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RELATIONS AND CORELATIONS BETWEEN LATTICES OF FUZZY SUBSETS

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Abstract

In this paper a theory of relations and corelations between the lattice of fuzzy subsets of a crisp set X and that of a crisp set Y is developed, based on the theory of relations and corelations between textures. In a series of examples it is shown that these notions generalize in a natural way the impotant concept of fuzzy relation from X to Y. Difunctions are also characterized and their relationship with known mappings between fuzzy sets is investigated.

Keywords: Hutton algebra, Hutton texture, Direlation, Difunction, Fuzzy relation, Isomorphism, Full embedding.

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1. Introduction

Throughout we consider fuzzy subsets of a crisp set X in the sense of Zadeh [11], that is $\mathbb{I}$-valued sets where $\mathbb{I}$ is the real unit interval $[0,1]$ endowed with the usual ordering $\leq$ and the order reversing involution $'$ defined by $s' = 1 - s$, $s \in \mathbb{I}$. Thus a fuzzy subset μ of X is identified with the function $\mu : X \to \mathbb{I}$.

We will denote by $F(X)$ the set of all fuzzy subsets of X endowed with the pointwise order $\mu \leq \nu \iff \mu(x) \leq \nu(x)\ \forall x \in X$ and the order reversing involution μ' given by $\mu'(x) = (\mu(x))' = 1 - \mu(x)\ \forall x \in X$.

It is well known that both $\mathbb{I}$ and $F(X)$ are *Hutton algebras*, that is complete, completely distributive lattices with an order reversing involution. In [3] it was shown that a texture is associated with an arbitrary Hutton algebra $\mathbb{L}$ in a natural way. Here we recall that a texture is a pair $(S, \mathcal{S})$, where S is a crisp set and $\mathcal{S}$ a collection of subsets of S, called a texturing of S, containing S and $\emptyset$, separating the points of S and for which $(\mathcal{S}, \subseteq)$ is a complete, completely distributive lattice with meet $\bigwedge$ coinciding with intersection

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and finite join $\vee$ coinciding with union. Here, to say that $\mathcal{S}$ separates the points of S means that given $s_1, s_2 \in S$ with $s_1 \neq s_2$ there exists $A \in \mathcal{S}$ with $s_1 \in A$, $s_2 \notin A$ or there exists $A \in \mathcal{S}$ with $s_2 \in A$, $s_1 \notin A$. To associate a texture with a Hutton algebra $\mathbb{L}$ we let $M_{\mathbb{L}}$ denote the set of molecules of $\mathbb{L}$, $m \in \mathbb{L}$ being a molecule if m is not the smallest element of $\mathbb{L}$ and $m \leq a \vee b,\ a, b \in \mathbb{L} \implies m \leq a$ or $m \leq b$. Now for $a \in \mathbb{L}$ we set $\hat{a} = \{m \in M_{\mathbb{L}} \mid m \leq a\}$ and $\mathcal{M}_{\mathbb{L}} = \{\hat{a} \mid a \in \mathbb{L}\}$. In [3] it is shown that $(M_{\mathbb{L}}, \mathcal{M}_{\mathbb{L}})$ is indeed a texture and that the mapping $a \mapsto \hat{a}$ is an isomorphism between the complete lattices $\mathbb{L}$ and $\mathcal{M}_{\mathbb{L}}$.

For the Hutton algebra $\mathbb{I}$ the molecules are just the non-zero elements, so $M_{\mathbb{I}} = (0, 1]$ and $\mathcal{M}_{\mathbb{I}} = \{(0, s] \mid s \in \mathbb{I}\}$, where $(0, 0]$ is interpreted as $\emptyset$. Generally this texture is denoted by $(L, \mathcal{L})$, so throughout this paper we let $M_{\mathbb{I}} = L$ and $\mathcal{M}_{\mathbb{I}} = \mathcal{L}$. When X is a crisp set the molecules of $F(X)$ are just the fuzzy points $x_s,\ x \in X,\ s \in L$. Hence the set $M_{F(X)}$ of molecules can be represented by the set $X \times L$. Likewise for $\mu \in F(X)$ we have $\hat{\mu} = \{(x, s) \in X \times L \mid s \leq \mu(x)\}$ and $\mathcal{M}_{F(X)} = \{\hat{\mu} \mid \mu \in F(X)\}$. As in [3] we denote the texture $(M_{F(X)}, \mathcal{M}_{F(X)})$ by $(W_X, \mathcal{W}_X)$, or just by $(W, \mathcal{W})$ if there is no danger of confusion. Textures of the form $(W_X, \mathcal{W}_X)$, X a crisp set, will be our main object of study in this paper.

Several aspects of the theory of textures have been studied in the literature [1–4,8]. Here we will be mainly concerned with relations and functions between textures and the necessary background will be included in the next section. To complete this introduction we will just mention some basic definitions and results which will be essential for an understanding of the material to follow.

Let $(S, \mathcal{S})$ be a texture. For $s \in S$, both the sets

$$P_s = \bigcap\{A \in \mathcal{S} \mid s \in A\} \text{ and } Q_s = \bigvee\{A \in \mathcal{S} \mid s \notin A\} = \bigvee\{P_u \mid P_s \not\subseteq P_u\}$$

belong to $\mathcal{S}$ and play an important role in the development of the theory. For the so called discrete texture $(X, \mathcal{P}(X))$, where X is a crisp set and $\mathcal{P}(X) = \{A \mid A \subseteq X\}$ we clearly have $P_x = \{x\}$, $Q_x = X \setminus \{x\}$ for $x \in X$, while for the texture $(L, \mathcal{L})$ and $s \in L$ we have $P_s = Q_s = (0, s]$. Despite the considerable variation in the actual values of these sets we have, for example,

1.1. Lemma. (cf. [4]) *If $(S, \mathcal{S})$ is a texture, $A, B \in \mathcal{S}$, then*

(1) $A = \bigvee\{P_s \mid A \not\subseteq Q_s\}$.
(2) $A = \bigcap\{Q_s \mid P_s \not\subseteq A\}$.
(3) $A \not\subseteq B \implies \exists\, s \in S$ *with* $A \not\subseteq Q_s$ *and* $P_s \not\subseteq B$.

Result (3) is particularly useful in proving inclusion by *reductio ad absurdum*, and will be used without comment in the sequel. It should also be noted that $A \not\subseteq Q_s$ implies that $P_s \subseteq A$, but not conversely in general.

If $(S, \mathcal{S})$, $(T, \mathcal{T})$ are textures, their product is the texture $(S \times T, \mathcal{S} \otimes \mathcal{T})$, where the product texturing $\mathcal{S} \otimes \mathcal{T}$ consists of arbitrary intersections of sets of the form $(A \times T) \cup (S \times B)$ for $A \in \mathcal{S}$ and $B \in \mathcal{T}$. This definition can be extended in the obvious way to give a product of any finite or infinite family of textures. We note in particular that

1.2. Lemma. (cf. [4]) *Let $(S, \mathcal{S})$, $(T, \mathcal{T})$ be textures. Then for $(s, t) \in S \times T$,*

(1) $P_{(s,t)} = P_s \times P_t$,
(2) $Q_{(s,t)} = (Q_s \times T) \cup (S \times Q_t)$,

with an obvious extension to more than two textures.

It is verified in [3] that the texturing $\mathcal{W}_X = \mathcal{M}_{P(X)}$ on $W_X = M_{P(X)} = X \times L$ is the product of the discrete texturing $\mathcal{P}(X)$ of X and the texturing $\mathcal{L}$ of L. In particular, by Lemma 1.2 we have $P_{(x,s)} = \{x\} \times (0, s]$ and $Q_{(x,s)} = ((X \setminus \{x\}) \times L) \cup (X \times (0, s])$ for all $(x,s) \in X \times L$. Hence, in particular, $P_{(x,s)} \not\subseteq Q_{(x',s')} \iff x = x'$ and $s' < s$, a fact which will be used without comment below. It will be useful to identify the elements of $F(X)$ corresponding to $P_{(x,s)}$ and $Q_{(x,s)}$.

1.3. Lemma. *For $(x,s) \in W_X$ we have*

(1) $P_{(x,s)} = \widehat{x_s}$, *where $x_s \in F(X)$ given by* $x_s(z) = \begin{cases} s & z = x \\ 0 & z \in X \setminus \{x\} \end{cases}$ *is the fuzzy point at x with value s.*

(2) $Q_{(x,s)} = \widehat{x^s}$, *where $x^s \in F(X)$ given by* $x^s(z) = \begin{cases} s & z = x \\ 1 & z \in X \setminus \{x\} \end{cases}$ *is called the fuzzy copoint at x with value s.*

Proof. Straightforward. □

If $(S, \mathcal{S})$ and $(T, \mathcal{T})$ are textures, a bijection $\varphi : S \to T$ is called a textural isomorphism [2] if $\varphi[A] \in \mathcal{T}$ for every $A \in \mathcal{S}$ and the mapping $A \mapsto \varphi[A]$ is a bijection between $\mathcal{S}$ and $\mathcal{T}$. The reader may easily verify that for textures $(S_1, \mathcal{S}_1)$, $(S_2, \mathcal{S}_2)$, $(S_3, \mathcal{S}_3)$ the mapping $((s_1, s_2), s_3) \mapsto (s_1, (s_2, s_3))$ from $(S_1 \times S_2) \times S_3$ to $S_1 \times (S_2 \times S_3)$ is a textural isomorphism between $((S_1 \times S_2) \times S_3, (\mathcal{S}_1 \otimes \mathcal{S}_2) \otimes \mathcal{S}_3)$ and $(S_1 \times (S_2 \times S_3), \mathcal{S}_1 \otimes (\mathcal{S}_2 \otimes \mathcal{S}_3))$. This enables us to identify these textures when convenient.

In general a texturing $\mathcal{S}$ need not be closed under set complementation. For example, this is the case for the texturings $\mathcal{L}$ and $\mathcal{W}$ defined above. On the other hand, in some cases there exists a mapping $\sigma : \mathcal{S} \to \mathcal{S}$, called a complementation on $(S, \mathcal{S})$, satisfying $\sigma(\sigma(A)) = A \ \forall A \in \mathcal{S}$ and $A \subseteq B \implies \sigma(B) \subseteq \sigma(A) \ \forall A, B \in \mathcal{S}$, which can be thought of as a tool to compensate for the lack of set complementation. When σ is a complementation on $(S, \mathcal{S})$, $(S, \mathcal{S}, \sigma)$ is referred to as a complemented texture.

For the discrete texturing $\mathcal{P}(X)$ of a set X, which is closed under set complementation, the usual complement $\pi_X : \mathcal{P}(X) \to \mathcal{P}(X)$, $A \mapsto X \setminus A$, is indeed a complementation on $(X, \mathcal{P}(X))$ in the above sense. For the texture $(L, \mathcal{L})$ there is a natural complementation $\lambda : \mathcal{L} \to \mathcal{L}$ defined by $\lambda((0,s]) = (0, 1-s]$, $s \in \mathbb{I}$, where again we set $(0,0] = \emptyset$. Likewise, there is a complementation ω_X on $\mathcal{W}_X$ defined by $\omega_X(\hat{\mu}) = \widehat{\mu'} = \{(x,s) \mid s \leq 1 - \mu(x)\} = \{(x,s) \mid \mu(x) \leq 1 - s\}$.

In [3] the product complementation on a product of complemented textures is defined and it is shown that the complementation ω on $(W_X, \mathcal{W}_X)$ is in fact the product of the complementations π_X and λ. By [3, Lemma 2.7] we deduce that $\omega(A \times L) = \pi_X(A) \times L$ and $\omega(X \times B) = X \times \lambda(B)$ for $A \subseteq X$, $B \in \mathcal{L}$, from which we obtain the following useful result:

1.4. Lemma. (cf. [4, Examples 3.11 (4)]) *In the complemented texture $(W_X, \mathcal{W}_X, \omega_X)$,*

(1) $\omega_X(P_{(x,s)}) = Q_{(x,1-s)}$, *and*

(2) $\omega_X(Q_{(x,s)}) = P_{(x,1-s)}$

for all $(x,s) \in W_X$. Here $Q_{(x,0)}$ is to be interpreted as $(X \setminus \{x\}) \times L$, and $P_{(x,0)}$ as $\emptyset$.

For notions from lattice theory not defined here the reader is referred to [6].

The original material in this paper forms part of the first author's PhD thesis [10].

