i G_i \in \tau$,

and the set of *closed sets* κ satisfies

(1) $S, \emptyset \in \kappa$,
(2) $K_1, K_2 \in \kappa \implies K_1 \cup K_2 \in \kappa$ and
(3) $K_i \in \kappa$, $i \in I \implies \bigcap K_i \in \kappa$.

Hence a ditopology is essentially a "topology" for which there is no *a priori* relation between the open and closed sets. The reader is referred to [1,3,4,6-8,10,11,12] for some results on ditopological texture spaces and their relation with fuzzy topologies. We shall be particularly interested in an appropriate notion of "compactness" for ditopological texture spaces, and recall the following definitions from [3].

1.3. Definition. Let (τ, κ) be a ditopology on the texture $(S, \mathcal{S})$. Then $(S, \mathcal{S}, \tau, \kappa)$ is called

(i) *Compact* if whenever $S = \bigvee_{i \in I} G_i$, $G_i \in \tau$, $i \in I$, there is a finite subset J of I with $\bigcup_{j \in J} G_j = S$.
(ii) *Cocompact* if whenever $\bigcap_{i \in I} F_i = \emptyset$, $F_i \in \kappa$, $i \in I$, there is a finite subset J of I with $\bigcap_{j \in J} F_j = \emptyset$.
(iii) *Stable* if every $K \in \kappa$ with $K \ne S$ is compact, i.e. whenever $K \subseteq \bigvee_{i \in I} G_i$, $G_i \in \tau$, $i \in I$, there is a finite subset J of I with $K \subseteq \bigcup_{j \in J} G_j$.

(iv) *Costable* if every $G \in \tau$ with $G \neq \emptyset$ is co-compact, i.e. whenever $\bigcap_{i\in I} F_i \subseteq G$, $F_i \in \kappa$, $i \in I$, there is a finite subset J of I with $\bigcap_{j\in J} F_j \subseteq G$.

We will refer to a ditopological texture space which has all four properties as a *dicompact* space (see [10], where a more detailed coverage of compactness properties in ditopological texture spaces is given). In order to state two important characterizations of dicompactness we recall the following concepts:

1.4. Definition. Let (τ, κ) be a ditopology on $(S, \mathcal{S})$.

(1) A set $\mathcal{D} \subseteq \mathcal{S} \times \mathcal{S}$ is called a *difamily* on $(S, \mathcal{S})$. A difamily $\mathcal{D}$ satisfying $\mathcal{D} \subseteq \tau \times \kappa$ is *open and co-closed*, one satisfying $\mathcal{D} \subseteq \kappa \times \tau$ is *closed and co-open.*

(2) A difamily $\mathcal{D}$ has the *finite exclusion property* (fep) if whenever $(F_i, G_i) \in \mathcal{D}$, $i = 1, 2, \ldots, n$ we have $\bigcap_{i=1}^n F_i \not\subseteq \bigcup_{i=1}^n G_i$.

(3) A closed, co-open difamily $\mathcal{D}$ with $\bigcap\{F \mid F \in \operatorname{dom} \mathcal{D}\} \not\subseteq \bigvee\{G \mid G \in \operatorname{ran} \mathcal{D}\}$ is said to be *bound* in $(S, \mathcal{S}, \tau, \kappa)$.

(4) A difamily $\mathcal{D} = \{(G_i, F_i) \mid i \in I\}$ is called a *dicover* of $(S, \mathcal{S})$ if for all partitions I_1, I_2 of I (including the trivial partitions) we have

$$\bigcap_{i\in I_1} F_i \subseteq \bigvee_{i\in I_2} G_i.$$

(5) A difamily $\mathcal{C}$ is called *finite* if the set $\operatorname{dom} \mathcal{C}$ is finite, and *cofinite* if the set $\operatorname{ran} \mathcal{C}$ is finite. In particular $\mathcal{C}$ is finite and cofinite if and only if the set $\mathcal{C}$ is finite.

We now have the following important theorem, which is proved in [3].

1.5. Theorem. *The following are equivalent for a ditopological texture $(S, \mathcal{S}, \tau, \kappa)$.*

(1) *$(S, \mathcal{S}, \tau, \kappa)$ is dicompact.*
(2) *Every closed, co-open difamily with the finite exclusion property is bound.*
(3) *Every open, coclosed dicover has a finite, cofinite subdicover.*

In view of the equivalence of (1) and (3), a ditopological texture space is dicompact if and only if it is dicover compact in the sense of [1, 3, 4].

1.6. Examples. (1) Consider the texture $(L, \mathcal{L})$ of Examples 1.2 (1) with the discrete, codiscrete ditopology $(\mathcal{L}, \mathcal{L})$ [7, 10]. Then, for example, $\{(0, 1 - \frac{1}{n}] \mid n = 2, 3, \ldots\}$ is an open cover of L with no finite subcover, so this ditopological texture space is not compact and hence not dicompact.

(2) Now consider the texture $(\mathbb{I}, \mathcal{I})$ of Examples 1.2 (2) with the natural ditopology

$$\tau_{\mathbb{I}} = \{[0, r) \mid r \in \mathbb{I}\} \cup \{\mathbb{I}\}, \quad \kappa_{\mathbb{I}} = \{[0, r] \mid r \in \mathbb{I}\} \cup \{\emptyset\}.$$

Any open cover of $\mathbb{I}$ must contain $\mathbb{I}$, and therefore have the finite subcover $\{\mathbb{I}\}$. Thus the ditopological texture space $(\mathbb{I}, \mathcal{I}, \tau_{\mathbb{I}}, \kappa_{\mathbb{I}})$ is compact. In exactly the same way it is stable, cocompact and costable, whence it is dicompact.

It will be noted that the proofs of the compactness, stability, cocompactness and costability of $(\mathbb{I}, \mathcal{I}, \tau_{\mathbb{I}}, \kappa_{\mathbb{I}})$ are trivial. On the other hand it may be shown that the condition that every open, coclosed dicover has a finite, cofinite subdicover is equivalent to the compactness of the unit interval under its usual topology. Hence the notion of dicover can be expected to occupy an important place in the study of ditopological texture spaces, and this is confirmed by its role in the theory of dicovering uniformities [11, 12]. Our study of completeness and total boundedness in di-uniformities on a texture in Section 3 will be stated almost exclusively in terms of dicovering uniformities, and the reader is referred to [11, 12] for the necessary details.

Finally we shall make occasional reference to difunctions and the notion of direlation on which they are based [2, 6], and to the notion of direlational uniformity on a texture which is shown in [11, 12] to be equivalent to that of dicovering uniformity. Results on direlations and difunctions between textures obtained over the past few years have been collected by the third author in the preprint [2], and much of this material appears with proofs in [6]. The necessary background may also be found in [11], and will not be repeated here.

Readers are referred to [9] for terms from lattice theory not defined here.

Much of the material described here may be found in the first author's Ph.D. thesis [12] on di-uniform spaces, while the notion of convergence of difilters also plays an important role in the second author's continuing Ph.D. studies on spaces of real bicontinuous difunctions and the concept of realcompactness for ditopological texture spaces.

2. Regular Difilters and Convergence

An appropriate notion of "filter" would seem to be that of difilter defined below.

2.1. Definition. Let $(S, \mathcal{S})$ be a texture.

(1) $\mathcal{F} \subseteq \mathcal{S}$ is called a $\mathcal{S}$*–filter*, or a *filter on* $(S, \mathcal{S})$, if $\mathcal{F} \neq \emptyset$ and satisfies:
  $\mathbf{F_1}$. $\emptyset \notin \mathcal{F}$,
  $\mathbf{F_2}$. $F \in \mathcal{F},\ F \subseteq F' \in \mathcal{S} \implies F' \in \mathcal{F}$, and
  $\mathbf{F_3}$. $F_1, F_2 \in \mathcal{F} \implies F_1 \cap F_2 \in \mathcal{F}$.

(2) $\mathcal{G} \subseteq \mathcal{S}$ is called a $\mathcal{S}$*–cofilter*, or a *cofilter on* $(S, \mathcal{S})$, if $\mathcal{G} \neq \emptyset$ and satisfies:
  $\mathbf{CF_1}$. $S \notin \mathcal{G}$,
  $\mathbf{CF_2}$. $G \in \mathcal{G},\ G \supseteq G' \in \mathcal{S} \implies G' \in \mathcal{G}$, and
  $\mathbf{CF_3}$. $G_1, G_2 \in \mathcal{G} \implies G_1 \cup G_2 \in \mathcal{G}$.

(3) If $\mathcal{F}$ is a $\mathcal{S}$-filter and $\mathcal{G}$ a $\mathcal{S}$-cofilter then $\mathcal{F} \times \mathcal{G}$ is called a $\mathcal{S}$-*difilter*, or a *difilter on* $(S, \mathcal{S})$.

The notion of difilter is very general, so we seek to impose a suitable "regularity" condition. To this end let us note the following.

2.2. Lemma. *The following are equivalent for a difilter* $\mathcal{F} \times \mathcal{G}$ *on* $(S, \mathcal{S})$.

(1) $\mathcal{F} \cap \mathcal{G} = \emptyset$.
(2) $\mathcal{F} \times \mathcal{G}$ *has the f.e.p.*
(3) $A \not\subseteq B$ *for all* $A \in \mathcal{F}$ *and* $B \in \mathcal{G}$.

Proof. (1) $\implies$ (2) If $\{(A_1, B_1), \dots (A_n, B_n)\} \subseteq \mathcal{F} \times \mathcal{G}$ then $\bigcap_{k=1}^n A_k \in \mathcal{F}$ by $\mathbf{F_3}$ and $\bigcup_{k=1}^n B_k \in \mathcal{G}$ by $\mathbf{CF_3}$. Since $\bigcap_{k=1}^n A_k \subseteq \bigcup_{k=1}^n B_k$ would give the contradiction $\bigcup_{k=1}^n B_k \in \mathcal{F} \cap \mathcal{G}$ by $\mathbf{F_2}$, we deduce that $\mathcal{F} \times \mathcal{G}$ has the f.e.p.

(2) $\implies$ (3) Clear.

(3) $\implies$ (1) $A \in \mathcal{F} \cap \mathcal{G}$ would contradict (3) since $A \subseteq A$. □

2.3. Definition. A difilter on $(S, \mathcal{S})$ is called *regular* if it satisfies the equivalent conditions of Lemma 2.2.