2. Representation of Relations and Corelations

The standard theory of binary relations between sets is largely inappropriate for the study of textures, and the second author has developed a theory of relations and corelations, and of pairs, called direlations, consisting of a relation and a corelation, between two textures. This work appears in the preprint [1], and a large part of it has been published in [4]. For the benefit of the reader we will repeat the basic definitions and results at the appropriate place in the text.

Let $(S, \mathcal{S})$ and $(T, \mathcal{T})$ be textures. Both relations and corelations from $(S, \mathcal{S})$ to $(T, \mathcal{T})$ are elements of the product texturing $\mathcal{P}(S) \otimes \mathcal{T}$. To avoid confusion we denote the p-sets and q-sets of this texture by $\overline{P}_{(s,t)}$, $\overline{Q}_{(s,t)}$ for $(s,t) \in S \times T$. Let X and Y be crisp sets. Then as noted above, the Hutton algebras $F(X)$ and $F(Y)$ are associated with the Hutton textures $(W_X, \mathcal{W}_X)$ and $(W_Y, \mathcal{W}_Y)$, respectively (for the moment we are ignoring the complementations). How can we represent a relation or corelation from $(W_X, \mathcal{W}_X)$ to $(W_Y, \mathcal{W}_Y)$ in terms of fuzzy sets? Due to the previous considerations, such a relation or corelation is an element of the product texturing $\mathcal{P}(W_X) \otimes \mathcal{W}_Y = \mathcal{P}(X \times L) \otimes (\mathcal{P}(Y) \otimes \mathcal{L})$. By the remarks following Lemma 1.3, we can identify this texturing with $(\mathcal{P}(X \times L) \otimes \mathcal{P}(Y)) \otimes \mathcal{L}$, which is just $\mathcal{P}(X \times L \times Y) \otimes \mathcal{L}$. However, this is precisely the Hutton texturing of the crisp set $X \times L \times Y$, whose elements correspond in a one to one way to the fuzzy subsets of $X \times L \times Y$.

Having set up a correspondence in this way between $\mathcal{P}(W_X) \otimes \mathcal{W}_Y$ and the set $F(X \times L \times Y)$, we are in a position to represent relations and corelations from $(W_X, \mathcal{W}_X)$ to $(W_Y, \mathcal{W}_Y)$ as elements of $F(X \times L \times Y)$ satisfying appropriate conditions. To determine these conditions, we recall the following:

2.1. Definition. [4] Let $(S, \mathcal{S})$, $(T, \mathcal{T})$ be textures. Then

(1) $r \in \mathcal{P}(S) \otimes \mathcal{T}$ is called a *relation from* $(S, \mathcal{S})$ *to* $(T, \mathcal{T})$ if it satisfies
  R1 $r \not\subseteq \overline{Q}_{(s,t)}, P_{s'} \not\subseteq Q_s \implies r \not\subseteq \overline{Q}_{(s',t)}$.
  R2 $r \not\subseteq \overline{Q}_{(s,t)} \implies \exists s' \in S$ such that $P_s \not\subseteq Q_{s'}$ and $r \not\subseteq \overline{Q}_{(s',t)}$.

(2) $R \in \mathcal{P}(S) \otimes \mathcal{T}$ is called a *corelation from* $(S, \mathcal{S})$ *to* $(T, \mathcal{T})$ if it satisfies
  CR1 $\overline{P}_{(s,t)} \not\subseteq R, P_s \not\subseteq Q_{s'} \implies \overline{P}_{(s',t)} \not\subseteq R$.
  CR2 $\overline{P}_{(s,t)} \not\subseteq R \implies \exists s' \in S$ such that $P_{s'} \not\subseteq Q_s$ and $\overline{P}_{(s',t)} \not\subseteq R$.

(3) A pair (r, R), where r is a relation and R a corelation from $(S, \mathcal{S})$ to $(T, \mathcal{T})$, is called a *direlation from* $(S, \mathcal{S})$ *to* $(T, \mathcal{T})$.

It will be noted that relations and corelations in the sense of Definition 2.1 are, in particular, binary relations from S to T in the classical sense. Where confusion is likely, the former are referred to as *textural relations* and *textural corelations*, respectively, or the latter are referred to as *point relations*.

Now let r be a relation from $(W_X, \mathcal{W}_X)$ to $(W_Y, \mathcal{W}_Y)$, and μ_r the corresponding element of $F(X \times L \times Y)$. Then $r = \widehat{\mu_r}$, where we are taking $\widehat{\mu_r} = \{((x,s),(y,t)) \mid t \leq \mu_r(x,s,y)\}$ in place of $\{((x,s,y),t) \mid t \leq \mu_r(x,s,y)\}$ in view of the identification mentioned above. Likewise, for a corelation R from $(W_X, \mathcal{W}_X)$ to $(W_Y, \mathcal{W}_Y)$, we also have $R = \widehat{\mu_R} = \{((x,s),(y,t)) \mid t \leq \mu_R(x,s,y)\}$, where μ_R is the corresponding element of $F(X \times L \times Y)$. The next lemma will be needed in the proof of Theorem 2.3. Its proof in straightforward, and is omitted.

2.2. Lemma. *Let r be a relation and R a corelation from $(W_X, \mathcal{W}_X)$ to $(W_Y, \mathcal{W}_Y)$. Then:*

(1) $r = \widehat{\mu_r} \not\subseteq \overline{Q}_{((x,s),(y,t))} \iff t < \mu_r(x,s,y)$,
(2) $\overline{P}_{((x,s),(y,t))} \not\subseteq \widehat{\mu_R} = R \iff t > \mu_R(x,s,y)$.

Now we have:

2.3. Theorem.

(1) *Take $r \in \mathcal{P}(W_X) \otimes \mathcal{W}_Y$ and let μ_r be the corresponding element of $F(X \times L \times Y)$. Then r is a relation from $(W_X, \mathcal{W}_X)$ to $(W_Y, \mathcal{W}_Y)$ if and only if for all $(x,y) \in$*

$X \times Y$ we have

$$\mu_r(x,s,y) = \bigvee \{\mu_r(x,s',y) \mid 0 < s' < s\} \ \forall\, s \in L.$$

(2) *Take $R \in \mathcal{P}(W_X) \otimes \mathcal{W}_Y$ and let μ_R be the corresponding element of $F(X \times L \times Y)$. Then R is a corelation from $(W_X, \mathcal{W}_X)$ to $(W_Y, \mathcal{W}_Y)$ if and only if for all $(x,y) \in X \times Y$ we have*

$$\mu_R(x,s,y) = \bigwedge \{\mu_R(x,s',y) \mid s < s' \le 1\} \ \forall\, s \in L.$$

In particular if r is a relation and R a corelation then $\mu_r(x,s,y)$, $\mu_R(x,s,y)$ are monotonically increasing functions of s for fixed x and y, while $\mu_R(x,1,y) = 1 \ \forall\,(x,y) \in X \times Y$.

Proof. (1) $\Longrightarrow$. Let r be a relation. Suppose first that there exists $x \in X$, $y \in Y$ and s', $0 < s' < s$, with $\mu_r(x,s,y) < \mu_r(x,s',y)$. If we let $t = \mu_r(x,s,y)$ then $r \not\subseteq \overline{Q}_{((x,s'),(y,t))}$ by Lemma 2.2 (1), while $P_{(x,s)} \not\subseteq Q_{(x,s')}$ since $s' < s$. By the condition *R1* we obtain $r \not\subseteq \overline{Q}_{((x,s),(y,t))}$, and hence the contradiction $t < \mu_r(x,s,y)$ by Lemma 2.2 (1). Thus

$$\bigvee \{\mu_r(x,s',y) \mid 0 < s' < s\} \le \mu_r(x,s,y).$$

Now suppose that the above inequality is strict and choose $t \in L$ satisfying $\bigvee\{\mu_r(x,s',y) \mid 0 < s' < s\} < t < \mu_r(x,s,y)$. Then $t < \mu_r(x,s,y)$ by Lemma 2.2 (1) and so there exists $s' \in L$ such that $P_{(x,s)} \not\subseteq Q_{(x,s')}$ and $r \not\subseteq \overline{Q}_{((x,s'),(y,t))}$ by condition *R2*. But now $0 < s' < s$ and so $t < \mu_r(x,s',y) \le \bigvee\{\mu_r(x,s',y) \mid 0 < s' < s\} < t$, which is a contradiction. Hence

$$\mu_r(x,s,y) = \bigvee \{\mu_r(x,s',y) \mid 0 < s' < s\}$$

for all $x \in X$, $y \in Y$ and $s \in L$.

$\Longleftarrow$. Take $r \in \mathcal{P}(X) \otimes \mathcal{L}$ with μ_r satisfying the stated equality. Suppose that $r \not\subseteq \overline{Q}_{((x,s),(y,t))}$, whence $t < \mu_r(x,s,y)$ by Lemma 2.2 (1).

First let $P_{(x,s')} \not\subseteq Q_{(x,s)}$. Then $s < s'$ so $t < \mu_r(x,s,y) \le \mu_r(x,s',y)$ since in particular μ_r is a monotonically increasing function of s, and we have $r \not\subseteq \overline{Q}_{((x,s'),(y,t))}$. Hence *R1* holds.

To establish *R2* note that from $t < \mu_r(x,s,y) = \bigvee\{\mu_r(x,s',y) \mid 0 < s' < s\}$ there exists $s' \in L$, $s' < s$ satisfying $t < \mu_r(x,s',y)$. This gives us $P_{(x,s)} \not\subseteq Q_{(x,s')}$ and $r \not\subseteq \overline{Q}_{((x,s'),(y,t))}$, as required.

(2). The proof is dual to (1), and is omitted.

If r is a relation and R a corelation, the fact that $\mu_r(x,s.y)$ and $\mu_R(x,s,y)$ are monotonically increasing functions of s for arbitrary $x \in X$, $y \in Y$ is an immediate consequence of the equalities proved in (1) and (2), respectively. Finally, $\mu_R(x,1,y) = \bigwedge\{\mu_R(x,s',y) \mid s < s' \le 1\} = 1$ since $\{\mu_R(x,s',y) \mid s < s' \le 1\} = \emptyset$ and 1 is the largest element of $(L, \le)$. □

2.4. Definition. Let ϕ, Φ be fuzzy subsets of $X \times L \times Y$.

(1) If $\phi(x,s,y) = \bigvee\{\phi(x,s',y) \mid 0 < s' < s\}$ for all $(x,s,y) \in X \times L \times Y$ we call ϕ a *textural fuzzy relation from $F(X)$ to $F(Y)$*.
(2) If $\Phi(x,s,y) = \bigwedge\{\Phi(x,s',y) \mid s < s' \le 1\}$ for all $(x,s,y) \in X \times L \times Y$ we call Φ a *textural fuzzy corelation from $F(X)$ to $F(Y)$*.
(3) If ϕ is a textural fuzzy relation and Φ a textural fuzzy corelation, (ϕ, Φ) is called a *textural fuzzy direlation from $F(X)$ to $F(Y)$*.

Throughout this paper we will abbreviate “textural fuzzy relation” to “relation,” and likewise for “corelation” and “direlation.” Since we will not be considering binary relations from $F(X)$ to $F(Y)$ in the classical sense at all, this will cause no confusion.

2.5. Example. Let φ be a fuzzy relation from X to Y, that is a fuzzy subset of $X \times Y$. There are many ways in which a direlation from $F(X)$ to $F(Y)$ may be associated with φ. We present one such direlation (ϕ, Φ) for which the above mentioned association will be found to have some useful properties, namely

$$\phi(x,s,y) = \varphi(x,y) \wedge s, \text{ and } \Phi(x,s,y) = (1 - \varphi(x,y)) \vee s$$

for all $(x,s,y) \in X \times L \times Y$. It is trivial to verify that ϕ is indeed a relation and Φ a corelation according to Definition 2.4.

2.6. Example. The identity direlation (i, I) on a texture $(S, \mathcal{S})$ is defined by

$$i = \bigvee \{\overline{P}_{(s,s)} \mid s \in S\} \text{ and } I = \bigcap \{\overline{Q}_{(s,s)} \mid s \in S^\flat\}.$$

If we consider the identity on $(W_X, \mathcal{W}_X)$, the corresponding direlation (μ_i, μ_I) on $F(X)$ is given by

$$\mu_i(x,s,y) = \begin{cases} 0 & x \neq y \\ s & x = y \end{cases}, \quad \mu_I(x,s,y) = \begin{cases} 1 & x \neq y \\ s & x = y \end{cases}.$$

The proof is straightforward, and is left to the interested reader. We will refer to (μ_i, μ_I) as the *identity direlation on* $F(X)$. Note that if φ is the identity (fuzzy) relation on X, that is for all $x, y \in X$

$$\varphi(x,y) = \begin{cases} 0 & x \neq y \\ 1 & x = y \end{cases},$$

then $\mu_i(x,s,y) = \varphi(x,y) \wedge s$ and $\mu_I(x,s,y) = (1 - \varphi(x,y)) \vee s$, so the correspondence defined in Example 2.5 preserves identities.