Note that Altay and Diker [1], who use closed, co-open difilters in their construction of a Wallman type compactification, use this term in the sense of regular difilter. Difilters may be compared by set inclusion in the obvious way.

2.4. Lemma. *For difilters* $\mathcal{F}_1 \times \mathcal{G}_1$, $\mathcal{F}_2 \times \mathcal{G}_2$ *on* $(S, \mathcal{S})$ *the following are equivalent:*

(1) $\mathcal{F}_1 \times \mathcal{G}_1 \subseteq \mathcal{F}_2 \times \mathcal{G}_2$.
(2) $\mathcal{F}_1 \subseteq \mathcal{F}_2$ *and* $\mathcal{G}_1 \subseteq \mathcal{G}_2$.

(3) *Given* $(A,B) \in \mathcal{F}_1 \times \mathcal{G}_1$ *there exists* $(A',B') \in \mathcal{F}_2 \times \mathcal{G}_2$ *with* $A' \subseteq A$ *and* $B \subseteq B'$.

Proof. Straightforward. □

Clearly, if $\mathcal{F}_1 \times \mathcal{G}_1 \subseteq \mathcal{F}_2 \times \mathcal{G}_2$ and $\mathcal{F}_2 \times \mathcal{G}_2$ is regular then so is $\mathcal{F}_1 \times \mathcal{G}_1$, but the converse will not be true in general as the next example shows.

2.5. Examples. (1) For $X = \{a,b,c\}$ consider the discrete texture $(X, \mathcal{P}(X))$ and define

$$\mathcal{F}_1 = \{\{a,b\}, X\},\ \mathcal{G}_1 = \{\emptyset, \{c\}\}$$

and

$$\mathcal{F}_2 = \{\{a\},\{a,b\},\{a,c\},X\},\ \mathcal{G}_2 = \{\emptyset,\{a\},\{c\},\{a,c\}\}.$$

Clearly $\mathcal{F}_1 \times \mathcal{G}_1$ is a regular difilter, $\mathcal{F}_1 \times \mathcal{G}_1 \subseteq \mathcal{F}_2 \times \mathcal{G}_2$ but $\mathcal{F}_2 \times \mathcal{G}_2$ is a difilter which is not regular.

(2) Let (τ,κ) be a ditopology on $(S,\mathcal{S})$. We recall [8] that a neighbourhood of $s \in S^\flat$ is an element N of $\mathcal{S}$ for which there exists $G \in \tau$ satisfying $P_s \subseteq G \subseteq N \not\subseteq Q_s$. The set of neighbourhoods (nhds) of s is denoted by $\eta(s)$. Dually, a coneighbourhood of s is an element M of $\mathcal{S}$ satisfying $P_s \not\subseteq M \subseteq K \subseteq Q_s$ for some $K \in \kappa$, and the set of coneighbourhoods (conhds) of s is denoted by $\mu(s)$.

It is easy to verify that $\mu(s)$ is a $\mathcal{S}$-cofilter, but in general $\eta(s)$ need not be a $\mathcal{S}$-filter. The reason is that if $N_1 \not\subseteq Q_s$ and $N_2 \not\subseteq Q_s$ it is not necessarily the case that $N_1 \cap N_2 \not\subseteq Q_s$ and so $\eta(s)$ need not satisfy $\mathbf{F_3}$. On the other hand if $(S,\mathcal{S})$ is plain then $N_1 \cap N_2 \not\subseteq Q_s$ is equivalent to $P_s \subseteq N_1 \cap N_2$, which clearly holds when $N_1 \not\subseteq Q_s$ and $N_2 \not\subseteq Q_s$, since then $P_s \subseteq N_1$ and $P_s \subseteq N_2$. Thus for a plain texture $(S,\mathcal{S})$ and ditopology (τ,κ) on $(S,\mathcal{S})$ the product $\eta(s) \times \mu(s)$ is a $\mathcal{S}$-difilter for all $s \in S^\flat = S$. Moreover, $\eta(s) \times \mu(s)$ is regular since $N \in \eta(s)$ satisfies $P_s \subseteq N$ and $M \in \mu(s)$ satisfies $P_s \not\subseteq M$, whence $N \not\subseteq M$.

(2) The open nhds of $s \in S^\flat$ generate the $\mathcal{S}$-filter

$$\eta^*(s) = \{A \in \mathcal{S} \mid \exists G_k \in \tau \text{ with } G_k \not\subseteq Q_s,\ 1 \le k \le n, \text{ and } G_1 \cap \ldots \cap G_n \subseteq A\}.$$

Likewise, the closed conhds of $s \in S$ give us the $\mathcal{S}$-cofilter

$$\mu^*(s) = \{A \in \mathcal{S} \mid \exists F_k \in \kappa \text{ with } P_s \not\subseteq F_k,\ 1 \le k \le n, \text{ and } A \subseteq F_1 \cup \ldots \cup F_n\}.$$

Clearly, $\mu^*(s) = \{B \in \mathcal{S} \mid \exists K \in \kappa \text{ with } P_s \not\subseteq K \text{ and } B \subseteq K\}$, and again $\eta^*(s) \times \mu^*(s)$ is clearly regular.

Let $(S,\mathcal{S},\tau,\kappa)$ be a ditopological texture space and $\mathcal{F} \times \mathcal{G}$ a difilter on $(S,\mathcal{S})$. On analogy with filters in classical topology it would be tempting to say that $\mathcal{F} \times \mathcal{G}$ converges to $s \in S^\flat$ if $\eta(s) \times \mu(s) \subseteq \mathcal{F} \times \mathcal{G}$. However in general this would seem to be too weak a concept to support a satisfactory theory, and we make the following tentative definitions.

2.6. Definition. Let $(S,\mathcal{S},\tau,\kappa)$ be a ditopological texture space, $\mathcal{F}$ a $\mathcal{S}$-filter and $\mathcal{G}$ a $\mathcal{S}$-cofilter.

1. We say $\mathcal{F}$ *converges to* $s \in S^\flat$, and write $\mathcal{F} \to s$, if $\eta^*(s) \subseteq \mathcal{F}$.
2. We say $\mathcal{G}$ *converges to* $s \in S$, and write $\mathcal{G} \to s$, if $\mu^*(s) \subseteq \mathcal{G}$.
3. The difilter $\mathcal{F} \times \mathcal{G}$ is said to be *diconvergent* if $\mathcal{F} \to s$ and $\mathcal{G} \to s'$ for some $s, s' \in S$ satisfying $P_{s'} \not\subseteq Q_s$. In this case, s is called a *limit* and s' a *colimit* of $\mathcal{F} \times \mathcal{G}$.

2.7. Lemma. *Let* $(S,\mathcal{S},\tau,\kappa)$ *be a ditopological texture space,* $\mathcal{F}$ *a* $\mathcal{S}$*-filter and* $\mathcal{G}$ *a* $\mathcal{S}$*-cofilter.*

1. $\mathcal{F} \to s$ *if and only if* $G \in \tau,\ G \not\subseteq Q_s \implies G \in \mathcal{F}$.
2. $\mathcal{G} \to s$ *if and only if* $K \in \kappa,\ P_s \not\subseteq K \implies K \in \mathcal{G}$.

Proof. Immediate. □

2.8. Proposition. *Let $\mathcal{F} \times \mathcal{G}$ be a difiter on the ditopological texture space $(S, \mathcal{S}, \tau, \kappa)$ with $(S, \mathcal{S})$ plain. Then the following are equivalent:*

(1) *$\mathcal{F} \times \mathcal{G}$ is diconvergent.*
(2) *$\eta(s) \times \mu(s) \subseteq \mathcal{F} \times \mathcal{G}$ for some $s \in S$.*

Proof. (1) $\Longrightarrow$ (2) Let $\mathcal{F} \to s$, $\mathcal{G} \to s'$ with $P_{s'} \not\subseteq Q_s$. Clearly $\mu^*(s) \subseteq \mu^*(s')$, while $\eta(s) = \eta^*(s)$, $\mu(s) = \mu^*(s)$ since $(S, \mathcal{S})$ is plain. Hence $\eta(s) \times \mu(s) \subseteq \mathcal{F} \times \mathcal{G}$, as required.

(2) $\Longrightarrow$ (1) From the equalities above we deduce $\mathcal{F} \to s$ and $\mathcal{G} \to s$, whence $\mathcal{F} \times \mathcal{G}$ is diconvergent since $P_s \not\subseteq Q_s$ for all $s \in S$ in a plain texture. □

This proposition shows that for plain textures, Definition 2.6 gives a notion of convergence which is a straightforward analogue of that for the convergence of filters in a topological space. This is not the case in general, however, as the next example illustrates.

2.9. Example. Consider the texture $(L, \mathcal{L})$ of Examples 1.2 (1) with the discrete, codiscrete ditopology $\tau = \kappa = \mathcal{L}$. For $0 < r < 1$ we have $\eta^*(r) = \eta(r) = \{(0, r'] \mid r < r'\}$ and $\mu^*(r) = \mu(r) = \{(0, r'] \mid r' < r\}$. Let $\mathcal{F} = \eta^*(r)$, $\mathcal{G} = \mu^*(r)$ so that $\eta^*(r) \times \mu^*(r) \subseteq \mathcal{F} \times \mathcal{G}$. On the other hand $\mathcal{F} \to s \iff \eta^*(s) \subseteq \eta^*(r) \iff r \le s$ and $\mathcal{G} \to s' \iff \mu^*(s') \subseteq \mu^*(r) \iff s' \le r$, so $s' \le s$ which would contradict $P_{s'} \not\subseteq Q_s$. Hence $\mathcal{F} \times \mathcal{G}$ is not diconvergent in the sense of Definition 2.6.

2.10. Lemma. *Let $\mathcal{F} \times \mathcal{G}$ be a regular difilter on the ditopological texture space $(S, \mathcal{S}, \tau, \kappa)$.*

(1) *If $\mathcal{F} \to s$ then $B \in \mathcal{G} \implies]B[\subseteq Q_s$.*
(2) *If $\mathcal{G} \to s$ then $A \in \mathcal{F} \implies P_s \subseteq [A]$.*

Proof. (1) Let $\mathcal{F} \to s$ and take $B \in \mathcal{G}$. Suppose that $]B[\not\subseteq Q_s$. Then $]B[\in \mathcal{F}$ by Lemma 2.7 (1), and $]B[\subseteq B$ now gives a contradiction to the regularity of $\mathcal{F} \times \mathcal{G}$. Hence $]B[\subseteq Q_s$, as required.

(2) Dual to (1). □

This leads to the following definition.

2.11. Definition. Let $(S, \mathcal{S}, \tau, \kappa)$ be a ditopological texture space, $\mathcal{F}$ a filter and $\mathcal{G}$ a cofilter on $(S, \mathcal{S})$.

(1) $s \in S$ is called a *cluster point of* $\mathcal{F}$ if $P_s \subseteq [A]$ for all $A \in \mathcal{F}$.
(2) $s \in S$ is called a *cluster point of* $\mathcal{G}$ if $]B[\subseteq Q_s$ for all $B \in \mathcal{G}$.
(3) The difilter $\mathcal{F} \times \mathcal{G}$ is said to be *diclustering in* $(S, \mathcal{S}, \tau, \kappa)$ if for some $s, s' \in S$ with $P_{s'} \not\subseteq Q_s$, s' is a cluster point of $\mathcal{F}$ and s a cluster point of $\mathcal{G}$. In this case, s' is known as a *cluster point of* $\mathcal{F} \times \mathcal{G}$ and s as a *cocluster point of* $\mathcal{F} \times \mathcal{G}$.

Altay and Diker use a slightly stronger concept of cluster point in [1], but it is easy to check that this would lead to the same notion of diclustering as given above. From Lemma 2.10 and the above definition we have at once:

2.12. Proposition. *A diconvergent regular difilter on a ditopological texture space $(S, \mathcal{S}, \tau, \kappa)$ is diclustering in $(S, \mathcal{S}, \tau, \kappa)$.*

In general the converse of Proposition 2.12 is false, as the following example shows.

2.13. Example. Consider the texture $(L, \mathcal{L})$ of Examples 1.6 (1) with the discrete, codiscrete ditopology $\tau = \kappa = \mathcal{L}$. Choose $0 < r_0 < r_1 < 1$ and define $\mathcal{F} = \{(0, r] \mid r_1 < r \le 1\}$, $\mathcal{G} = \{(0, r] \mid 0 < r < r_0\}$. Clearly $\mathcal{F} \times \mathcal{G}$ is a regular difilter. Suppose $\mathcal{F} \to s$, $\mathcal{G} \to s'$ with $P_{s'} \not\subseteq Q_s$, that is $s < s'$. Choose $s' < a < s$ and let $A = (0, a]$. then

$$A \not\subseteq Q_s, A \in \tau \implies A \in \eta^*(s) \subseteq \mathcal{F},\ P_{s'} \not\subseteq A, A \in \kappa \implies A \in \mu^*(s') \subseteq \mathcal{G}$$

which gives the contradiction $A \in \mathcal{F} \cap \mathcal{G} \neq \emptyset$. Hence $\mathcal{F} \times \mathcal{G}$ is not diconvergent. On the other hand $(0, r] \in \mathcal{F} \implies P_{r_1} \subseteq (0, r]$ so r_1 is a cluster point of $\mathcal{F}$, and likewise r_0 is a cluster point of $\mathcal{G}$. Since $P_{s_1} \not\subseteq Q_{r_0}$ we see that $\mathcal{F} \times \mathcal{G}$ is diclustering.

Let us consider the situation for *maximal regular difilters*, that is for regular difilters which are maximal elements in the set of all regular difilters on $(S, \mathcal{S}, \tau, \kappa)$ ordered by set inclusion. We will require a characterization of maximal regular difilters, and the following concepts will be of interest.

2.14. Definition. Let $\mathcal{F}$ be a filter and $\mathcal{G}$ a cofilter on $(S, \mathcal{S})$:

(1) $\mathcal{F}$ is called *prime* if $A_1, A_2 \in \mathcal{S}$, $A_1 \cup A_2 \in \mathcal{F} \implies A_1 \in \mathcal{F}$ or $A_2 \in \mathcal{F}$.
(2) $\mathcal{G}$ is called *prime* if $B_1, B_2 \in \mathcal{S}$, $B_1 \cap B_2 \in \mathcal{G} \implies B_1 \in \mathcal{G}$ or $B_2 \in \mathcal{G}$.

Now we may give:

2.15. Proposition. *The following are equivalent for a regular difilter $\mathcal{F} \times \mathcal{G}$ on $(S, \mathcal{S})$.*

(1) *$\mathcal{F} \times \mathcal{G}$ is a maximal regular difilter.*
(2) *$\mathcal{F} \cup \mathcal{G} = \mathcal{S}$.*
(3) *$\mathcal{F}$ is a prime $\mathcal{S}$-filter and $\mathcal{G} = \mathcal{S} \setminus \mathcal{F}$.*
(4) *$\mathcal{G}$ is a prime $\mathcal{S}$-cofilter and $\mathcal{F} = \mathcal{S} \setminus \mathcal{G}$.*

Proof. (1) $\implies$ (2) Suppose that $\mathcal{F} \cup \mathcal{G} \neq \mathcal{S}$. Then there exists $C \in \mathcal{S}$ with $C \notin \mathcal{F}$ and $C \notin \mathcal{G}$. We claim that

$$C \cap F \not\subseteq G \ \forall (F, G) \in \mathcal{F} \times \mathcal{G} \quad \text{or} \quad F \not\subseteq C \cup G \ \forall (F, G) \in \mathcal{F} \times \mathcal{G}.$$

Suppose that this is not the case. Then

(1) $\qquad \exists (F_1, G_1) \in \mathcal{F} \times \mathcal{G}$ satisfying $C \cap F_1 \subseteq G_1$

and

(2) $\qquad \exists (F_2, G_2) \in \mathcal{F} \times \mathcal{G}$ satisfying $F_2 \subseteq C \cup G_2$.

Now $F_1 \cap F_2 \in \mathcal{F}$, $G_1 \cup G_2 \in \mathcal{G}$ and $\mathcal{F} \times \mathcal{G}$ is regular so $F_1 \cap F_2 \not\subseteq G_1 \cup G_2$ (Lemma 2.2 (3)). If we take $s \in F_1 \cap F_2$ with $s \notin G_1 \cup G_2$ then $s \in C$ by the inclusion (2) since $s \notin G_2$, and now we have the contradiction $s \in G_1$ by inclusion (1). This establishes our claim so we have the following two cases:

Case 1 $C \cap F \not\subseteq G \ \forall (F, G) \in \mathcal{F} \times \mathcal{G}$. Now define $\mathcal{F}^* = \{A \in \mathcal{S} \mid C \cap F \subseteq A$ for some $F \in \mathcal{F}\}$. It is clear that $\mathcal{F}^*$ is a $\mathcal{S}$-filter and that $\mathcal{F}^* \times \mathcal{G}$ is a regular difilter on $(S, \mathcal{S})$. Moreover $\mathcal{F} \times \mathcal{G} \subseteq \mathcal{F}^* \times \mathcal{G}$ and $C \in \mathcal{F}^* \setminus \mathcal{F}$ so $\mathcal{F} \times \mathcal{G} \subset \mathcal{F}^* \times \mathcal{G}$, which contradicts the maximality.

Case 2 $F \not\subseteq C \cup G \ \forall (F, G) \in \mathcal{F} \times \mathcal{G}$. A contradiction may be obtained by an argument dual to the above, and we omit the details.

This establishes $\mathcal{F} \cup \mathcal{G} = \mathcal{S}$, as required.

(2) $\implies$ (3) Since $\mathcal{F} \cup \mathcal{G} = \mathcal{S}$ and $\mathcal{F} \cap \mathcal{G} = \emptyset$ by regularity (Lemma 2.2 (1)) we deduce that $\mathcal{G} = \mathcal{S} \setminus \mathcal{F}$. Suppose that $\mathcal{F}$ is not prime. Then for some $A_1, A_2 \in \mathcal{S}$ we have $A_1 \cup A_2 \in \mathcal{F}$ but $A_1, A_2 \notin \mathcal{F}$. Now $A_1, A_2 \in \mathcal{S} \setminus \mathcal{F} = \mathcal{G}$, whence $A_1 \cup A_2 \in \mathcal{G}$ since $\mathcal{G}$ is a $\mathcal{S}$-cofilter. But then we have $A_1 \cup A_2 \in \mathcal{F} \cap \mathcal{G} \neq \emptyset$, which contradicts the regularity of $\mathcal{F} \times \mathcal{G}$.

(3) $\implies$ (4) Since $\mathcal{G} = \mathcal{S} \setminus \mathcal{F}$ we deduce $\mathcal{F} \cup \mathcal{G} = \mathcal{S}$, and an argument dual to the above easily establishes (4).

(4) $\implies$ (1) Let $\mathcal{F}' \times \mathcal{G}'$ be a regular difilter with $\mathcal{F} \times \mathcal{G} \subseteq \mathcal{F}' \times \mathcal{G}'$. Then $\mathcal{F} \subseteq \mathcal{F}'$ and $\mathcal{G} \subseteq \mathcal{G}'$ by Lemma 2.4, and $\mathcal{F}' \cap \mathcal{G}' = \emptyset$ by Lemma 2.2. Now $A \in \mathcal{F}' \implies A \notin \mathcal{G}' \implies A \notin \mathcal{G} \implies A \in \mathcal{S} \setminus \mathcal{G} = \mathcal{F}$ by (4). Hence $\mathcal{F}' \subseteq \mathcal{F}$ and we have $\mathcal{F}' = \mathcal{F}$. Likewise, $\mathcal{G}' = \mathcal{G}$ and we have established that $\mathcal{F} \times \mathcal{G}$ is a maximal regular difilter on $(S, \mathcal{S})$. $\square$

2.16. Corollary. *There exists a one to one correspondence between the set of maximal regular difilters on* $(S, \mathcal{S})$*, the set of prime filters on* $(S, \mathcal{S})$*, and the set of prime cofilters on* $(S, \mathcal{S})$*.*