2.7. Lemma. *Let (c, C), (d, D) be direlations from $(W_X, \mathcal{W}_X)$ to $(W_Y, \mathcal{W}_Y)$ and (μ_c, μ_C), (μ_d, μ_D) the corresponding direlations from $F(X)$ to $F(Y)$. Then:*

(1) $c \subseteq d \iff \mu_c \leq \mu_d$.
(2) $C \subseteq D \iff \mu_C \leq \mu_D$.
(3) $(c, C) \sqsubseteq (d, D) \iff \mu_c \leq \mu_d$ *and* $\mu_D \leq \mu_C$.

Proof. (1) Immediate from Lemma 2.2 (1).

(2) Immediate from Lemma 2.2 (2).

(3) By definition $(c, C) \sqsubseteq (d, D)$ means $c \subseteq d$ and $D \subseteq C$. Hence this result follows at once from (1) and (2). □

2.8. Definition. Let (ϕ, Φ), (ψ, Ψ) be direlations from $F(X)$ to $F(Y)$. Then we write $(\phi, \Phi) \sqsubseteq (\psi, \Psi)$ to denote $\phi \leq \psi$ and $\Psi \leq \Phi$.

Let us next recall the following concepts from [4].

2.9. Definition. Let r be a relation, R a corelation from $(S, \mathcal{S})$ to $(T, \mathcal{T})$, and define

$$r^{\leftarrow} = \bigcap \{\overline{Q}_{(t,s)} \mid r \not\subseteq \overline{Q}_{(s,t)}\}, \quad R^{\leftarrow} = \bigvee \{\overline{P}_{(t,s)} \mid \overline{P}_{(s,t)} \not\subseteq R\}.$$

Then $r^{\leftarrow}$ is a corelation and $R^{\leftarrow}$ a relation from $(T, \mathcal{T})$ to $(S, \mathcal{S})$, so that $(r, R)^{\leftarrow} = (R^{\leftarrow}, r^{\leftarrow})$ is a direlation from $(T, \mathcal{T})$ to $(S, \mathcal{S})$. Each of $r^{\leftarrow}$, $R^{\leftarrow}$ and $(r, R)^{\leftarrow}$ is known as the *inverse* of r, R and (r, R), respectively.

A direlation on $(S, \mathcal{S})$ is known as *symmetric* if $(r, R)^{\leftarrow} = (r, R)$.

Now consider a direlation (r, R) from $(W_X, \mathcal{W}_X)$ to $(W_Y, \mathcal{W}_Y)$, and let (μ_r, μ_R) be the corresponding direlation from $F(X)$ to $F(Y)$. The following theorem gives formulae for calculating the direlation $(\mu_{R^\leftarrow}, \mu_{r^\leftarrow})$ from $F(Y)$ to $F(X)$ corresponding to the inverse $(r, R)^\leftarrow$ from $(W_Y, \mathcal{W}_Y)$ to $(W_X, \mathcal{W}_X)$.

2.10. Theorem. *With the notation above, for $(y, t, x) \in Y \times L \times X$ we have:*

$$\mu_{r^\leftarrow}(y, t, x) = \bigvee \{s \in L \mid \mu_r(x, s, y) \le t\}, \text{ and}$$
$$\mu_{R^\leftarrow}(y, t, x) = \bigwedge \{s \in L \mid t \le \mu_R(x, s, y)\}.$$

Proof. By [4, Lemma 2.4 (1)] we have $r \not\subseteq \overline{Q}_{((x,s),(y,t))} \iff \overline{P}_{((y,t),(x,s))} \not\subseteq r^\leftarrow$, so $t < \mu_r(x, s, y) \iff s > \mu_{r^\leftarrow}(y, t, x)$ by Lemma 2.2. Thus $\mu_r(x, s, y) \le t \iff s \le \mu_{r^\leftarrow}(y, t, x)$, from which the first equality follows immediately.

The proof of the second equality is dual to the above, and is omitted. □

2.11. Definition. Let (ϕ, Φ) be a direlation from $F(X)$ to $F(Y)$ and for $(y, t, x) \in Y \times L \times X$ set

$$\phi^\leftarrow(y, t, x) = \bigvee \{s \in L \mid \phi(x, s, y) \le t\},\ \Phi^\leftarrow(y, t, x) = \bigwedge \{s \in L \mid t \le \Phi(x, s, y)\}.$$

Then the corelation $\phi^\leftarrow$, the relation $\Phi^\leftarrow$ and the direlation $(\phi, \Phi)^\leftarrow = (\Phi^\leftarrow, \phi^\leftarrow)$ from $F(Y)$ to $F(X)$ is known as the *inverse* of ϕ, Φ and (ϕ, Φ), respectively.

If a direlation (ϕ, Φ) on $F(X)$ satisfies $(\phi, \Phi)^\leftarrow = (\phi, \Phi)$ it is called *symmetric*.

2.12. Examples.

1. Let the direlation (ϕ, Φ) from $F(X)$ to $F(Y)$ correspond to the fuzzy relation φ from X to Y as in Example 2.5. Then it is easy to verify that $(\phi, \Phi)^\leftarrow = (\Phi^\leftarrow, \phi^\leftarrow)$ is given by

$$\Phi^\leftarrow(y, t, x) = \begin{cases} 0 & \varphi(x, y) \le 1 - t \\ t & 1 - t < \varphi(x, y) \end{cases}, \quad \phi^\leftarrow(y, t, x) = \begin{cases} 1 & \varphi(x, y) \le t \\ t & t < \varphi(x, y) \end{cases}.$$

   It is clear from the form of these equalities that $\Phi^\leftarrow$, $\phi^\leftarrow$ do not in general correspond to a fuzzy relation from Y to X under the correspondence described in Example 2.5. Hence, in general, the inverses in the sense of Definition 2.11 are independent of any notion of inverse for fuzzy relations under the given correspondence.
2. Since for any texture $(S, \mathcal{S})$ the identity direlation on $(S, \mathcal{S})$ is symmetric [4] it follows that the identity direlation (μ_i, μ_I) on $F(X)$ is symmetric. This may also be observed directly on taking φ to be the identity (fuzzy) relation in (1).

It is known from [4, Lemma 2.4] that if (r, R) is a direlation from $(S, \mathcal{S})$ to $(T, \mathcal{T})$ then $(r^\leftarrow)^\leftarrow = r$ and $(R^\leftarrow)^\leftarrow = R$. Hence:

2.13. Proposition. *Let (ϕ, Φ) be a direlation from $F(X)$ to $F(Y)$. Then $(\phi^\leftarrow)^\leftarrow = \phi$ and $(\Phi^\leftarrow)^\leftarrow = \Phi$.*

Proof. For some direlation (r, R) from $(W_X, \mathcal{W}_X)$ to $W_Y, \mathcal{W}_Y)$ we have $\phi = \mu_r$ and $\Phi = \mu_R$ by Theorem 2.3. Hence

$$(\phi^\leftarrow)^\leftarrow = (\mu_r^\leftarrow)^\leftarrow = (\mu_{r^\leftarrow})^\leftarrow = \mu_{(r^\leftarrow)^\leftarrow} = \mu_r = \phi$$

by the result mentioned above. The proof of $(\Phi^\leftarrow)^\leftarrow = \Phi$ is similar. □

2.14. Note. Properties of relations, corelations and direlations in the sense of Definition 2.4 which depend on known results for relations, corelations or direlations between general textures, as in the above proposition, will be given without proof in what follows. Generally speaking only the counterparts of the more major results will be mentioned explicitly and the interested reader may consult [1] or [4] for further information.

The counterparts of the remaining results from [4, Lemma 2.4] will also be useful. They are given in the following proposition.

2.15. Proposition.

(1) *For a direlation (ϕ, Φ) from $F(X)$ to $F(Y)$, $x \in X, y \in Y, s, t \in L$ we have*

$$t < \phi(x, s, y) \iff \phi^{\leftarrow}(y, t, x) < s \text{ and } \Phi(x, s, y) < t \iff s < \Phi^{\leftarrow}(y, t, x).$$

(2) *Let (ϕ_1, Φ_1), (ϕ_2, Φ_2) be direlations from $F(X)$ to $F(Y)$. Then*

$$\phi_1 \le \phi_2 \iff \phi_2^{\leftarrow} \le \phi_1^{\leftarrow} \text{ and } \Phi_1 \le \Phi_2 \iff \Phi_2^{\leftarrow} \le \Phi_1^{\leftarrow}.$$

One of the most important concepts for point relations is that of composition. We recall the following analogues for relations, corelations and direlations between textures [4, Definition 2.13].

2.16. Definition. Let $(S, \mathcal{S})$, $(T, \mathcal{T})$, $(U, \mathcal{U})$ be textures.

(1) If c is a relation from $(S, \mathcal{S})$ to $(T, \mathcal{T})$ and d a relation from $(T, \mathcal{T})$ to $(U, \mathcal{U})$ then their *composition* is the relation $d \circ c$ from $(S, \mathcal{S})$ to $(U, \mathcal{U})$ defined by

$$d \circ c = \bigvee\{\overline{P}_{(s,u)} \mid \exists\, t \in T \text{ with } c \not\subseteq \overline{Q}_{(s,t)} \text{ and } d \not\subseteq \overline{Q}_{(t,u)}\}.$$

(2) If C is a corelation from $(S, \mathcal{S})$ to $(T, \mathcal{T})$ and D a corelation from $(T, \mathcal{T})$ to $(U, \mathcal{U})$ then their *composition* is the corelation $D \circ C$ from $(S, \mathcal{S})$ to $(U, \mathcal{U})$ defined by

$$D \circ C = \bigcap\{\overline{Q}_{(s,u)} \mid \exists\, t \in T \text{ with } \overline{P}_{(s,t)} \not\subseteq C \text{ and } \overline{P}_{(t,u)} \not\subseteq D\}.$$

(3) With c, d; C, D as above, the *composition* of the direlations (c, C), (d, D) is the direlation $(d, D) \circ (c, C) = (d \circ c, D \circ C)$.

Now we may give:

2.17. Theorem. *Let (c, C) be a direlation from $(W_X, \mathcal{W}_X)$ to $(W_Y, \mathcal{W}_Y)$ and (d, D) a direlation from $(W_Y, \mathcal{W}_Y)$ to $(W_Z, \mathcal{W}_Z)$. If (μ_c, μ_C), (μ_d, μ_D) and $(\mu_{d\circ c}, \mu_{D\circ C})$ are the corresponding direlations from $F(X)$ to $F(Y)$, $F(Y)$ to $F(Z)$ and $F(X)$ to $F(Z)$, respectively, then for $(x, s, z) \in X \times L \times Z$,*

(1) $\mu_{d\circ c}(x, s, z) = \bigvee\{\mu_d(y, t, z) \mid y \in Y,\ 0 < t < \mu_c(x, s, y)\}$, *and*
(2) $\mu_{D\circ C}(x, s, z) = \bigwedge\{\mu_D(y, t, z) \mid y \in Y,\ \mu_C(x, s, y) < t \le 1\}$.

In particular, the first equality may be written in the form

$$\mu_{d\circ c}(x, s, z) = \bigvee\{\mu_d(y,\ \mu_c(x, s, y),\ z) \mid y \in Y \text{ with } \mu_c(x, s, y) > 0\}.$$

Proof. (1) Lemma 2.2 (1), together with Definition 2.16 (1), lead easily to the equivalence

$$w < \mu_{d\circ c}(x, s, z) \iff \exists\, y \in Y,\ t \in L \text{ with } t < \mu_c(x, s, y) \text{ and } w < \mu_d(y, t, z)$$

for all $w \in L$. The required equality is now immediate.

(2) The proof is dual to the above, and is omitted.