Proof. Let $\mathcal{F}$ be a prime filter on $(S, \mathcal{S})$ and set $\mathcal{G} = \mathcal{S} \setminus \mathcal{F}$. Then $\mathcal{G}$ is a cofilter on $(S, \mathcal{S})$. Indeed $\emptyset \notin \mathcal{F} \implies \emptyset \in \mathcal{G}$ which shows that $\mathcal{G} \neq \emptyset$, while $S \in \mathcal{F} \implies S \notin \mathcal{G}$ so $\mathcal{G}$ satisfies $\mathbf{CF_1}$. For $\mathbf{CF_2}$ take $G \in \mathcal{G}$ and $G \supseteq G' \in \mathcal{S}$. Then $G' \notin \mathcal{G}$ would give $G' \in \mathcal{F}$ and hence the contradiction $G \in \mathcal{F}$ by $\mathbf{F_2}$. Thus $G' \in \mathcal{G}$. Finally, suppose $G_1, G_2 \in \mathcal{G}$ but $G_1 \cup G_2 \notin \mathcal{G}$. Then $G_1 \cup G_2 \in \mathcal{F}$, and we have the contradiction $G_1 \in \mathcal{F}$ or $G_2 \in \mathcal{F}$ since $\mathcal{F}$ is prime. This establishes $\mathbf{CF_3}$, and hence $\mathcal{G}$ is a cofilter on $(S, \mathcal{S})$ as claimed. Since $\mathcal{F} \cap \mathcal{G} = \emptyset$ we see that $\mathcal{F} \times \mathcal{G}$ is a regular difilter, and $\mathcal{F} \cup \mathcal{G} = \mathcal{S}$ shows that $\mathcal{F} \times \mathcal{G}$ is a maximal regular difilter by Proposition 2.15. Hence

$$\mathcal{F} \mapsto \mathcal{F} \times (\mathcal{S} \setminus \mathcal{F})$$

is a mapping between the prime filters on $(S, \mathcal{S})$ and the maximal regular difilters on $(S, \mathcal{S})$. Since for prime filters $\mathcal{F}, \mathcal{F}'$ on $(S, \mathcal{S})$ we have $\mathcal{F} \times (\mathcal{S} \setminus \mathcal{F}) = \mathcal{F}' \times (\mathcal{S} \setminus \mathcal{F}') \implies \mathcal{F} = \mathcal{F}'$ this mapping is injective, while by Proposition 2.15 (3) every maximal regular difilter on $(S, \mathcal{S})$ has the form $\mathcal{F} \times (\mathcal{S} \setminus \mathcal{F})$ with $\mathcal{F}$ prime, so it is also surjective.

This argument verifies that the above mapping is a bijection between the prime filters and the maximal regular difilters, and a bijection between the prime cofilters and the maximal regular difilters may be obtained in the same way. □

2.17. Example. Let us use Proposition 2.15 to characterize the maximal regular difilters on the texture $(L, \mathcal{L})$.

Let $\mathcal{F} \times \mathcal{G}$ be a maximal regular difilter on $(L, \mathcal{L})$ and define $r_0 = \inf\{r \in L \mid (0, r] \in \mathcal{F}\}$. There are three cases to consider:

Case 1: $r_0 = 0$. This clearly implies $\mathcal{F} = \mathcal{L} \setminus \{\emptyset\}$ and $\mathcal{G} = \{\emptyset\}$.

Case 2: $r_0 = 1$. This clearly implies $\mathcal{F} = \{L\}$ and $\mathcal{G} = \mathcal{S} \setminus \{L\}$.

Case 3: $0 < r_0 < 1$. In this case there are two possibilities

a) $\mathcal{F} = \{(0, r] \mid r_0 < r \leq 1\}$ and $\mathcal{G} = \{(0, r] \mid 0 < r \leq r_0\}$, or

b) $\mathcal{F} = \{(0, r] \mid r_0 \leq r \leq 1\}$ and $\mathcal{G} = \{(0, r] \mid 0 < r < r_0\}$.

Conversely, with $\mathcal{F}$ and $\mathcal{G}$ defined as above, $\mathcal{F}$ is a prime filter (and $\mathcal{G}$ a prime cofilter) on $(L, \mathcal{L})$, $\mathcal{F} \cap \mathcal{G} = \emptyset$ and $\mathcal{F} \cup \mathcal{G} = \mathcal{L}$, so $\mathcal{F} \times \mathcal{G}$ is a maximal regular difilter on $(L, \mathcal{L})$ in each case.

With regard to the existence of maximal regular difilters under the axiom of choice we have the following.

2.18. Proposition. *Let* $\mathcal{F} \times \mathcal{G}$ *be a regular difilter on* $(S, \mathcal{S})$*. Then there exists a maximal regular difilter* $\mathcal{H} \times \mathcal{K}$ *satisfying* $\mathcal{F} \times \mathcal{G} \subseteq \mathcal{H} \times \mathcal{K}$*.*

Proof. Let $\mathcal{F}_j \times \mathcal{G}_j$, $j \in J$, be a chain of regular difilters satisfying $\mathcal{F} \times \mathcal{G} \subseteq \mathcal{F}_j \times \mathcal{G}_j$ for all $j \in J$. Let $\mathcal{F}' = \bigcup_{j \in J} \mathcal{F}_j$. We claim that $\mathcal{F}'$ is a $\mathcal{S}$-filter. Since $\emptyset \notin \mathcal{F}_j$ for all $j \in J$ we have $\emptyset \notin \mathcal{F}'$ so $\mathbf{F_1}$ holds. $\mathbf{F_2}$ is also trivial, so we verify $\mathbf{F_3}$. Take $A_1, A_2 \in \mathcal{F}'$. Then $A_1 \in \mathcal{F}_j$, $A_2 \in \mathcal{F}_k$ for some $j, k \in J$. Without loss of generality we may suppose $\mathcal{F}_j \subseteq \mathcal{F}_k$ and so $A_1 \cap A_2 \in \mathcal{F}_k \subseteq \mathcal{F}'$, whence $\mathbf{F_3}$ holds also. In just the same way $\mathcal{G}' = \bigcup_{j \in J} \mathcal{G}_j$ is a $\mathcal{S}$-cofilter, and a similar argument to the above shows that $\mathcal{F}' \times \mathcal{G}'$ is regular. Hence $\mathcal{F}' \times \mathcal{G}'$ is an upper bound for the chain $\mathcal{F}_j \times \mathcal{G}_j$, $j \in J$ in the set of regular difilters on $(S, \mathcal{S})$ containing $\mathcal{F} \times \mathcal{G}$ and ordered by inclusion. By Zorn's Lemma we see that this partially ordered set of regular difilters contains a maximal element $\mathcal{H} \times \mathcal{K}$. Hence $\mathcal{F} \times \mathcal{G} \subseteq \mathcal{H} \times \mathcal{K}$

and it is clear that $\mathcal{H} \times \mathcal{K}$ is in fact maximal in the set of all regular difilters on $(S, \mathcal{S})$ ordered by inclusion. $\square$

2.19. Proposition. *Let $\mathcal{F} \times \mathcal{G}$ be a maximal regular difilter on $(S, \mathcal{S}, \tau, \kappa)$. Then $\mathcal{F} \times \mathcal{G}$ is diconvergent on $(S, \mathcal{S}, \tau, \kappa)$ if and only if it diclustering.*

Proof. In view of Proposition 2.12 we only have to prove sufficiency. Hence for $P_{s'} \not\subseteq Q_s$ let s' be a cluster point and s a cocluster point of $\mathcal{F} \times \mathcal{G}$. Take $G \in \tau$ with $G \not\subseteq Q_s$. Suppose that $G \in \mathcal{G}$. Then $]G[\, \subseteq Q_s$ since s is a cluster point of $\mathcal{G}$, and this is a contradiction since $]G[\, = G$. Hence $G \in \mathcal{S} \setminus \mathcal{G} = \mathcal{F}$ by Proposition 2.15, and we have established $\mathcal{F} \to s$ by Lemma 2.7 (1). A dual argument gives $\mathcal{G} \to s'$, whence s is a limit and s' a colimit of $\mathcal{F} \times \mathcal{G}$. Since $P_{s'} \not\subseteq Q_s$ this shows that $\mathcal{F} \times \mathcal{G}$ is diconvergent on $(S, \mathcal{S}, \tau, \kappa)$. $\square$

We end this section by giving characterizations of dicompactness [3, 10] of ditopological texture spaces in terms of the concepts given here (c.f. [1]).

2.20. Theorem. *The following are equivalent for a ditopological texture space $(S, \mathcal{S}, \tau, \kappa)$:*

(1) *$(S, \mathcal{S}, \tau, \kappa)$ is dicompact.*
(2) *Every regular difilter on $(S, \mathcal{S}, \tau, \kappa)$ is diclustering.*
(3) *Every maximal regular difilter on $(S, \mathcal{S}, \tau, \kappa)$ is diconvergent.*

Proof. (1) $\Longrightarrow$ (2) Let $\mathcal{F} \times \mathcal{G}$ be a regular difilter and consider the family $\mathcal{D} = \{([A],]B[) \mid A \in \mathcal{F},\ B \in \mathcal{G}\} \subseteq \kappa \times \tau$. Clearly $\mathcal{D}$ has the fep because $\mathcal{F} \times \mathcal{G}$ has the fep by Lemma 2.2 (2), $A \subseteq [A]$ and $]B[\, \subseteq B$. In view of Theorem 1.5 (2), dicompactness gives $\bigcap \operatorname{dom} \mathcal{D} \not\subseteq \bigvee \operatorname{ran} \mathcal{D}$, so we may choose $s, s' \in S$ satisfying $\bigcap \operatorname{dom} \mathcal{D} \not\subseteq Q_{s'}$, $P_{s'} \not\subseteq Q_s$, and $P_s \not\subseteq \bigvee \operatorname{ran} \mathcal{D}$. Hence, $A \in \mathcal{F} \implies [A] \in \operatorname{dom} \mathcal{D} \implies P_{s'} \subseteq [A]$ and $B \in \mathcal{G} \implies]B[\, \in \operatorname{ran} \mathcal{D} \implies]B[\, \subseteq Q_s$, which establishes that s' is a cluster point of $\mathcal{F}$ and s a cluster point of $\mathcal{G}$. Thus $\mathcal{F} \times \mathcal{G}$ is diclustering in $(S, \mathcal{S}, \tau, \kappa)$, as required.