To establish the final equality, note first that the function μ_d is monotonically increasing in its second argument (Theorem 2.3), so $t < \mu_c(x, s, y) \implies \mu_d(y, t, z) \le \mu_d(y,\ \mu_c(x, s, y),\ z)$. Hence

$$\mu_{d\circ c}(x, s, z) \le \bigvee\{\mu_d(y,\ \mu_c(x, s, y),\ z) \mid y \in Y,\ \mu_c(x, s, y) > 0\}.$$

On the other hand, if $w < \bigvee\{\mu_d(y,\ \mu_c(x,s,y),\ z) \mid y \in Y \text{ with } \mu_c(x,s,y) > 0\}$ there exists $y \in Y$ with $w < \mu_d(y,\ \mu_c(x,s,y),\ z)$, and then by Theorem 2.3 (1) there exists $t \in L$ with $w < \mu_d(y,t,z)$ and $t < \mu_c(x,s,y)$. Hence $w < \mu_{d\circ c}(x,s,z)$ by (1), which proves $\bigvee\{\mu_d(y,\ \mu_c(x,s,y),\ z) \mid y \in Y \text{ with } \mu_c(x,s,y) > 0\} \leq \mu_{d\circ c}(x,s,z)$. □

2.18. Definition. Let (ϕ,Φ) be a direlation from $F(X)$ to $F(Y)$ and (θ,Θ) a direlation from $F(Y)$ to $F(Z)$. Then the *composition* $(\theta,\Theta)\circ(\phi,\Phi)$ of (ϕ,Φ) and (θ,Θ) is the direlation $(\theta\circ\phi,\Theta\circ\Phi)$ from $F(X)$ to $F(Y)$ defined by the equalities

(1) $(\theta\circ\phi)(x,s,z) = \bigvee\{\theta(y,t,z) \mid y \in Y,\ 0 < t < \phi(x,s,y)\}$,
(2) $(\Theta\circ\Phi)(x,s,z) = \bigwedge\{\Theta(y,t,z) \mid y \in Y,\ \Phi(x,s,y) < t \leq 1\}$.

As in Theorem 2.17, an alternate formula for $\theta\circ\phi$ is

$$(\theta\circ\phi)(x,s,z) = \bigvee\{\theta(y,\ \phi(x,s,y),\ z) \mid y \in Y,\ \phi(x,s,y) > 0\}.$$

2.19. Definition. A relation ϕ on $F(X)$ is called *transitive* if $\phi\circ\phi \leq \phi$ and a corelation Φ on $F(X)$ is *transitive* if $\Phi \leq \Phi\circ\Phi$. In this case the direlation (ϕ,Φ) in $F(X)$ is also said to be *transitive*.

In view of [4, Proposition 2.17] we have:

2.20. Theorem.

(1) *The identity direlation (μ_i,μ_I) (Example 2.6) is the identity for composition. In particular, (μ_i,μ_I) is transitive.*
(2) *The operation of taking the composition of relations, corelations and direlations is associative.*
(3) *With the notation of Definition 2.16,*
$(\theta\circ\phi)^{\leftarrow} = \phi^{\leftarrow}\circ\theta^{\leftarrow}$ *and* $(\Theta\circ\Phi)^{\leftarrow} = \Phi^{\leftarrow}\circ\Theta^{\leftarrow}$.

2.21. Example. Let $\varphi \in F(X\times Y)$, $\vartheta \in F(Y\times Z)$ be fuzzy relations and (ϕ,Φ) (θ,Θ) the corresponding direlations under the correspondence given in Example 2.5. Then for $(x,s,z) \in X\times L\times Z$ we have

$$\begin{aligned}(\theta\circ\phi)(x,s,z) &= \bigvee\{\theta(y,\ \phi(x,s,y),\ z) \mid y \in Y,\ \phi(x,s,y) > 0\}\\ &= \bigvee\{\vartheta(y,z)\wedge\phi(x,s,y) \mid y \in Y,\ \phi(x,s,y) > 0\}\\ &= \bigvee\{\vartheta(y,z)\wedge\varphi(x,y)\wedge s \mid y \in Y\}\\ &= \big(\bigvee\{\vartheta(y,z)\wedge\varphi(x,y) \mid y \in Y\}\big)\wedge s\\ &= (\vartheta\circ\varphi)(x,z)\wedge s.\end{aligned}$$

Hence $\theta\circ\phi$ corresponds to the composition $\vartheta\circ\varphi$ of the fuzzy relations φ and ϑ, taken with respect to the min t-norm [7]. Likewise it may be verified that

$$(\Theta\circ\Phi)(x,s,z) = \big(1 - \bigvee\{\vartheta(y,z)\wedge\varphi(x,y) \mid y \in Y\}\big)\vee s,$$

whence $(\Theta\circ\Phi) = (1 - (\theta\circ\phi)(x,z))\vee s$, and so $\Theta\circ\Phi$ also corresponds to $\vartheta\circ\varphi$ under the correspondence given in Example 2.5.

Next we turn to a study of sections of relations and corelations.

2.22. Definition. [4, Definition 2.5], Let $(S,\mathcal{S})$, $(T,\mathcal{T})$ be textures, (r,R) a direlation from $(S,\mathcal{S})$ to $(T,\mathcal{T})$ and $A \in \mathcal{S}$.

(1) The *A–section of* r is the element $r^{\rightarrow}A$ of $\mathcal{T}$ defined by

$$r^{\rightarrow}A = \bigcap\{Q_t \mid \forall s,\ r \not\subseteq \overline{Q}_{(s,t)} \implies A \subseteq Q_s\}.$$

(2) The *A–section of R* is the element $R^{\to}A$ of $\mathcal{T}$ defined by

$$R^{\to}A = \bigvee\{P_t \mid \forall\, s,\ \overline{P}_{(s,t)} \not\subseteq R \implies P_s \subseteq A\}.$$

2.23. Theorem. *Let (r, R) be a direlation from $(W_X, \mathcal{W}_X)$ to $(W_Y, \mathcal{W}_Y)$, (μ_r, μ_R) the direlation from $F(X)$ to $F(Y)$ corresponding to (r, R), and $\alpha \in F(X)$. Then for $\beta \in F(Y)$ we have:*

(1) $r^{\to}\widehat{\alpha} = \widehat{\beta} \iff \beta(y) = \bigwedge\{t \in L \mid s < \alpha(x) \implies \mu_r(x, s, y) \le t\}\ \forall\, y \in Y,$
(2) $R^{\to}\widehat{\alpha} = \widehat{\beta} \iff \beta(y) = \bigvee\{t \in L \mid \alpha(x) < s \implies t \le \mu_R(x, s, y)\}\ \forall\, y \in Y,$

where the implications on the right are taken over all $x \in X$ and $s \in L$.

Proof. (1) By Definition 2.22,

$$\begin{aligned} r^{\to}\widehat{\alpha} &= \bigcap\{\overline{Q}_{(y,t)} \mid (x, s) \in X \times L,\ r \not\subseteq \overline{Q}_{((x,s),(y,t))} \implies \widehat{\alpha} \subseteq \overline{Q}_{(x,s)}\} \\ &= \bigcap\{\overline{Q}_{(y,t)} \mid t < \mu_r(x, s, y) \implies \alpha(x) \le s\} \\ &= \bigcap\{\overline{Q}_{(y,t)} \mid (x, s) \in X \times L,\ s < \alpha(x) \implies \mu_r(x, s, y) \le t\}, \end{aligned}$$

whence $f^{\to}\widehat{\alpha} = \widehat{\beta} \iff (w \le \beta(y) \iff (s < \alpha(x) \implies \mu_r(x, s, y) \le t))$. The required equality now follows easily.

(2) The proof is dual to (1), and is left to the interested reader. □

2.24. Definition. Let (ϕ, Φ) be a direlation from $F(X)$ to $F(Y)$ and $\alpha \in F(X)$. Then:

(1) $\beta \in F(Y)$ defined by $\beta(y) = \bigwedge\{t \in L \mid s < \alpha(x) \implies \phi(x, s, y) \le t\},\ y \in Y$, is called the *$\alpha$-section of the relation* ϕ and is denoted by $\beta = \phi^{\to}\alpha$,
(2) $\beta \in F(Y)$ defined by $\beta(y) = \bigvee\{t \in L \mid \alpha(x) < s \implies t \le \Phi(x, s, y)\},\ y \in Y$, is called the *$\alpha$-section of the corelation* Φ and is denoted by $\beta = \Phi^{\to}\alpha$.

The element $\phi^{\to}\alpha$ of $F(Y)$ may also be called the *image*, and $\Phi^{\to}\alpha$ the *co-image* of α under (ϕ, Φ), although we normally reserve this terminology for difunctions (see Definition 3.3). Clearly $\phi^{\to}$, $\Phi^{\to}$ define mappings (actually, functors) from $F(X)$ to $F(Y)$. The following gives some basic properties of these mappings.

2.25. Theorem. *For direlations $(\phi, \Phi), (\phi_1, \Phi_1), (\phi_2, \Phi_2)$ from $F(X)$ to $F(Y)$, $\alpha, \alpha_1, \alpha_2 \in F(X)$ we have:*

(1) $\alpha_1 \le \alpha_2 \implies \phi^{\to}\alpha_1 \le \phi^{\to}\alpha_2$ *and* $\Phi^{\to}\alpha_1 \le \Phi^{\to}\alpha_2$.
(2) $t < \phi(x, s, y) \iff \phi^{\to}x_s \not\le y^t$ *and* $\Phi(x, s, y) < t \iff y_t \not\le \Phi^{\to}x^s$.
(3) (a) $(\phi_1 \le \phi_2) \iff (\phi_1^{\to}\alpha \le \phi_2^{\to}\alpha\ \forall\, \alpha \in F(X))$,
(b) $(\Phi_1 \le \Phi_2) \iff (\Phi_1^{\to}\alpha \le \Phi_2^{\to}\alpha\ \forall\, \alpha \in F(X))$.
(4) *For any $\alpha_j \in F(X)$, $j \in J$, we have*

$$\phi^{\to}(\bigvee_{j\in J} \alpha_j) = \bigvee_{j\in J} \phi^{\to}\alpha_j \text{ and } \Phi^{\to}(\bigwedge_{j\in J} \alpha_j) = \bigwedge_{j\in J} \Phi^{\to}\alpha_j.$$

(5) *If (θ, Θ) is a direlation from $F(Y)$ to $F(Z)$ then*

$$(\theta \circ \phi)^{\to}\alpha = \theta^{\to}(\phi^{\to}\alpha) \text{ and } (\Theta \circ \Phi)^{\to}\alpha = \Theta^{\to}(\Phi^{\to}\alpha).$$

Proof. (1) is trivial and (2), (3) follow from [4, Lemma 2.6, Lemma 2.7], respectively. Finally, (4) is a consequence of [4, Corollary 2.12], and (5) follows from [4, Lemma 2.16 (1)]. □

2.26. Example. In case ϕ corresponds to the fuzzy relation φ, let us verify that for $\alpha \in F(X)$ we have

$$(\phi^{\to}\alpha)(y) = \bigvee\{\varphi(x, y) \wedge \alpha(x) \mid x \in X\},\ y \in Y.$$

Suppose first that $(\phi^{\rightarrow}\alpha)(y) < w < \bigvee\{\varphi(x,y) \wedge \alpha(x) \mid x \in X\}$. Then we have $t < w$ satisfying $s < \alpha(x) \implies \varphi(x,y) \wedge s \leq t$, and also $x \in X$ with $w < \varphi(x,y) \wedge \alpha(x)$. Clearly $w < \alpha(x)$ and $\varphi(x,y) \wedge w = w$, so taking $s = w$ in the above implication gives the contradiction $w \leq t$.

On the other hand suppose that $\bigvee\{\varphi(x,y) \wedge \alpha(x) \mid x \in X\} < w < (\phi^{\rightarrow}\alpha)(y)$. The right hand inequality now gives $x \in X$ and $s \in L$ satisfying $s < \alpha(x)$ and $\varphi(x,y) \wedge s > w$. However we now have $\varphi(x,y) \wedge \alpha(x) > w$, and hence the contradiction $\bigvee\{\varphi(x,y) \wedge \alpha(x) \mid x \in X\} > w$. This completes the proof of the above formula for $(\phi^{\rightarrow}\alpha)$. Hence, in case ϕ is obtained from φ as given in Example 2.5, we see that the α-section is non other than the full image $\varphi''\alpha$ of α under φ, calculated using the min t-norm [7, Definition 2.9].

The reader may easily verify the dual formula

$$(\Phi^{\rightarrow}\alpha)(y) = \bigwedge\{(1 - \varphi(x,y)) \vee \alpha(x) \mid x \in X\},\ y \in Y,$$

for $\Phi^{\rightarrow}\alpha$, where $\alpha \in F(X)$ and Φ is the corelation corresponding to φ.