(2) $\Longrightarrow$ (1) Take $\mathcal{D} \subseteq \kappa \times \tau$ satisfying the fep and define

$$\mathcal{F} = \{A \in \mathcal{S} \mid \exists\, K_1, \dots, K_n \in \operatorname{dom} \mathcal{D} \text{ with } K_1 \cap \dots \cap K_n \subseteq A\},$$
$$\mathcal{G} = \{B \in \mathcal{S} \mid \exists\, G_1, \dots, G_m \in \operatorname{ran} \mathcal{D} \text{ with } B \subseteq G_1 \cup \dots \cup G_m\}.$$

It is trivial to verify that $\mathcal{F} \times \mathcal{G}$ is a regular difilter, so there exists $s, s' \in S$ satisfying $P_{s'} \not\subseteq Q_s$ for which s' is a cluster point of $\mathcal{F}$ and s a cluster point of $\mathcal{G}$. Since $\mathcal{D} \subseteq \mathcal{F} \times \mathcal{G}$, for each $(K, G) \in \mathcal{D}$ we have $P_{s'} \subseteq K$ and $G \subseteq Q_s$. Hence $P_{s'} \subseteq \bigcap \operatorname{dom} \mathcal{D}$, $\bigvee \operatorname{ran} \mathcal{D} \subseteq Q_s$, so $\bigcap \operatorname{dom} \mathcal{D} \not\subseteq \bigvee \operatorname{ran} \mathcal{D}$. Hence $\mathcal{D}$ is bound and so $(S, \mathcal{S}, \tau, \kappa)$ is dicompact.

(2) $\Longleftrightarrow$ (3) Immediate from Proposition 2.19. $\square$

3. Completeness of Di-uniformities

Let us first define the notion of Cauchy difilter and completeness for dicovering uniformities [11, 12]. We recall that the uniform ditopology (τ, κ) of υ [11, Definition 4.5] is given by:

$$G \in \tau \iff G \not\subseteq Q_s \implies \exists\, \mathcal{C} \in \upsilon \text{ with } \operatorname{St}(\mathcal{C}, P_s) \subseteq G,$$
$$K \in \kappa \iff P_s \not\subseteq K \implies \exists\, \mathcal{C} \in \upsilon \text{ with } K \subseteq \operatorname{CSt}(\mathcal{C}, Q_s).$$

In what follows, convergence and clustering of difilters on $(S, \mathcal{S}, \upsilon)$ will be with respect to the uniform ditopology (τ, κ).

3.1. Definition. Let $(S, \mathcal{S})$ be a texture and υ a dicovering uniformity on $(S, \mathcal{S})$.

(1) A difilter $\mathcal{F} \times \mathcal{G}$ on $(S, \mathcal{S})$ is said to be *Cauchy* if $(\mathcal{F} \times \mathcal{G}) \cap \mathcal{C} \neq \emptyset$ for all $\mathcal{C} \in \upsilon$.
(2) $(S, \mathcal{S}, \upsilon)$ is called *dicomplete* if every regular Cauchy difilter is diconvergent.

3.2. Proposition. *A difilter which is diconvergent for the uniform ditopology of a dicovering uniformity υ on $(S, \mathcal{S})$ is Cauchy.*

Proof. Let $\mathcal{F} \times \mathcal{G}$ be a diconvergent difilter on $(S, \mathcal{S})$ and $\mathcal{C} \in \upsilon$. Since by [11, Proposition 4.8] the di-uniformity υ has a base of open, coclosed dicovers we may choose $\mathcal{D} \in \upsilon$ open, coclosed so that $\mathcal{D} \prec \mathcal{C}$. On the other hand, by ([11, Definition 3.6], the di-uniformity has a base of anchored dicovers so we have an anchored dicover $\mathcal{E} \in \upsilon$ with $\mathcal{E} \prec \mathcal{D}$.

On the other hand we have $s, s' \in S$ satisfying $P_{s'} \not\subseteq Q_s$ for which $\mathcal{F} \to s$ and $\mathcal{G} \to s'$. Choose $u \in S$ with $P_{s'} \not\subseteq Q_u$ and $P_u \not\subseteq Q_s$. Since $\mathcal{E}$ is anchored we have in particular $\mathcal{P} \prec \mathcal{E}$, so there exists $A\, \mathcal{E}\, B$ with $P_u \subseteq A$ and $B \subseteq Q_u$. Now we have $G\, \mathcal{D}\, F$ with $A \subseteq G$, $F \subseteq B$ and so $G \not\subseteq Q_s$ and $P_{s'} \not\subseteq F$. Since $G \in \tau$, $F \in \kappa$ we obtain $G \in \eta^*(s) \subseteq \mathcal{F}$ since $F \to s$ and $F \in \mu^*(s') \subseteq \mathcal{G}$ since $\mathcal{G} \to s'$. Finally we have $C\, \mathcal{C}\, D$ with $G \subseteq C$, $D \subseteq F$, whence $(C, D) \in \mathcal{C} \cap (\mathcal{F} \times \mathcal{G}) \neq \emptyset$. Thus, $\mathcal{F} \times \mathcal{G}$ is Cauchy as required. □

3.3. Proposition. *A diclustering Cauchy difilter is diconvergent.*

Proof. Let $\mathcal{F} \times \mathcal{G}$ be a Cauchy difilter on $(S, \mathcal{S}, \upsilon)$ which is diclustering in $(S, \mathcal{S}, \tau, \kappa)$. Then we have $P_{s'} \not\subseteq Q_s$ with s' a cluster point and s a cocluster point of $\mathcal{F} \times \mathcal{G}$. Take $u', u \in S$ satisfying $P_{s'} \not\subseteq Q_{u'}$, $P_{u'} \not\subseteq Q_u$ and $P_u \not\subseteq Q_s$. We prove that $\mathcal{F} \to u$. To this end take $G \in \tau$ with $G \not\subseteq Q_u$. By the above we now have $\mathcal{C} \in \upsilon$ satisfying $\mathrm{St}(\mathcal{C}, P_u) \subseteq G$, and without loss of generality we may suppose that $\mathcal{C}$ is closed and coopen [11, Proposition 4.8]. Since $\mathcal{F} \times \mathcal{G}$ is Cauchy there exists $(A, B) \in (\mathcal{F} \times \mathcal{G}) \cap \mathcal{C}$, whence $B =]B[\subseteq Q_s$ since s is a cluster point of $\mathcal{G}$. This gives $P_u \not\subseteq B$, whence $A \subseteq \mathrm{St}(\mathcal{C}, P_u) \subseteq G$ and we obtain $G \in \mathcal{F}$ as claimed. A dual argument may be used to show that $\mathcal{G} \to u'$, and hence $\mathcal{F} \times \mathcal{G}$ is diconvergent in $(S, \mathcal{S}, \tau, \kappa)$ since $P_{u'} \not\subseteq Q_u$. □

3.4. Corollary. *Every dicovering uniformity with a dicompact uniform ditopology is dicomplete.*

Proof. Let $(S, \mathcal{S}, \upsilon)$ have a dicompact uniform ditopology and let $\mathcal{F} \times \mathcal{G}$ be a regular Cauchy difilter. By Theorem 2.20, $\mathcal{F} \times \mathcal{G}$ is diclustering. Hence $\mathcal{F} \times \mathcal{G}$ diconverges by Proposition 3.3, which establishes dicompleteness. □

Now we generalize the notion of total boundedness to dicovering uniformities.

3.5. Definition. The dicovering uniformity υ on $(S, \mathcal{S})$ is said to be *totally bounded* if each $\mathcal{C} \in \upsilon$ has a finite, cofinite sub-dicover.

3.6. Proposition. *Every dicovering uniformity with a dicompact uniform ditopology is totally bounded.*

Proof. It suffices to note that by [11, Proposition 4.8], the dicovering uniformity υ has a base of open, coclosed dicovers. □

3.7. Proposition. *Suppose that υ is a totally bounded dicovering uniformity on $(S, \mathcal{S})$ and let $\mathcal{F} \times \mathcal{G}$ be a maximal regular difilter. Then $\mathcal{F} \times \mathcal{G}$ is Cauchy.*

Proof. Take $\mathcal{C} \in \upsilon$. Since υ is totally bounded we have $A_j \mathcal{C} B_j$, $j \in J = \{1, 2, \ldots, n\}$, for which $\{(A_j, B_j) \mid j \in J\}$ is a dicover of $(S, \mathcal{S})$. Define $J_1 = \{j \in J \mid B_j \notin \mathcal{G}\}$ and $J_2 = J \setminus J_1$. By the definition of dicover we have $\bigcap_{j \in J_1} B_j \subseteq \bigcup_{j \in J_2} A_j$.

Now $j \in J_1 \implies B_j \notin \mathcal{G} \implies B_j \in \mathcal{F}$ by Proposition 2.15 (2) since $\mathcal{F} \times \mathcal{G}$ is a maximal regular difilter, whence $\bigcap_{j \in J_1} B_j \in \mathcal{F}$ and so $\bigcup_{j \in J_2} A_j \in \mathcal{F}$ by the above inclusion. However, by Proposition 2.15 (3), $\mathcal{F}$ is prime and so there exists $k \in J_2$ satisfying $A_k \in \mathcal{F}$. On the other hand since $k \notin J_1$ we have $B_k \in \mathcal{G}$, which gives $(A_k, B_k) \in (\mathcal{F} \times \mathcal{G}) \cap \mathcal{C} \neq \emptyset$. This verifies that $\mathcal{F} \times \mathcal{G}$ is Cauchy. □

Now we may give:

3.8. Theorem. *Let υ be a dicovering uniformity on $(S,\mathcal{S})$ with uniform ditopology (τ,κ). Then $(S,\mathcal{S},\tau,\kappa)$ is dicompact if and only if υ is totally bounded and dicomplete.*

Proof. In view of Corollary 3.4 and Proposition 3.6 we need only prove the sufficiency. Hence, let υ be totally bounded and dicomplete. If $\mathcal{F}\times\mathcal{G}$ is a maximal regular difilter on $(S,\mathcal{S})$ it is Cauchy by Proposition 3.7, and hence diconvergent. This shows that (τ,κ) is dicompact by Theorem 2.20 (3). $\square$

The following concepts will be useful in dealing with finite, cofinite dicovers.