We now give a definition of the presections of a relation and corelation based on [4, Definition 2.8].

2.27. Definition. Let (ϕ, Φ) be a direlation from $F(X)$ to $F(Y)$, and $\beta \in F(Y)$. Then

(1) The *β-presection of* ϕ is the β-section of the corelation $\phi^{\leftarrow}$ from $F(Y)$ to $F(X)$.
(2) The *β-presection of* Φ is the β-section of the relation $\Phi^{\leftarrow}$ from $F(Y)$ to $F(X)$.

The presection mappings (or functors) are therefore $(\phi^{\leftarrow})^{\rightarrow}$ and $(\Phi^{\leftarrow})^{\rightarrow}$. We will normally abbreviate the β-presection $(\phi^{\leftarrow})^{\rightarrow}\beta$ to $\phi^{\leftarrow}\beta$, and the β-presection $(\Phi^{\leftarrow})^{\rightarrow}\beta$ to $\Phi^{\leftarrow}\beta$, in conformity with standard mathematical practice. Again, $\phi^{\leftarrow}\beta$ may also be called the *inverse image* and $\Phi^{\leftarrow}\beta$ the *inverse co-images* of β under (ϕ, Φ), terms which, however, are normally reserved for difunctions.

The following gives formulae for calculating the presections directly from ϕ and Φ.

2.28. Lemma. *Let (ϕ, Φ) be a direlation from $F(X)$ to $F(Y)$ and $\beta \in F(Y)$. Then*

(1) $(\phi^{\leftarrow}\beta)(x) = \bigvee\{s \in L \mid \phi(x,s,y) \leq \beta(y)\ \forall\, y \in Y\}$.
(2) $(\Phi^{\leftarrow}\beta)(x) = \bigwedge\{s \in L \mid \beta(y) \leq \Phi(x,s,y)\ \forall\, y \in Y\}$.

Proof. (1) Applying Definition 2.23 (2) to the corelation $\phi^{\leftarrow}$ from $F(Y)$ to $F(X)$ gives

$$(\phi^{\leftarrow}\beta)(x) = \bigvee\{s \in L \mid \beta(y) < t \implies s \leq \phi^{\leftarrow}(y,t,x)\},$$

where $\phi^{\leftarrow}(y,t,x) = \bigvee\{z \in L \mid \phi(x,z,y) \leq t\}$ by Definition 2.11. Suppose first that $\beta(y) < t \implies s \leq \phi^{\leftarrow}(y,t,x)$ but that $\phi(x,s,y) \not\leq \beta(y)$. Then we may choose t satisfying $\beta(y) < t < \phi(x,s,y)$. Since ϕ is a relation, by Definition 2.4 (1) we have $0 < s' < s$ with $t < \phi(x,s',y)$. On the other hand, since $\beta(y) < t$ the above implication gives $s' < s \leq \phi^{\leftarrow}(y,t,x)$, so there exists z with $s' < z$ and $\phi(x,z,y) \leq t$. By monotonicity this gives $\phi(x,s',y) \leq t$, which is a contradiction.

Conversely suppose $\phi(x,s,y) \leq \beta(y)$ for all $y \in Y$ and suppose that for some $y \in Y$ and $t \in L$ we have $\beta(y) < t$ and $s > \phi^{\leftarrow}(y,t,x)$. Now $\phi(x,s,y) > t$, which gives the contradiction $\phi(x,s,y) > \beta(y)$. This completes the proof of equality (1).

(2) The proof is dual to (1) and is left to the interested reader. □

2.29. Example. Let ϕ be the relation from $F(X)$ to $F(Y)$ corresponding to the fuzzy relation φ. Then we obtain $(\phi^{\leftarrow}\beta)(x) = \bigvee\{s \in L \mid \varphi(x,y) \wedge s \leq \beta(y)\ \forall\, y \in Y\}$ for $\beta \in F(Y)$. Hence $\phi^{\leftarrow}\beta$ is non other than the fuzzy set $\varphi \downarrow \beta$ [7, Definition 2.9], calculated with respect to the min t-norm (see also [9] where $\varphi \downarrow \alpha$ corresponds to the @-operation).

In case Φ is the corelation corresponding to φ, then for $\beta \in F(Y)$ we have $(\Phi^{\leftarrow}\beta)(x) = \bigwedge\{s \in L \mid \beta(y) \leq (1-\varphi(x,y)) \vee s \ \forall y \in Y\}$.

Since presections are just sections of the inverse, the following is an immediate consequence of Theorem 2.25, Proposition 2.13, Proposition 2.15 (1) and Theorem 2.20 (3).

2.30. Theorem. *For direlations $(\phi,\Phi), (\phi_1,\Phi_1), (\phi_2,\Phi_2)$ from $F(X)$ to $F(Y)$, $\beta, \beta_1, \beta_2 \in F(Y)$ we have:*

(1) $\beta_1 \leq \beta_2 \implies \phi^{\leftarrow}\beta_1 \leq \phi^{\leftarrow}\beta_2$ *and* $\Phi^{\leftarrow}\beta_1 \leq \Phi^{\leftarrow}\beta_2$.

(2) $t < \phi(x,s,y) \iff x_s \not\leq \phi^{\leftarrow}y^t$ *and* $\Phi(x,s,y) < t \iff \Phi^{\leftarrow}y_t \not\leq x^s$.

(3) (a) $(\phi_2 \leq \phi_1) \iff (\phi_1^{\leftarrow}\beta \leq \phi_2^{\leftarrow}\beta \ \forall \beta \in F(Y))$,
(b) $(\Phi_2 \leq \Phi_1) \iff (\Phi_1^{\leftarrow}\beta \leq \Phi_2^{\leftarrow}\beta \ \forall \beta \in F(Y))$.

(4) *For any $\beta_j \in F(Y)$, $j \in J$, we have*

$$\phi^{\leftarrow}\Big(\bigwedge_{j\in J}\beta_j\Big) = \bigwedge_{j\in J}(\phi^{\leftarrow}\beta_j) \text{ and } \Phi^{\leftarrow}\Big(\bigvee_{j\in J}\beta_j\Big) = \bigvee_{j\in J}(\Phi^{\leftarrow}\beta_j).$$

(5) *If (θ,Θ) is a direlation from $F(Y)$ to $F(Z)$, $\gamma \in F(Z)$ then*

$$(\theta\circ\phi)^{\leftarrow}\gamma = \phi^{\leftarrow}(\theta^{\leftarrow}\gamma) \text{ and } (\Theta\circ\Phi)^{\leftarrow}\gamma = \Phi^{\leftarrow}(\Theta^{\leftarrow}\gamma).$$

There are also important results relating sections and presections [4, Lemma 2.9].

2.31. Theorem. *Let (ϕ,Φ) be a direlation from $F(X)$ to $F(Y)$.*

(1) $\phi^{\rightarrow}\mathbf{0} = \mathbf{0}$, *where $\mathbf{0}$ denotes the zero fuzzy subset of X. Also,*

$$\alpha \leq \phi^{\leftarrow}(\phi^{\rightarrow}\alpha) \ \forall \alpha \in F(X) \text{ and } \phi^{\rightarrow}(\phi^{\leftarrow}\beta) \leq \beta \ \forall \beta \in F(Y).$$

(2) $\Phi^{\rightarrow}\mathbf{1} = \mathbf{1}$, *where $\mathbf{1}$ denotes the unit fuzzy subset of X. Also,*

$$\Phi^{\leftarrow}(\Phi^{\rightarrow}\alpha) \leq \alpha \ \forall \alpha \in F(X) \text{ and } \beta \leq \Phi^{\rightarrow}(\Phi^{\leftarrow}\beta) \ \forall \beta \in F(Y).$$

(3) *For the identity direlation (μ_i,μ_I) on $F(X)$ we have*

$$\mu_i^{\rightarrow}\alpha = \mu_I^{\rightarrow}\alpha = \alpha,\ \mu_i^{\leftarrow}\alpha = \mu_I^{\leftarrow}\alpha = \alpha,\ \forall \alpha \in F(X).$$

We conclude this section by considering the complemented textures $(W, \mathcal{W}, \omega)$. Let us first recall the following notions of complement for direlations in general.

2.32. Definition. [4, Definition 2.18] Let $(S, \mathcal{S}, \sigma)$ and $(T, \mathcal{T}, \theta)$ be complemented textures and (r, R) a direlation from $(S, \mathcal{S})$ to $(T, \mathcal{T})$. Then:

(1) The *complement r' of the relation r* is the corelation

$$r' = \bigcap\{\overline{Q}_{(s,t)} \mid \exists\, u, v \text{ with } r \not\subseteq \overline{Q}_{(u,v)},\ \sigma(Q_s) \not\subseteq Q_u \text{ and } P_v \not\subseteq \theta(P_t)\}.$$

(2) The *complement R' of the corelation R* is the relation

$$R' = \bigvee\{\overline{P}_{(s,t)} \mid \exists\, u, v \text{ with } \overline{P}_{(u,v)} \not\subseteq R,\ P_u \not\subseteq \sigma(P_s) \text{ and } \theta(Q_t) \not\subseteq Q_v\}.$$

(3) The *complement $(r,R)'$ of the direlation (r,R)* is the direlation

$$(r,R)' = (R', r').$$

The direlation (r,R) is said to be *complemented* if $(r,R)' = (r,R)$.

2.33. Theorem. *Let (r,R) be a direlation from $(W_X, \mathcal{W}_X, \omega_X)$ to $(W_Y, \mathcal{W}_Y, \omega_Y)$, $(r,R)' = (R', r')$ the complement and (μ_r, μ_R), $(\mu_{R'}, \mu_{r'})$ the corresponding direlations from $F(X)$ to $F(Y)$. Then, for $(x,s,y) \in X \times L \times Y$ we have*

$$\mu_{R'}(x,s,y) = \begin{cases} 1-\mu_R(x,1-s,y) & s \neq 1 \\ \bigvee\{1-\mu_R(x,u,y) \mid u \in L\} & s = 1 \end{cases}$$

and

$$\mu_{r'}(x,s,y) = \begin{cases} 1-\mu_r(x,1-s,y) & s \neq 1 \\ 1 & s = 1 \end{cases}$$

Proof. We establish the result for $\mu_{R'}$, leaving the essentially dual proof of the equality for $\mu_{r'}$ to the interested reader.

By Definition 2.32 (2) we have

$$\widehat{\mu_{R'}} = \bigvee\{\overline{P}_{((x,s),(y,t))} \mid \exists\,(z,u),(w,v) \text{ with } \overline{P}_{((z,u),(w,v))} \not\subseteq R,\\ P_{(z,u)} \not\subseteq \omega_X(P_{(x,s)}) \text{ and } \omega_Y(Q_{(y,t)}) \not\subseteq Q_{(w,v)}\}.$$

By Lemma 1.4 we see that $P_{(z,u)} \not\subseteq \omega_X(P_{(x,s)})$ if and only if $z = x$, $1 - s < u$, while $\omega_Y(Q_{(y,t)}) \not\subseteq Q_{(w,v)}$ if and only if $w = y$ and $v < 1 - t$. Hence,

$$t' < \mu_{R'}(x,s,y) \iff \exists\, t' < t < 1, u, v \text{ with } \mu_R(x,u,y) < v,\ 1-s < u,\ v < 1-t.$$

Assume that $s \neq 1$, so that $1 - s \in L$. Let $t' < \mu_{R'}(x,s,y)$. Then for t, u and v as above, $\mu_R(x,1-s,y) \leq \mu_R(x,u,y) < v < 1-t < 1-t'$ by Theorem 2.3 and so $t' < 1 - \mu_R(x,1-s,y)$. Thus $\mu_{R'}(x,s,y) \leq 1 - \mu_R(x,1-s,y)$. If this inequality were proper we could choose t' with $\mu_{R'}(x,s,y) < t' < 1 - \mu_R(x,1-s,y)$, and then t and v satisfying $\mu_R(x,1-s,y) < v < 1-t < 1-t'$. By Theorem 2.3 (2) we could then choose u with $1-s < u$ and $\mu_R(x,u,y) < v$, which would give the contradiction $t' < \mu_{R'}(x,s,y)$ by the above equivalence.

This establishes the required equality for $s \neq 1$. The case $s = 1$ involves basically the same steps, and the details are left to the reader. □

2.34. Definition. Let (ϕ, Φ) be a direlation from $F(X)$ to $F(Y)$.

(1) The corelation ϕ' from $F(X)$ to $F(Y)$ given by

$$\phi'(x,s,y) = \begin{cases} 1 - \phi(x,1-s,y) & s \neq 1 \\ 1 & s = 1 \end{cases}$$

is known as the *complement* of ϕ.