3.9. Definition. Let $\mathcal{C}=\{(A_j,B_j)\mid j\in J\}$ be a dicover of $(S,\mathcal{S})$.

(1) $\mathcal{C}$ is said to be *excluding* if $A_j\not\subseteq B_j$ for all $j\in J$.
(2) (A_k,B_k), $k\in J$, is said to be *essential in* $\mathcal{C}$ if $\{(A_j,B_j)\mid j\in J\setminus\{k\}\}$ is not a dicover of $(S,\mathcal{S})$.
(3) $\mathcal{C}$ is called *essential* if every (A_j,B_j), $j\in J$, is essential in $\mathcal{C}$.

3.10. Lemma. *Let $(S,\mathcal{S})$ be a texture.*

(1) *A finite, cofinite dicover has an essential subdicover.*
(2) *An essential finite, cofinite dicover is excluding.*
(3) *A finite excluding dicover is anchored.*

Proof. (1) Since the number of elements of a finite, cofinite dicover is finite, non-essential elements may be removed until an essential subdicover is obtained.

(2) Let $\mathcal{C}=\{(A_j,B_j)\mid j\in J\}$, J finite, be essential. Then $\mathcal{C}\setminus\{(A_k,B_k)\}$ is not a dicover for each $k\in J$, so there exists a partition K_1, K_2 of $K=J\setminus\{k\}$ satisfying $\bigcap_{j\in K_1}B_j\not\subseteq\bigcup_{j\in K_2}A_j$. Take $s\in\bigcap_{j\in K_1}B_j\setminus\bigcup_{j\in K_2}A_j$. Since $\{K_1\cup\{k\},\ K_2\}$ is a partition of J and $\mathcal{C}$ is a dicover, $\bigcap_{j\in K_1\cup\{k\}}B_j\subseteq\bigcup_{j\in K_2}A_j$. Hence $s\notin B_k$. Likewise, considering the partition $\{K_1,\ K_2\cup\{k\}\}$ leads to $s\in A_k$, and so $A_k\not\subseteq B_k$ as required.

(3) Let $\mathcal{C}$ be a finite excluding dicover of $(S,\mathcal{S})$. To show that $\mathcal{C}$ is anchored we must first establish $\mathcal{P}\prec\mathcal{C}$, so take $s\in S^\flat$. Suppose that $P_s\not\subseteq A_j$ or $B_j\not\subseteq Q_s$ for all $j\in J$, and define $J_1=\{j\in J\mid P_s\subseteq B_j\}$, $J_2=J\setminus J_1$. Since $\mathcal{C}$ is a dicover $P_s\subseteq\bigcap_{j\in J_1}B_j\subseteq\bigvee_{j\in J_2}A_j$, whence $P_s\subseteq\bigcup_{j\in J_2}A_j$ since by finiteness the family $\{A_j\mid j\in J\}$ is finite. Hence, $P_s\subseteq A_j$ for some $j\in J_2$. On the other hand $j\in J_2\implies P_s\not\subseteq B_j\implies B_j\subseteq Q_s\implies P_s\not\subseteq A_j$ by hypothesis. This contradiction establishes $\mathcal{P}\prec\mathcal{C}$.

Secondly, take $j\in J$. Then $A_j\not\subseteq B_j$, whence $A_j\not\subseteq Q_s, P_s\not\subseteq B_j$ for some $s\in S$. Associate this s with (A_j,B_j). To show [11, Definition 2.1 (2 a)] take $A_j\not\subseteq Q_u$. Clearly, setting $(A',B')=(A_j,B_j)$ gives $A'\mathcal{C}B'$ for which $A'\not\subseteq Q_u$ and $P_s\not\subseteq B'$. Dually, [11, Definition 2.1 (2 b)] may be established in the same way. $\square$

It is well known that in the classical case, a totally bounded covering uniformity actually has a base of finite covers. It is not known if the corresponding result holds for dicovering uniformities on a texture, but we may establish the following result.

3.11. Proposition. *A totally bounded dicovering uniformity has a base of finite, cofinite excluding dicovers if and only if it has a base of excluding dicovers.*

Proof. Necessity is clear, so suppose that the totally bounded dicovering uniformity υ has a base of excluding dicovers. Then for $\mathcal{C}\in\upsilon$ we may take an excluding dicover $\mathcal{D}\in\upsilon$ with $\mathcal{D}\prec_{(\star)}\mathcal{C}$.

Since υ is totally bounded there exists a dicover $\mathcal{E}=\{(A_j,B_j)\mid j\in J\}\subseteq\mathcal{D}$ with J finite, and for each $j\in J$ we may choose $A_j^*\mathcal{C}B_j^*$ with $\mathrm{St}(\mathcal{D},A_j)\subseteq A_j^*$ and $B_j^*\subseteq$

$\mathrm{CSt}(\mathcal{D}, B_j)$. Since $A_j \subseteq A_j^*$, $B_j^* \subseteq B_j$ and $A_j \not\subseteq B_j$ we have $A_j^* \not\subseteq B_j^*$, so $\mathcal{E}^* = \{(A_j^*, B_j^*) \mid j \in J\}$ is a finite, cofinite excluding subdicover of $\mathcal{C}$. Since $\mathcal{E}^* \prec \mathcal{C}$ it will suffice to show that $\mathcal{D} \prec \mathcal{E}^*$, whence $\mathcal{E}^* \in \upsilon$. To this end take $U\mathcal{D}V$ and suppose that $U \subseteq B_j$ or $A_j \subseteq V$ for all $j \in J$. Let $J_1 = \{j \in J \mid U \subseteq B_j\}$ and $J_2 = J \setminus J_1$. Since $\mathcal{E}$ is a dicover we have $U \subseteq \bigcap_{j \in J_1} B_j \subseteq \bigcup_{j \in J_2} A_j \subseteq V$, which is a contradiction. Hence there exists $j \in J$ with $U \not\subseteq B_j$ and $A_j \not\subseteq V$, so $U \subseteq \mathrm{St}(\mathcal{D}, A_j) \subseteq A_j^*$, $B_j^* \subseteq \mathrm{CSt}(\mathcal{D}, B_j) \subseteq V$. This establishes $\mathcal{D} \prec \mathcal{E}^*$, as required. □

3.12. Example. Consider the unit interval texture $(\mathbb{I}, \mathcal{I})$ with the usual dicovering uniformity $\upsilon_{\mathbb{I}}$ [11, Example 3.8]. We recall that the family $\{\mathcal{D}_\epsilon \mid \epsilon > 0\}$, where $\mathcal{D}_\epsilon = \{([0, r+\epsilon), [0, r-\epsilon]) \mid r \in \mathbb{I}\}$, is a base for $\upsilon_{\mathbb{I}}$.

The uniform ditopology of $(\mathbb{I}, \mathcal{I}, \upsilon_{\mathbb{I}})$ is the usual ditopology of $(\mathbb{I}, \mathcal{I})$. Indeed, take $0 < r < 1$ and $[0, r) \not\subseteq Q_s$. Then $s < r$, and if we set $\epsilon = \frac{1}{2}(r - s) > 0$ it is trivial to verify $\mathrm{St}(\mathcal{D}_\epsilon, P_s) \subseteq [0, r)$, whence $[0, r)$ is open for the uniform ditopology. On the other hand $[0, r] \not\subseteq Q_r$ and for any $\epsilon > 0$ we have $[0, r+\epsilon) \subseteq \mathrm{St}(\mathcal{D}_\epsilon, P_r)$, whence $\mathrm{St}(\mathcal{D}_\epsilon, P_r) \not\subseteq [0, r]$ and so $[0, r]$ is not open. Hence the only open sets in the uniform ditopology are $\emptyset$, $\mathbb{I}$ and $[0, r)$, $0 < r < 1$. Likewise a dual argument shows that the only closed sets are $\mathbb{I}$, $\emptyset$ and $[0, r]$, $0 < r < 1$.

As shown in Examples 1.6 (2), the usual ditopology $(\tau_{\mathbb{I}}, \kappa_{\mathbb{I}})$ on $(\mathbb{I}, \mathcal{I})$ is dicompact, whence $(\mathbb{I}, \mathcal{I}, \upsilon_{\mathbb{I}})$ is dicomplete and totally bounded by Theorem 3.8. Since the basic dicovers $\mathcal{D}_\epsilon$, $\epsilon > 0$, are clearly excluding we see from Proposition 3.11 that $(\mathbb{I}, \mathcal{I}, \upsilon_{\mathbb{I}})$ has a base of finite, cofinite excluding dicovers. More specifically, let us show that the family $\mathfrak{E}$ of all finite, cofinite open, coclosed excluding dicovers actually forms a base for $\upsilon_{\mathbb{I}}$.

To this end take $\mathcal{E} = \{(A_j, B_j) \mid 1 \le j \le n\} \in \mathfrak{E}$. Since the union of the sets A_j is $\mathbb{I}$, one of these sets must equal $\mathbb{I}$, and we assume without loss of generality that $A_1 = [0, 1]$. Hence $B_1 = [0, t_1]$ with $t_1 < 1$ since $A_1 \not\subseteq B_1$ and B_1 is closed. Likewise, the intersection of the sets B_j is empty, so without loss of generality we have $A_2 = [0, s_2)$, $s_2 > 0$, and $B_2 = \emptyset$. Finally, the remaining elements[†] of $\mathcal{C}$ have the form $A_k = [0, s_k)$, $B_k = [0, t_k]$ where $s_k > t_k$ for $k = 3, \dots, n$. Now define $\epsilon > 0$ by

$$\epsilon = \frac{1}{2} \min\{1 - s_1, t_2, s_3 - t_3, \dots, s_n - t_n\}.$$

If we let $u_1 = \frac{1}{2}(1 + s_1)$, $u_2 = \frac{1}{2} t_2$ and $u_k = \frac{1}{2}(s_k + t_k)$ for $k = 3, \dots, n$ then it is trivial to verify that $\mathcal{C} = \{([0, u_j + \epsilon), [0, u_j - \epsilon]) \mid 1 \le j \le n\} \prec \mathcal{E}$. On the other hand $\mathcal{C} \prec \mathcal{D}_\epsilon$, and since $\mathcal{D}_{\epsilon/2} \prec_{(\star)} \mathcal{D}_\epsilon$ an argument very similar to that used in the proof of Proposition 3.11 gives $\mathcal{D}_{\epsilon/2} \prec \mathcal{E}$, whence $\mathcal{E} \in \upsilon_{\mathbb{I}}$.

Now take any $\mathcal{C} \in \upsilon_{\mathbb{I}}$. By [11, Proposition 4.8] there exists an open, coclosed dicover $\mathcal{D} \in \upsilon_{\mathbb{I}}$ with $\mathcal{D} \prec \mathcal{C}$, and since $\upsilon_{\mathbb{I}}$ is totally bounded we have a finite, cofinite subdicover $\mathcal{E}$ of $\mathcal{D}$. Without loss of generality we may assume $\mathcal{E}$ is essential by Lemma 3.10 (1), and then $\mathcal{E}$ is excluding by Lemma 3.10 (2). Hence $\mathcal{E} \in \mathfrak{E}$, while $\mathcal{E} \prec \mathcal{C}$, so $\mathfrak{E}$ is a base of $\upsilon_{\mathbb{I}}$.

3.13. Proposition. *An initial dicovering uniformity of totally bounded dicovering uniform spaces is totally bounded.*

Proof. Let υ be the initial dicovering uniformity on $(S, \mathcal{S})$ generated by the totally bounded dicovering uniform spaces $(T_j, \mathcal{T}_j, \nu_j)$, $j \in J$ and the difunctions $(f_j, F_j) : (S, \mathcal{S}) \to (T_j, \mathcal{T}_j)$, $j \in J$. By [11, Definition 5.23], a base for υ is given by dicovers of the form $\left(\bigwedge_{k=1}^{n} (f_{j_k}, F_{j_k})^{-1} (\mathcal{C}_{j_k})^{\Delta} \right)^{\Delta}$, $\mathcal{C}_{j_k} \in \nu_{j_k}$. Since ν_{j_k} is totally bounded, $\mathcal{C}_{j_k}$ has a

[†]For simplicity we are assuming that only one A_j equals $\mathbb{I}$ and that only one B_j equals $\emptyset$. It is easy to modify the argument if this is not the case

finite, cofinite subdicover $\mathcal{D}_{j_k}$. Clearly $(f_{j_k}, F_{j_k})^{-1}(\mathcal{D}_{j_k})$ is also finite and cofinite, so by Lemma 3.10 (1) we may choose an essential finite, cofinite subdicover $\mathcal{E}_{j_k}$. Hence

$$\mathcal{E}_{j_k} \prec (f_{j_k}, F_{j_k})^{-1}(\mathcal{D}_{j_k}) \prec (f_{j_k}, F_{j_k})^{-1}(\mathcal{C}_{j_k}),$$

and so

$$\mathcal{E}_{j_k} \prec_{(\Delta)} (f_{j_k}, F_{j_k})^{-1}(\mathcal{D}_{j_k})^{\Delta} \prec (f_{j_k}, F_{j_k})^{-1}(\mathcal{C}_{j_k})^{\Delta}.$$

By Lemma 3.10 (2,3) the dicover $\mathcal{E}_{j_k}$ is anchored, so by [11, Lemma 2.2 (3 i)] we have $\mathcal{E}_{j_k} \prec (f_{j_k}, F_{j_k})^{-1}(\mathcal{D}_{j_k})^{\Delta}$. Hence

$$\bigwedge_{k=1}^{n} \mathcal{E}_{j_k} \prec \bigwedge_{k=1}^{n} (f_{j_k}, F_{j_k})^{-1}(\mathcal{D}_{j_k})^{\Delta} \prec \bigwedge_{k=1}^{n} (f_{j_k}, F_{j_k})^{-1}(\mathcal{C}_{j_k})^{\Delta}.$$

Again $\bigwedge_{k=1}^{n} \mathcal{E}_{j_k}$ is finite and cofinite so we have an essential finite, cofinite subdicover $\mathcal{E}$. Hence, by the same argument as used above we obtain

$$\mathcal{E} \prec \Big(\bigwedge_{k=1}^{n} \mathcal{E}_{j_k} \Big)^{\Delta} \prec \Big(\bigwedge_{k=1}^{n} (f_{j_k}, F_{j_k})^{-1}(\mathcal{C}_{j_k})^{\Delta} \Big)^{\Delta}$$

which shows that υ is totally bounded. □

3.14. Theorem. *Every completely biregular ditopological texture space* ([8]*, see also* [11, Definition 4.11]) *has a compatible totally bounded dicovering uniformity.*

Proof. According to [11, Theorem 5.16], the initial direlational uniformity generated by the space $(\mathbb{I}, \mathcal{I}, \mathcal{U}_{\mathbb{I}})$ and the family of bicontinuous difunctions from the completely biregular ditopological texture space $(S, \mathcal{S}, \tau, \kappa)$ to $(\mathbb{I}, \mathcal{I}, \tau_{\mathbb{I}}, \kappa_{\mathbb{I}})$ is compatible with the ditopology (τ, κ). In view of the equivalence between direlational uniformities and dicovering uniformities established in [11, 12], the initial dicovering uniformity υ generated by the space $(\mathbb{I}, \mathcal{I}, \upsilon_{\mathbb{I}})$ and the same family of difunctions is also compatible. However $(\mathbb{I}, \mathcal{I}, \upsilon_{\mathbb{I}})$ is totally bounded (Example 3.12), so υ is totally bounded by Proposition 3.13. Thus $(S, \mathcal{S}, \tau, \kappa)$ has a compatible totally bounded dicovering uniformity. □

3.15. Example. Consider the texture $(L, \mathcal{L})$ with the discrete, codiscrete ditopology $(\mathcal{L}, \mathcal{L})$. Note that $(L, \mathcal{L}, \mathcal{L}, \mathcal{L})$ is completely biregular. Indeed, take $G \in \mathcal{L}$ and $s \in L$ with $G \not\subseteq Q_s$, whence $G = (0, r]$ and $s < r$. Define $\varphi : L \to \mathbb{I}$ by

$$\varphi(t) = \begin{cases} 0 & t \le r \\ 1 & r < t \end{cases}.$$

Clearly φ satisfies the condition of [6, Lemma 3.4], so we have a difunction $(f, F) : (L, \mathcal{L}) \to (\mathbb{I}, \mathcal{I})$ for which $f^{\leftarrow}B = F^{\leftarrow}B = \psi^{\leftarrow}B$ for all $B \in \mathcal{I}$, where $\psi^{\leftarrow}B = \bigvee\{P_s \mid P_{\psi(u)} \subseteq B \,\forall\, u \in L \text{ with } P_s \not\subseteq Q_u\}$. We deduce easily that $P_s \subseteq f^{\leftarrow}P_0$ and $F^{\leftarrow}Q_1 \subseteq G$, while (f, F) is clearly bicontinuous, which verifies that $(L, \mathcal{L}, \mathcal{L}, \mathcal{L})$ is completely regular. The proof of complete coregularity is dual, and is omitted.

We deduce that the initial dicovering uniformity υ generated by the space $(\mathbb{I}, \mathcal{I}, \upsilon_{\mathbb{I}})$ and the bicontinuous difunctions between $(L, \mathcal{L}, \mathcal{L}, \mathcal{L})$ and $(\mathbb{I}, \mathcal{I}, \tau_{\mathbb{I}}, \kappa_{\mathbb{I}})$ is compatible with $(\mathcal{L}, \mathcal{L})$. Since by Theorem 3.14, $(L, \mathcal{L}, \upsilon)$ is totally bounded, this di-uniform space cannot be dicomplete because the uniform ditopology $(\mathcal{L}, \mathcal{L})$ is not dicompact by Examples 1.6 (1).

To obtain more specific information about υ let us consider the embedding $\psi : L \to \mathbb{I}$, $r \mapsto r$ for $0 < r \le 1$. If $P_r \not\subseteq Q_{r'}$ in $(L, \mathcal{L})$ then $r' < r$, whence $\psi(r') < \psi(r)$ and so $P_{\psi(r')} \not\subseteq Q_{\psi(r)}$ in $(\mathbb{I}, \mathcal{I})$. Hence ψ satisfies the condition of [6, Lemma 3.4], and as above we have a difunction $(f, F) : (L, \mathcal{L}) \to (\mathbb{I}, \mathcal{I})$ for which $f^{\leftarrow}B = F^{\leftarrow}B = \psi^{\leftarrow}B$ for all $B \in \mathcal{I}$. It is easy to verify that for $B = [0, r]$ and for $B = [0, r)$, $0 < r < 1$, we have $\psi^{\leftarrow}B = [0, r]$.

In particular the difunction (f, F) is bicontinuous. Take $\mathcal{E} \in \mathfrak{E}$, where $\mathfrak{E}$ is the base of $\upsilon_{\mathbb{I}}$ described in Example 3.12. Then $\mathcal{C} = (f, F)^{-1}(\mathcal{E}) = \{(C_j, D_j) \mid 1 \leq j \leq n\}$, where $C_1 = (0, 1]$, $D_1 = (0, t_1]$, $t_1 < 1$; $C_2 = (0, s_2]$, $D_2 = \emptyset$, $s_2 > 0$; $C_k = (0, s_k]$, $D_k = (0, t_k]$, $s_k > t_k$, $k = 3, \dots, n$. Hence $\mathcal{C}$ is a finite, cofinite excluding dicover of $(L, \mathcal{L})$, so $\mathcal{C}$ is anchored by Lemma 3.10 (2,3). We now obtain $\mathcal{C} \in \upsilon$. Indeed if $\mathcal{E}_1 \in \mathfrak{E}$ satisfies $\mathcal{E}_1 \prec_{(\star)} \mathcal{E}$ then $\mathcal{C}_1 = (f, F)^{-1}(\mathcal{E}_1)$ satisfies $\mathcal{C}_1 \prec_{(\star)} \mathcal{C}$ because the inverse (co)image of a difunction preserves inclusion. Since $\mathcal{C}_1$, like $\mathcal{C}$, is anchored, by [11, Lemma 2.2 (2)] we have $\mathcal{C}_1^{\Delta} \prec \mathcal{C}$ and $\mathcal{C} \in \upsilon$ follows because $\mathcal{C}_1^{\Delta} \in \upsilon$ by [11, Definition 5.23]. This argument clearly establishes that υ contains the set $\mathfrak{C}$ of all finite, cofinite excluding dicovers of $(L, \mathcal{L})$.