(2) The relation Φ' from $F(X)$ to $F(Y)$ given by

$$\Phi'(x,s,y) = \begin{cases} 1 - \Phi(x,1-s,y) & s \neq 1 \\ \bigvee\{1 - \Phi(x,u,y) \mid u \in L\} & s = 1 \end{cases}$$

is known as the *complement* of Φ.

(3) The direlation $(\phi,\Phi)' = (\Phi',\phi')$ is called the *complement* of (ϕ,Φ).

A direlation satisfying $(\phi,\Phi)' = (\phi,\Phi)$ is said to be *complemented*.

2.35. Example. Let φ be a fuzzy relation from X to Y and (ϕ,Φ) the corresponding direlation from $F(X)$ to $F(Y)$, as in Example 2.5. Then, for $s \neq 1$,

$$\begin{aligned} \phi'(x,s,y) &= 1 - \phi(x,1-s,y) \\ &= 1 - (\varphi(x,y) \wedge (1-s)) \\ &= (1 - \varphi(x,y)) \vee s \\ &= \Phi(x,s,y). \end{aligned}$$

This formula also clearly holds for $s = 1$, so $\phi' = \Phi$. Likewise, $\Phi' = \phi$, and we have shown that $(\phi,\Phi)' = (\phi,\Phi)$. Hence the direlation corresponding to a fuzzy relation is complemented.

Finally, we may note the following properties of complementation corresponding to [4, Proposition 2.21 and Lemma 2.20].

2.36. Theorem. *Let (ϕ,Φ) be a direlation from $F(X)$ to $F(Y)$, $(\phi,\Phi)' = (\Phi',\phi')$ its complement. Then:*

(1) $(\phi')' = \phi$ *and* $(\Phi')' = \Phi$. *That is, the operations of taking the complement are idempotent.*
(2) $(\phi')^{\leftarrow} = (\phi^{\leftarrow})'$ *and* $(\Phi')^{\leftarrow} = (\Phi^{\leftarrow})'$. *That is, the inverse operations commute with complementation.*
(3) $(\phi')^{\rightarrow}\alpha = (\phi^{\rightarrow}\alpha')'$ *and* $(\Phi')^{\rightarrow}\alpha = (\phi^{\rightarrow}\alpha')'$ *for all* $\alpha \in F(X)$.
(4) $(\phi')^{\leftarrow}\beta = (\phi^{\leftarrow}\beta')'$ *and* $(\Phi')^{\leftarrow}\beta = (\Phi^{\leftarrow}\beta')'$ *for all* $\beta \in F(Y)$.
(5) *If* (θ, Θ) *is a direlation from* $F(Y)$ *to* $F(Z)$, $(\theta, \Theta)' = (\Theta', \theta')$ *its complement, then* $(\theta \circ \phi)' = \theta' \circ \phi'$ *and* $(\Theta \circ \Phi)' = \Theta' \circ \Phi'$.
(6) *The identity direlation* (μ_i, μ_I) *on* $F(X)$ *is complemented.*

3. Difunctions between Fuzzy Sets

A difunction between textures is defined as a special type of direlation, much as a point function (i.e., function in the classical sense) is a special type of binary point relation.

3.1. Definition. [4] Let (f, F) be a direlation from $(S, \mathcal{S})$ to $(T, \mathcal{T})$. Then (f, F) is called a *difunction from* $(S, \mathcal{S})$ *to* $(T, \mathcal{T})$ if it satisfies the following two conditions.

DF1 For $s, s' \in S$, $P_s \not\subseteq Q_{s'} \implies \exists\, t \in T$ with $f \not\subseteq \overline{Q}_{(s,t)}$ and $\overline{P}_{(s',t)} \not\subseteq F$.

DF2 For $t, t' \in T$ and $s \in S$, $f \not\subseteq \overline{Q}_{(s,t)}$ and $\overline{P}_{(s,t')} \not\subseteq F \implies P_{t'} \not\subseteq Q_t$.

It is clear that (i_S, I_S) is a difunction on $(S, \mathcal{S})$. In this context it is called the *identity difunction*.

3.2. Theorem. *Let* (f, F) *be a direlation from* $(W_X, \mathcal{W}_X)$ *to* $(W_Y, \mathcal{W}_Y)$*, and* (μ_f, μ_F) *the corresponding direlation from* $F(X)$ *to* $F(Y)$. *Then* (f, F) *is a difunction if and only* (μ_f, μ_F) *satisfies the following conditions:*

(i) $\forall\, x \in X$, $\forall\, s, s' \in L$, $s' < s \implies \exists y \in Y$ *with* $\mu_F(x, s', y) < \mu_f(x, s, y)$.
(ii) $\mu_f(x, s, y) \le \mu_F(x, s, y)\ \forall\, (x, s, y) \in X \times L \times Y$.
(iii) *For* $x \in X$, $s \in L$ *and* $y, y' \in Y$ *with* $y \ne y'$ *we have* $\mu_f(x, s, y) = 0$ *or* $\mu_F(x, s, y') = 1$.

Proof. First suppose that (f, F) is a difunction. We verify (i)–(iii).

(i) Take $x \in X$, $s, s' \in L$ with $s' < s$. Then $P_{(x,s)} \not\subseteq Q_{(x,s')}$, so by *DF1* there exists $(y, t) \in Y \times L$ for which $\widehat{\mu_f} \not\subseteq \overline{Q}_{((x,s),(y,t))}$ and $\overline{P}_{((x,s'),(y,t))} \not\subseteq \widehat{\mu_F}$. For this $y \in Y$ we obtain $\mu_F(x, s', y) < t < \mu_f(x, s, y)$ by Lemma 2.2.

(ii) Take $(x, s, y) \in X \times L \times Y$ and suppose that $\mu_f(x, s, y) \not\le \mu_F(x, s, y)$. Now we may take $t, t' \in L$ with $\mu_F(x, s, y) < t' = t < \mu_f(x, s, y)$, which gives $\widehat{\mu_f} \not\subseteq \overline{Q}_{((x,s),(y,t))}$, $\overline{P}_{((x,s),(y,t'))} \not\subseteq \widehat{\mu_F}$ but $P_{(y,t')} \subseteq Q_{(y,t)}$, so contradicting *DF2*.

(iii) Suppose that there exist $x \in X$, $s \in L$ and $y, y' \in L$ with $y \ne y'$ for which the stated conclusion does not hold. Now we may choose $t, t' \in L$ for which $t < \mu_f(x, s, y)$ and $\mu_F(x, s, y') < t'$, and we have $\widehat{\mu_f} \not\subseteq \overline{Q}_{((x,s),(y,t))}$, $\overline{P}_{((x,s),(y,t'))} \not\subseteq \widehat{\mu_F}$ but $P_{(y',t')} \subseteq Q_{(y,t)}$, which again contradicts *DF2*.

Conversely, let (μ_f, μ_F) satisfy (i)–(iii). We verify *DF1, DF2* for (f, F).

DF1 Take $x \in X$, $s, s' \in L$ with $P_{(x,s)} \not\subseteq Q_{(x,s')}$. Then $s' < s$ and so by (i) we have $y \in Y$ with $\mu_F(x, s', y) < \mu_f(x, s, y)$. If we choose $\mu_F(x, s', y) < t < \mu_f(x, s, y)$ then $(y, t) \in Y \times L$ and clearly $f \not\subseteq \overline{Q}_{((x,s),(y,t))}$, $\overline{P}_{((x,s'),(y,t))} \not\subseteq F$.

DF2 Take (y, t), $(y', t') \in T \times L$ and $(x, s) \in X \times L$ satisfying $f \not\subseteq \overline{Q}_{((x,s),(y,t))}$, $\overline{P}_{((x,s),(y',t'))} \not\subseteq F$. Hence, $t < \mu_f(x, s, y)$ and $t' > \mu_F(x, s, y')$. In particular $\mu_f(x, s, y) \ne 0$ and $\mu_F(x, s, y') \ne 1$, so $y = y'$ by (iii) and we have $t < \mu_f(x, s, y) \le \mu_F(x, s, y') < t'$ by (ii). This verifies that $P_{(y',t')} \not\subseteq Q_{(y,t)}$, as required. $\square$

This result enables us to give the following intrinsic definition of a difunction from $F(X)$ to $F(Y)$.

3.3. Definition. A direlation (ϕ, Φ) from $F(X)$ to $F(Y)$ is called a *textural fuzzy difunction* if it satisfies:

(i) $\forall x \in X, \forall s, s' \in L, s' < s \implies \exists y \in Y$ with $\Phi(x, s', y) < \phi(x, s, y)$.
(ii) $\phi(x, s, y) \leq \Phi(x, s, y)\ \forall (x, s, y) \in X \times L \times Y$.
(iii) For $x \in X$, $s \in L$, $y, y' \in Y$ with $y \neq y'$ we have $\phi(x, s, y) = 0$ or $\Phi(x, s, y') = 1$.

Again, we will abbreviate textural fuzzy difunction to difunction throughout the remainder of this paper.

3.4. Examples. (1). The identity direlation (μ_i, μ_I) on $F(X)$ (Example 2.6) is a difunction on $F(X)$.

(2) Let φ be a fuzzy relation from X to Y, and (ϕ, Φ) the corresponding direlation from $F(X)$ to $F(Y)$ as defined in Example 2.5. We seek necessary and sufficient conditions under which (ϕ, Φ) is a difunction.

Let us note that for all $(x, s, y) \in X \times L \times Y$ we have

$$\phi(x, s, y) = \varphi(x, y) \wedge s \leq s \leq (1 - \varphi(x, y)) \vee s = \Phi(x, s, y).$$

Hence, (ii) is automatically satisfied by (ϕ, Φ).

First suppose that (ϕ, Φ) is a difunction and take $x \in X$. We claim that there is a unique $y \in Y$ for which $\varphi(x, y) \neq 0$. To prove existence take $s, s' \in L$ with $s' < s < 1$ and apply (i) to give $y \in Y$ for which $\Phi(x, s', y) < \phi(x, s, y)$. In particular this gives $\phi(x, s, y) = \varphi(x, y) \wedge s \neq 0$, and so $\varphi(x, y) \neq 0$ for this y. Now take any $y' \in Y$ with $y \neq y'$. By (iii) we have $\Phi(x, s, y') = 1$ since $\phi(x, s, y) \neq 0$, so $(1 - \varphi(x, y')) = 1$ because $s < 1$, whence $\varphi(x, y') = 0$ as required. We denote this unique value of y by $g(x)$, thereby setting up a point function $g : X \to Y$ with the property that $\varphi(x, y) \neq 0 \iff y = g(x)$.

Finally, let us verify that $\varphi(x, g(x)) = 1$ for all $x \in X$. To this end suppose this is false for some $x \in X$ and choose $s, s' \in L$ satisfying $\varphi(x, g(x)) \leq s' < s \leq 1$. Then for $y = g(x)$ we have $\Phi(x, s', y) \geq s' \geq \varphi(x, g(x)) = \phi(x, s, y)$, while for $y \neq g(x)$, $\Phi(x, s', y) > 0 = \phi(x, s, y)$, which contradicts (i). We therefore deduce that if (ϕ, Φ) is a difunction then φ is a crisp function from X to Y.

Conversely, if φ is a crisp function from X to Y it is straightforward to verify that (ϕ, Φ) is a difunction. The details are left to the interested reader.

It is natural to ask if the crisp function that occurs for the special type of difunction discussed in Examples 3.4 (2) arises also in the general case. In general a difunction from a texture $(S, \mathcal{S})$ to a texture $(T, \mathcal{T})$ does not correspond to a point function from S to T, and likewise a point function from S to T need not define a difunction from $(S, \mathcal{S})$ to $(T, \mathcal{T})$. However, under certain special conditions such a correspondence does exist. In particular, this is the case for difunctions between simple textures [4]. Let us recall that a texture $(S, \mathcal{S})$ is simple if every molecule is a p-set. This is easily seen to be true of textures of the type $(W_X, \mathcal{W}_X)$ and so each difunction from $(W_X, \mathcal{W}_X)$ to $(W_Y, \mathcal{W}_Y)$ corresponds to a point function $\varphi : X \times L \to Y \times L$. The conditions that must be satisfied by φ are given in [4, Example 3.11 (4)], and we repeat these conditions below for the benefit of the reader.

3.5. Theorem. *The difunctions from $(W_X, \mathcal{W}_X)$ to $(W_Y, \mathcal{W}_Y)$ are in one to one correspondence with point functions $\varphi : W_X \to W_Y$ satisfying the conditions*

(i) $\varphi(x, r) = (\varphi_1(x), \varphi_2(x, r))$, *where* $\varphi_1 : X \to Y$ *and* $\varphi_2 : X \times L \to L$.
(ii) $r, s \in L$, $r < s \implies \varphi_2(x, r) < \varphi_2(x, s)\ \forall x \in X$.

(iii) $x \in X,\ s, t \in L,\ t < \varphi_2(x,s) \implies \exists\, r \in L$ *with* $r < s$ *and* $t < \varphi_2(x,r)$.

Moreover, (f,F) is complemented if and only if φ satisfies in addition

(iv) (A) $\varphi_2(x,s) + \varphi_2(x,1-s) \leq 1\ \forall\, x \in X,\ \forall\, s \in L \setminus \{1\}$.

(B) $\varphi_2(x,v) + \varphi_2(x,1-s) > 1\ \forall\, x \in X,\ \forall\, s, v \in L,\ v > s$.

Here the correspondence between (f,F) and φ is given explicitly by the equality $(f,F) = (f_\varphi, F_\varphi)$, where

$$f_\varphi = \bigvee \{\overline{P}_{((x,s),(\varphi_1(x),t))} \mid x \in X, s,\ t \in L, t < \varphi_2(x,s)\},\ \textit{and}$$

$$F_\varphi = \bigcap \{\overline{Q}_{((x,s),(\varphi_1(x),t))} \mid x \in X, s, t \in L,\ \exists\, v \in L \textit{ with } s < v, \varphi_2(x,v) < t\}.$$

We are now in a position to give a useful characterization of difunctions between $F(X)$ and $F(Y)$.

3.6. Theorem. *A direlation (ϕ, Φ) from $F(X)$ to $F(Y)$ is a difunction if and only if there exists a function $\varphi : W_X \to W_Y$ satisfying the conditions (i)–(iii) of Theorem 3.5 for which*

$$\phi(x,s,y) = \begin{cases} \varphi_2(x,s) & \textit{if } y = \varphi_1(x), \\ 0 & \textit{otherwise} \end{cases},$$

$$\Phi(x,s,y) = \begin{cases} \bigwedge\{t \in L \mid \exists s < v \in L,\ \varphi_2(x,v) < t\} & \textit{if } y = \varphi_1(x), \\ 1 & \textit{otherwise.} \end{cases}$$

The difunction (ϕ, Φ) is complemented if and only if φ also satisfies condition (iv) of Theorem 3.5.

Proof. Straightforward from the formulae for (f_φ, F_φ) given in Theorem 3.5. □

If (ϕ, Φ) is a difunction from $F(X)$ to $F(Y)$ and $\alpha \in F(X)$ then $\phi^{\to}\alpha$ will be called the *image* and $\Phi^{\to}\alpha$ the *co-image* of α under (ϕ, Φ). Likewise, for $\beta \in F(Y)$, $\phi^{\leftarrow}\beta$ will be known as the *inverse image* and $\Phi^{\leftarrow}\beta$ as the *inverse co-image* of β under (ϕ, Φ). It is known that for difunctions between textures the inverse image always coincides with the inverse co-image [4, Theorem 2.24], and indeed that this characterizes difunctions among the direlations. Hence, the same will be true of difunctions between Hutton algebras of fuzzy subsets.

Let us give formulae for calculating these fuzzy sets.

3.7. Theorem. *Let (ϕ, Φ) be a difunction from $F(X)$ to $F(Y)$ and $\varphi = \langle \varphi_1, \varphi_2 \rangle$ the corresponding point function from $X \times L$ to $Y \times L$. Then for $\alpha \in F(X)$, $y \in Y$ we have:*

$$\begin{aligned}(\phi^{\to}\alpha)(y) &= \bigvee \{\varphi_2(x,s) \mid (x,s) \in X \times L,\ y = \varphi_1(x),\ s < \alpha(x)\} \\ &= \bigvee \{\varphi_2(x,\alpha(x)) \mid x \in X,\ y = \varphi(x),\ \alpha(x) > 0\},\ \textit{and} \\ (\Phi^{\to}\alpha)(y) &= \bigwedge \{\varphi_2(x,s) \mid (x,s) \in X \times L,\ y = \varphi_1(x),\ \alpha(x) < s\}.\end{aligned}$$

Likewise, for $\beta \in F(Y)$, $x \in X$ we have:

$$\begin{aligned}(\phi^{\leftarrow}\beta)(x) &= \bigvee \{s \in L \mid \varphi_2(x,s) \leq \beta(\varphi_1(x))\} \\ &= (\Phi^{\leftarrow}\beta)(x).\end{aligned}$$

Proof. To prove the first equality for $(\phi^{\to}\alpha)(y)$ suppose first that $\bigvee\{\varphi_2(x,s) \mid (x,s) \in X \times L,\ y = \varphi_1(x),\ s < \alpha(x)\} \not\leq (\phi^{\to}\alpha)(y)$. Now we have $(x_0, s_0) \in X \times L$ satisfying $y = \varphi_1(x_0)$, $s_0 < \alpha(x_0)$ and $\varphi_2(x_0, s_0) \not\leq (\phi^{\to}\alpha)(y)$. By Definition 2.24 (1) there exists $t \in L$ with $s < \alpha(x) \implies \phi(x,s,y) \leq t$ for which $\varphi_2(x_0,s_0) \not\leq t$. On the other hand, by the above implication with $(x, s) = (x_0, s_0)$ we obtain $\phi(x_0, s_0, y) \leq t$, whence by Theorem 3.6 we obtain the contradiction

$$\varphi_2(x_0, s_0) = \phi(x_0, s_0, \varphi_1(x_0)) = \phi(x_0, s_0, y) \leq t.$$

Suppose now that there exists $w \in L$ with

$$\bigvee\{\varphi_2(x, s) \mid (x, s) \in X \times L,\ y = \varphi_1(x),\ s < \alpha(x)\} < w < (\phi^{\rightarrow}\alpha)(y).$$

Using the right-hand inequality there exists $(x_1, s_1) \in X \times L$ satisfying $s_1 < \alpha(x_1)$ and $\phi(x_1, s_1, y) \not\leq w$. It follows that $\phi(x_1, s_1, y) \neq 0$ and so by Theorem 3.6 we have $y = \varphi_1(x_1)$ and $\phi(x_1, s_1, y) = \varphi_2(x_1, s_1)$, whence $w < \varphi_2(x_1, s_1)$. However, we now have

$$w < \varphi_2(x_1, s_1) \leq \bigvee\{\varphi_2(x, s) \mid (x, s) \in X \times L,\ y = \varphi_1(x),\ s < \alpha(x)\} < w,$$

which is a contradiction.

This establishes the first equality, and the second equality for $(\phi^{\rightarrow}\alpha)(y)$ follows easily from this and we omit the details.

The proof of the equality for $(\Phi^{\rightarrow}\alpha)(y)$ is essentially dual to that for $\phi^{\rightarrow}\alpha)(y)$, and is left to the interested reader.

Finally we note that $(\phi^{\leftarrow}\beta)(x) = (\Phi^{\leftarrow}\beta)(x)$ for all β and all x by the equality of the inverse image and inverse co-image mentioned above. Hence, it will be sufficient to prove the equality for $(\phi^{\leftarrow}\beta)(x)$. To this end suppose that $(\phi^{\leftarrow}\beta)(x) \not\leq \bigvee\{s \in L \mid \varphi_2(x, s) \leq \beta(\varphi_1(x))\}$. By Lemma 2.28 (1) we now have $s \in L$ with $s \not\leq \bigvee\{s \in L \mid \varphi_2(x, s) \leq \beta(\varphi_1(x))\}$ for which $\phi(x, s, y) \leq \beta(y)$ for all $y \in Y$. In particular for $y = \varphi_1(x)$ we obtain $\varphi_2(x, s) \leq \phi(x, s, \varphi_1(x))$, which gives the contradiction $s \leq \bigvee\{s \in L \mid \varphi_2(x, s) \leq \beta(\varphi_1(x))\}$.

Conversely, suppose that $\bigvee\{s \in L \mid \varphi_2(x, s) \leq \beta(\varphi_1(x))\} \not\leq (\phi^{\leftarrow}\beta)(x)$. Then we have $s \in L$ with $s \not\leq (\phi^{\leftarrow}\beta)(x)$ for which $\varphi_2(x, s) \leq \beta(\varphi_1(x))$. Hence, by Theorem 3.6, if $y = \varphi_1(x)$ then $\phi(x, s, y) = \varphi_2(x, s) \leq \beta(y)$, while if $y \neq \varphi_1(x)$ then $\phi(x, s, y) = 0 \leq \beta(y)$, whence we have the contradiction $s \leq \bigvee\{s \in L \mid \phi(x, s, y) \leq \beta(y)\ \forall\, y \in L\} = (\phi^{\leftarrow}\beta)(x)$ by Lemma 2.28 (1). □

The following example treats a very important special case.

3.8. Example. Let $g : X \to Y$ be a point function. Then $\varphi_1 = g$, $\varphi_2 = \iota_L$, where ι_L is the identity on L, defines a function $\varphi_g = \langle g, \iota_L\rangle$ which clearly satisfies the conditions (i)–(iv) of Theorem 3.6 and hence defines a complemented difunction from $F(X)$ to $F(Y)$ that we will denote by (γ, Γ). Hence, by Theorem 3.6, we have

$$\gamma(x, s, y) = \begin{cases} s & \text{if } y = g(x), \\ 0 & \text{otherwise;} \end{cases} \qquad \Gamma(x, s, y) = \begin{cases} s & \text{if } y = g(x), \\ 1 & \text{otherwise.} \end{cases}$$

See also Examples 3.4 (2). The equalities in Theorem 3.7 now lead trivially to the following:

(1) For $\alpha \in F(X)$,

$$(\gamma^{\rightarrow}\alpha)(y) = \begin{cases} \bigvee\{\alpha(x) \mid y = g(x)\} & \text{if } g^{-1}(y) \neq \emptyset, \\ 0 & \text{otherwise, and} \end{cases}$$

$$(\Gamma^{\rightarrow}\alpha)(y) = \begin{cases} \bigwedge\{\alpha(x) \mid y = g(x)\} & \text{if } g^{-1}(y) \neq \emptyset, \\ 1 & \text{otherwise.} \end{cases}$$

(2) For $\beta \in F(Y)$,

$$\gamma^{\leftarrow}\beta = \Gamma^{\leftarrow}\beta = \beta \circ \alpha.$$

It will be noted that the mapping $\gamma^{\to}$, $\alpha \mapsto \gamma^{\leftarrow}\alpha$ is non-other than the mapping from $F(X)$ to $F(Y)$ usually denoted by g, while the mappings $(\gamma^{\leftarrow})^{\to}$ and $(\Gamma^{\leftarrow})^{\to}$ each give the mapping $\beta \mapsto \beta \circ g$ from $F(Y)$ to $F(X)$ which is often denoted by g^{-1}. So far as the authors are aware, the mapping $\Gamma^{\to}$, $\alpha \to \Gamma^{\to}\alpha$ from $F(X)$ to $F(Y)$ has not been considered in the context of $\mathbb{I}$-valued sets before, although it does occur in the study of intuitionistic fuzzy sets [5].

The following are important properties relating the (co)-images and inverse (co)-images of difunctions [4, Theorem 2.24]. We have already noted the equality $\phi^{\leftarrow}\beta = \Phi^{\leftarrow}\beta$ of the inverse image and inverse co-image of $\beta \in F(Y)$.

3.9. Theorem. *Let (ϕ, Φ) be a difunction from $F(X)$ to $F(Y)$, $\alpha \in F(X)$ and $\beta \in F(Y)$. Then*

$$\phi^{\leftarrow}(\Phi^{\to}\alpha) \leq \alpha \leq \Phi^{\leftarrow}(\phi^{\to}\alpha),$$
$$\phi^{\to}(\Phi^{\leftarrow}\beta) \leq \beta \leq \Phi^{\to}(\phi^{\leftarrow}\beta).$$

Again, these inequalities for all α and β actually characterize (ϕ, Φ) as a difunction among the direlations from $F(X)$ to $F(Y)$. The reader is referred to [4] for additional details.