Now take $\mathcal{D} \in \upsilon$. Since υ is totally bounded we have a finite, cofinite subdicover $\mathcal{C}$ of $\mathcal{D}$, and without loss of generality we may assume that $\mathcal{C}$ is essential and hence excluding by Lemma 3.10. Thus $\mathcal{C} \in \mathfrak{C}$ and $\mathcal{C} \prec \mathcal{D}$, so $\mathfrak{C}$ is a base of υ.

We have found it convenient to work in terms of dicovering uniformities in this section, but we will close by giving characterizations of the Cauchy property and of total boundedness in terms of direlational uniformities.

3.16. Proposition. *Let υ be a dicovering uniformity and $\mathcal{F} \times \mathcal{G}$ a difilter on $(S, \mathcal{S})$. Then $\mathcal{F} \times \mathcal{G}$ is Cauchy with respect to υ if and only if given $(d, D) \in \Delta(\upsilon)$ there exists $s \in S$ with $d[s] \in \mathcal{F}$ and $D[s] \in \mathcal{G}$.*

Proof. First let $\mathcal{F} \times \mathcal{G}$ be Cauchy and take $(d, D) \in \Delta(\upsilon)$. Now we have $\mathcal{C} \in \upsilon$ with $\delta(\mathcal{C}) \sqsubseteq (d, D)$. By [11, Theorem 3.7 (4)] we have $\gamma(\delta(\mathcal{C})) \in \Gamma(\Delta(\upsilon)) = \upsilon$ and so $\gamma(\delta(\mathcal{C})) \cap (\mathcal{F} \times \mathcal{G}) \neq \emptyset$, whence there exists $s \in S$ with $(d(\mathcal{C})[s], D(\mathcal{C})[s]) \in \mathcal{F} \times \mathcal{G}$. But $d(\mathcal{C}) \subseteq d$, $D \subseteq D(\mathcal{C})$ so $d(\mathcal{C})[s] \subseteq d[s]$ and $D[s] \subseteq D(\mathcal{C})[s]$. This verifies $d[s] \in \mathcal{F}$ and $D[s] \in \mathcal{G}$, as required.

Conversely let $\mathcal{F} \times \mathcal{G}$ satisfy the stated condition and take $\mathcal{C} \in \upsilon$. Since, as above, $\mathcal{C} \in \Gamma(\Delta(\upsilon))$ there exists $(d, D) \in \Delta(\upsilon)$ with $\gamma(d, D) \prec \mathcal{C}$. By hypothesis there exists $s \in S$ with $d[s] \in \mathcal{F}$ and $D[s] \in \mathcal{G}$. Since $(d[s], D[s]) \in \gamma(d, D)$ there exists $A\mathcal{C}B$ with $d[s] \subseteq A$, $B \subseteq D[s]$, whence $(A, B) \in \mathcal{C} \cap (\mathcal{F} \times \mathcal{G}) \neq \emptyset$. This verifies that $\mathcal{F} \times \mathcal{G}$ is Cauchy. □

In view of the above the following definition is compatible with Definition 3.1 (1).

3.17. Definition. Let $\mathcal{U}$ be a direlational uniformity and $\mathcal{F} \times \mathcal{G}$ a difilter on $(S, \mathcal{S})$. Then $\mathcal{F} \times \mathcal{G}$ is said to be *Cauchy* if for $(d, D) \in \mathcal{U}$ there exists $s \in S$ with $d[s] \in \mathcal{F}$ and $D[s] \in \mathcal{G}$.

3.18. Proposition. *Let υ be a dicovering uniformity on $(S, \mathcal{S})$. Then $(S, \mathcal{S}, \upsilon)$ is totally bounded if and only if for each $(d, D) \in \Delta(\upsilon)$ there exists $s_1, s_2, \dots, s_n \in S$ for which $\{(d[s_1], D[s_1]), (d[s_2], D[s_2]), \dots, (d[s_n], D[s_n])\}$ is a dicover of $(S, \mathcal{S})$.*

Proof. First suppose that υ is totally bounded and take $(d, D) \in \Delta(\upsilon)$. Now we have $\mathcal{C} \in \upsilon$ with $\delta(\mathcal{C}) \sqsubseteq (d, D)$ and by hypothesis there exists a finite, cofinite subdicover $\{(A_1, B_1), (A_2, B_2), \dots, (A_n, B_n)\}$ of $\mathcal{C}$. By Lemma 3.10 (1) we may assume that this dicover is essential, so by Lemma 3.10 (2) we have $A_j \not\subseteq B_j$ for all $1 \leq j \leq n$. Hence, for each j we have $s_j \in S$ satisfying $A_j \not\subseteq Q_{s_j}$ and $P_{s_j} \not\subseteq B_j$. We claim that $A_j \subseteq d(\mathcal{C})[s_j]$. To show this assume the contrary and take $u \in S$ satisfying $A_j \not\subseteq Q_u$, $P_u \not\subseteq d(\mathcal{C})[s_j]$, and then $u' \in S$ with $A_j \not\subseteq Q_{u'}$, $P_{u'} \not\subseteq Q_u$. We recall from [11, statement of Proposition 2.5] that

$$d(\mathcal{C}) = \bigvee \{\overline{P}_{(s,t)} \mid \exists\, j \in J \text{ with } A_j \not\subseteq Q_t \text{ and } P_s \not\subseteq B_j\},$$

whence $\overline{P}_{(s_j, u')} \subseteq d(\mathcal{C})$. Hence $d(\mathcal{C}) \not\subseteq \overline{Q}_{(s_j, u)}$, from which we deduce $d(\mathcal{C})[s_j] \not\subseteq Q_u$ by [6, Lemma 2.6 (1)]. However, this contradicts $P_u \not\subseteq d(\mathcal{C})[s_j]$ and we have established

$A_j \subseteq d(\mathcal{C})[s_j] \subseteq d[s_j]$. A dual argument shows that $D[s_j] \subseteq D(\mathcal{C})[s_j] \subseteq B_j$, so the dicover $\{(A_1, B_1), (A_2, B_2), \dots, (A_n, B_n)\}$ refines $\{(d[s_1], D[s_1]), (d[s_2], D[s_2]), \dots, (d[s_n], D[s_n])\}$, showing that the latter is a dicover of $(S, \mathcal{S})$ as required.

Conversely, suppose that $\Delta(\upsilon)$ has the stated property and take $\mathcal{C} \in \upsilon$. Now there exists $\mathcal{D} \in \upsilon$ with $\mathcal{D} \prec_{(\star)} \mathcal{C}$, and without loss of generality we may assume that $\mathcal{D}$ is anchored. Since $\delta(\mathcal{D}) \in \Delta(\upsilon)$ there exists $s_1, s_2, \dots, s_n \in S$ for which $\mathcal{E} = \{(d(\mathcal{D})[s_1], D(\mathcal{D})[s_1]), (d(\mathcal{D})[s_2], D(\mathcal{D})[s_2]), \dots, (d(\mathcal{D})[s_n], D(\mathcal{D})[s_n])\}$ is a dicover of $(S, \mathcal{S})$. By definition $\mathcal{E} \subseteq \gamma(\delta(\mathcal{D}))$, and $\gamma(\delta(\mathcal{D})) = \mathcal{D}^{\Delta}$ by [11, Theorem 2.7 (2)]. Finally, since $\mathcal{D}$ is anchored and hence satisfies $\mathcal{P} \prec \mathcal{D}$ we have $\mathcal{D} \prec_{(\Delta)} \mathcal{C}$ by [4, Lemma 4.7 (2)], that is $\mathcal{D}^{\Delta} \prec \mathcal{C}$. Hence $\mathcal{E} \prec \mathcal{C}$ and for each j we may choose $A_j \mathcal{C} B_j$ satisfying $d(\mathcal{D})[s_j] \subseteq A_j$ and $B_j \subseteq D(\mathcal{D})[s_j]$. Clearly $\{(A_1, B_1), (A_2, B_2), \dots (A_n, B_n)\}$ is a finite, cofinite subdicover of $\mathcal{C}$, so υ is totally bounded. □

In view of the above result, the following is consistent with Definition 3.5.

3.19. Definition. Let $\mathcal{U}$ be a direlational uniformity on $(S, \mathcal{S})$. Then $(S, \mathcal{S}, \mathcal{U})$ is called *totally bounded* if for each $(d, D) \in \mathcal{U}$ there exists $s_1, s_2, \dots, s_n \in S$ for which the family $\{(d[s_1], D[s_1]), (d[s_2], D[s_2]), \dots, (d[s_n], D[s_n])\}$ is a dicover of $(S, \mathcal{S})$.

These equivalent definitions enable the results for dicovering uniformities given above to be reformulated for direlational uniformities in the natural way, and we omit the details.

4. Conclusion

As mentioned earlier, the definitions relating to convergence given here should be regarded as tentative. It is therefore worthwhile seeing how far the results obtained meet our expectations for a workable theory.

On the positive side it would appear that the general notion of difilter is a suitable construct on which to base such a theory. The regularity condition also occurs naturally, and has interesting consequences with respect to maximality as is shown by Proposition 2.15. Also, Proposition 2.12 and Proposition 2.19 show that for regular difilters the relation between diconvergence and diclustering echoes that between convergence and clustering of filters in general topology, while Theorem 2.20 gives a characterization of dicompactness analogous to that for compactness. Further, in connection with dicovering uniformities, Proposition 3.2 and Proposition 3.6 give the expected relation between diconvergence, diclustering and the Cauchy property for regular difilters, while Theorem 3.8 shows that a relation analogous to that between compactness, completeness and total boundedness in General Topology holds between dicompactness, dicompleteness and total boundedness.

We may conclude that the notions of diconvergence and diclustering for regular difilters is appropriate for the study of dicompact spaces, and for the study of dicompleteness of di-uniformities, particularly in the presence of total boundedness. Moreover, in view of Proposition 3.8, these concepts should be appropriate for plain textures in the general case. However, the characterization of maximal regular difilters on $(L, \mathcal{L})$ in Example 2.17 and the formulae for $\eta^*(s)$, $\mu^*(s)$ on this texture with the discrete, codiscrete ditopology given in Example 2.9, lead easily to the conclusion that this space has no diconvergent (maximal) regular difilters. This shows very strikingly that the concept of diconvergence for regular difilters on general non-plain ditopological texture spaces can be excessively restrictive. It remains an open problem to develop appropriate concepts related to convergence in the context of such spaces.

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