3.10. Definition. [4] Let (f, F) be a difunction from $(S, \mathcal{S})$ to $(T, \mathcal{T})$. Then (f, F) is called *surjective* if it satisfies the condition

SUR. For $t, t' \in T$, $P_t \not\subseteq Q_{t'} \implies \exists\, s \in S$ with $f \not\subseteq \overline{Q}_{(s,t')}$ and $\overline{P}_{(s,t)} \not\subseteq F$.

Likewise, (f, F) is called *injective* if it satisfies the condition

INJ. For $s, s' \in S$ and $t \in T$, $f \not\subseteq \overline{Q}_{(s,t)}$ and $\overline{P}_{(s',t)} \not\subseteq F \implies P_s \not\subseteq Q_{s'}$.

This leads easily to the following equivalent definition of injectivity and surjectivety of difunctions between Hutton algebras of fuzzy subsets:

3.11. Definition. Let (ϕ, Φ) be a difunction from $F(X)$ to $F(Y)$. Then (ϕ, Φ) is called *surjective* if it satisfies the condition

SUR. For $t, t' \in L$, $y \in Y$,

$$t' < t \implies \exists\, (x, s) \in X \times L \text{ with } t' < \phi(x, s, y) \text{ and } \Phi(x, s, y) < t.$$

Likewise, (ϕ, Φ) is called *injective* if it satisfies the condition

INJ. For $(x, s), (x', s') \in X \times L$ and $y \in Y$,

$$\Phi(x', s', y) < \phi(x, s, y) \implies x = x' \text{ and } s' < s.$$

We note the following counterparts of [4, Corollary 2.33 and Proposition 2.34].

3.12. Proposition. *Let (ϕ, Φ) be a difunction from $F(X)$ to $F(Y)$.*

(1) *If (ϕ, Φ) is surjective then $\Phi^{\to}(\phi^{\leftarrow}\beta) = \beta = \phi^{\to}(\Phi^{\leftarrow}\beta)$ for all $\beta \in F(Y)$. In particular*
  (a) $\Phi^{\to}\alpha \leq \phi^{\to}\alpha$, $\forall \alpha \in F(X)$, *and*
  (b) $\forall \beta_1, \beta_2 \in F(Y)$, $\phi^{\leftarrow}\beta_1 \leq \phi^{\leftarrow}\beta_2 \implies \beta_1 \leq \beta_2$.

(2) *If (ϕ, Φ) is injective then $\Phi^{\leftarrow}(\phi^{\to}\alpha) = \alpha = \phi^{\leftarrow}(\Phi^{\to}\alpha)$ for all $\alpha \in F(X)$. In particular*
  (a) $\phi^{\to}\alpha \leq \Phi^{\to}\alpha$, $\forall \alpha \in F(X)$, *and*
  (b) $\forall \alpha_1, \alpha_2 \in F(X)$, $\Phi^{\to}\alpha_1 \leq \Phi^{\to}\alpha_2 \implies \alpha_1 \leq \alpha_2$.

3.13. Proposition. *Let (ϕ, Φ) be a difunction from $F(X)$ to $F(Y)$.*

(1) *(ϕ, Φ) is surjective if and only if $(\phi, \Phi)'$ is surjective.*
(2) *(ϕ, Φ) is injective if and only if $(\phi, \Phi)'$ is injective.*

Now we give characterizations of injectivity and surjectivity in terms of the corresponding point function.

3.14. Theorem. *Let (ϕ, Φ) be a difunction from $F(X)$ to $F(Y)$ which corresponds to the point function $\varphi = \langle \varphi_1, \varphi_2 \rangle$ as in Theorem 3.6.*

(1) *(ϕ, Φ) is injective if and only if the point function $\varphi_1 : X \to Y$ is one-to-one.*

(2) *(ϕ, Φ) is surjective if and only if given $y \in Y$ and $t, t' \in L$ with $t' < t$ there exists $(x, s) \in X \times L$ satisfying $y = \varphi_1(x)$ and $t' < \varphi_2(x, s) < t$.*

In particular if (ϕ, Φ) is surjective then the point function $\varphi_1 : X \to Y$ is onto, but the converse is not true in general.

Proof. (1) Suppose that (ϕ, Φ) is injective, but that φ_1 is not one-to-one. Then we have $x, x' \in X$ with $x \neq x'$ and $\varphi_1(x) = \varphi_1(x')$. Let us show first that $\varphi_2(x', s) = \varphi_2(x, s)$ for all $s \in L$. Suppose this is not so, then for some $s \in L$ we have $\varphi_2(x', s) \neq \varphi_2(x, s)$, and without loss of generality we may suppose $\varphi_2(x', s) < \varphi_2(x, s)$. By Theorem 3.5 (iii) we have $r \in L$ with $r < s$ satisfying $\varphi_2(x', s) < \varphi_2(x, r)$. Let $y = \varphi_1(x) = \varphi_1(x')$. Since $x' \neq x$ and $r \not< r$, by Theorem 3.6, *INJ* gives

$$\varphi_2(x, r) = \phi(x, r, y) \leq \Phi(x', r, y) \leq \varphi_2(x', s)$$

since $\varphi_2(x', r) < \varphi_2(x', s)$ by Theorem 3.5 (ii). This contradicts the above assumption.

Now choose $s' \in L$ with $s' < s$. By Theorem 3.5 (ii) we deduce $\varphi(x', s') < \varphi_2(x', s) = \varphi_2(x, s)$, which gives the contradiction $x = x'$ by *INJ*.

Conversely, suppose that $\varphi_1 : X \to Y$ is one to one. Take $(x, s), (x', s') \in X \times L$ and $y \in Y$ with $\Phi(x', s', y) < \phi(x, s, y)$. Clearly $\phi(x, s, y) \neq 0$ so by Theorem 3.6 we have $y = \varphi_1(x)$ and $\phi(x, s, y) = \varphi_2(x, s)$. Likewise, $\Phi(x', s', y) \neq 1$ so by the same theorem we have $y = \varphi_1(x')$ and $\Phi(x', s', y) = \bigwedge\{t \in L \mid \exists s' < v \in L,\ \varphi_2(x', v) < t\}$. Since φ_1 is one-to-one we obtain $x = x'$. On the other hand there exists $s' < v \in L$ and $t \in L$ satisfying $\varphi_2(x', v) < t < \varphi_2(x, s)$, whence $\varphi_2(x, s') = \varphi_2(x', s') < \varphi_2(x', v) < \varphi_2(x, s)$ by Theorem 3.5 (ii). Clearly $s \neq s$, while $s < s'$ would give a contradiction by Theorem 3.5 (ii). Hence $s' < s$ and we have established that (ϕ, Φ) is injective.

(2) The given condition is clearly equivalent to surjectivity, and so surjectivity of (ϕ, Φ) certainly implies that φ_1 is onto. To see that this is not sufficient, take $Y = X$, let φ_1 be the identity on X and consider the function

$$\varphi_2(x, r) = \begin{cases} r/2, & 0 < r \leq 1/3 \\ r/2 + 1/4, & 1/3 < r \leq 2/3 \\ r/2 + 1/2, & 2/3 < r \leq 1. \end{cases}$$

considered in [4, Example 3.11 (4)]. Then $\varphi = \langle \varphi_1, \varphi_2 \rangle$ satisfies (i)–(iv), and φ_1 is certainly onto, but the corresponding difunction is not surjective. Indeed, take $x \in X$, $t' = (1/3)/2 = 1/6$, $t = (1/3)/2 + 1/4 = 5/12$. Then $t' < t$ and $x = \varphi_1(x)$ but there is no $s \in L$ for which $t' < \varphi_2(x, s) < t$. □

3.15. Corollary. *The difunction (ϕ, Φ) corresponding to the point function $\varphi = \langle \varphi_1, \varphi_2 \rangle$ is an isomorphism if and only if φ_1 is one-to-one and onto and given $y \in Y$, $t, t' \in L$ with $t' < t$ there exists $(x, s) \in X \times L$ satisfying $y = \varphi_1(x)$ and $t' < \varphi_2(x, s) < t$.*

Proof. By [4, Proposition 3.14 (5)], (ϕ, Φ) is an isomorphism if and only if it is injective and surjective. The result now follows from Theorem 3.14. □

The example given in the proof of Theorem 3.14 also shows that a difunction for which φ_1 is one-to-one and onto need not be an isomorphism.

Finally, let us consider the composition of difunctions. By [4, Proposition 2.28] and Theorem 2.20 we have:

3.16. Theorem. *The composition of two difunctions is a difunction. The identity difunction is the identity for composition and the operation of composition is associative.*

It follows from this result that we may define a category whose objects are the Hutton algebras $F(X)$ for $X \in \mathrm{Ob}\,\mathbf{Set}$, and whose morphisms are difunctions between these objects. We shall denote this category by $\mathbb{I}$-**dfSet**. If we take the morphisms to be complemented difunctions then we will denote the corresponding category by $\mathbb{I}$-**cdfSet**. Recalling [4] that the category of simple textures and difunctions is denoted by **dfSTex**, we may set up a functor $\mathfrak{E} : \mathbb{I}\text{-}\mathbf{dfSet} \to \mathbf{dfSTex}$ by setting $\mathfrak{E}(F(X)) = (W_X, \mathcal{W}_X)$ and $\mathfrak{E}(\phi, \Phi) = (f, F)$ where $\phi = \mu_f$ and $\Phi = \mu_F$. Likewise, we may set up a functor $\mathfrak{E}_c$ from $\mathbb{I}$-**cdfSet** to the category **cdSTex** of complemented simple textures and complemented difunctions by setting $\mathfrak{E}_c(F(X)) = (W_X, \mathcal{W}_X, \omega_X)$ and $\mathfrak{E}_c(\phi, \Phi) = (f, F)$ for complemented difunctions $(\phi, \Phi) = (\mu_f, \mu_F)$. It is immediate that:

3.17. Theorem. *$\mathfrak{E}$ is a full embedding of $\mathbb{I}$-**dfSet** in **dfSTex** and $\mathcal{E}_c$ a full embedding of $\mathbb{I}$-**cdfSet** in **cdfSTex**.*

We recall from [4] that **fSTex** is the construct whose objects are simple textures and whose morphisms are point functions between the base sets satisfying certain conditions (a) and (b). The construct **fSTex** is isomorphic with **dfSText** under the functor $\mathcal{D}_s$ that is the identity on objects and takes a **fStex** morphism $\varphi : S \to T$ to the corresponding difunction (f_φ, F_φ) from $(S, \mathcal{S})$ to $(T, \mathcal{T})$ [4, Proposition 3.6]. This correspondence between the morphisms of **fSTex** and those of **dfSTex** parallels that between a point function between $X \times L$ and $Y \times L$ satisfying (i)–(iii), and the corresponding difunction between $F(X)$ and $F(Y)$. It follows that we have a category, which we will denote by $\mathbb{I}$-**fSet**, whose objects are the Hutton algebras $F(X)$, $X \in \mathrm{Ob}\,\mathbf{Set}$, and whose morphisms are point functions from $X \times L$ to $Y \times L$ satisfying (i)–(iii), and an isomorphism $\mathfrak{D}^{\mathbb{I}}$ from $\mathbb{I}$-**fSet** to $\mathbb{I}$-**dfSet** which is the identity on objects and takes a $\mathbb{I}$-**fSet** morphism $\varphi : X \times L \to Y \times L$ to the corresponding difunction from $F(X)$ to $F(Y)$. This gives the left-hand commutative diagram below, where $\mathfrak{G}$ is the full embedding of $\mathbb{I}$-**fSet** in **fSTex**. The right-hand commutative diagram treats the complemented case. Here the morphisms of $\mathbb{I}$-**cfSet** satisfy (iv) in addition to (i)–(iii), and the difunctions involved are complemented.

$$
\begin{tikzcd}
\mathbb{I}\text{-}\mathbf{fSet} \arrow[r, "\mathfrak{G}"] \arrow[d, "\mathfrak{D}^{\mathbb{I}}"] & \mathbf{fSTex} \arrow[d, "\mathfrak{D}_s"] \\
\mathbb{I}\text{-}\mathbf{dfSet} \arrow[r, "\mathfrak{E}"] & \mathbf{dfSTex}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{I}\text{-}\mathbf{cfSet} \arrow[r, "\mathfrak{G}_c"] \arrow[d, "\mathfrak{D}^{\mathbb{I}}_c"] & \mathbf{cfSTex} \arrow[d, "\mathfrak{D}_{cs}"] \\
\mathbb{I}\text{-}\mathbf{cdfSet} \arrow[r, "\mathfrak{E}_c"] & \mathbf{cdfSTex}
\end{tikzcd}
$$

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