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Approximation of generalized left derivations in modular spaces

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Abstract

In this paper, we define modular spaces, and introduce some properties of them. Moreover, we present a fixed point method to prove superstability of generalized left derivations from an algebra into a modular space.

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1. Introduction

Let $\mathcal{A}$ be an algebra over the real or complex field $\mathbb{F}$ and let $\mathcal{X}$ be an $\mathcal{A}$-module. An additive mapping $d : \mathcal{A} \to \mathcal{X}$ is said to be a left derivation if the functional equation $d(xy) = xd(y) + yd(x)$ holds for all $x, y \in \mathcal{A}$. Moreover, if $d(\alpha x) = \alpha d(x)$ is valid for all $x \in \mathcal{A}$ and for all $\alpha \in \mathbb{F}$, then d is called a linear left derivation. An additive mapping $D : \mathcal{A} \to \mathcal{X}$ is said to be a generalized left derivation if there exists a left derivation $d : \mathcal{A} \to \mathcal{X}$ such that $D(xy) = xD(y) + yd(x)$ holds for all $x, y \in \mathcal{A}$. Furthermore, if $D(\alpha x) = \alpha D(x)$ is valid for all $x \in \mathcal{A}$ and for all $\alpha \in \mathbb{F}$, then D is called a linear generalized left derivation.

In 1940, Ulam [21] posed the first stability problem of functional equations, concerning the stability of group homomorphisms, was solved in the case of the additive mapping by Hyers [4] in the next year. Subsequently, Aoki [1] extended Hyers' theorem for approximately additive mappings and for approximately linear mappings was presented by Rassias [18]. The stability result concerning derivations between operator algebras was first obtained by Semrl [20]. Also Badora [2] present the Hyers-Ulam stability and the superstability of derivations. The equation is called *superstable* if each its approximate solution is an exact solution. Various stability and superstability results for derivations

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have been investigated by a number of mathematicians [3, 5, 11, 12, 16, 17, 19]. In this paper, we define modular spaces, and introduce some properties of them. Moreover, we prove the superstability of generalized left derivations from an algebra with unit into a modular space by using a fixed point method. The theory of modular spaces were founded by Nakano [14] and were intensively developed by Luxemburg [9], Koshi and Shimogaki [7] and Yamamuro [22] and their collaborators. In the present time the theory of modulars and modular spaces is extensively applied, in particular, in the study of various Orlicz spaces [15] and interpolation theory [8, 10], which in their turn have broad applications [13].

1.1. Definition. Let $\mathcal{X}$ be an arbitrary vector space.
(a) A functional $\rho : \mathcal{X} \to [0, \infty]$ is called a modular if for arbitrary $x, y \in \mathcal{X}$,
(i) $\rho(x) = 0$ if and only if $x = 0$,
(ii) $\rho(\alpha x) = \rho(x)$ for every scaler α with $|\alpha| = 1$,
(iii) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ if and only if $\alpha + \beta = 1$ and $\alpha, \beta \geq 0$,
(b) if (iii) is replaced by
(iii)$'$ $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$ if and only if $\alpha + \beta = 1$ and $\alpha, \beta \geq 0$,
then we say that ρ is a convex modular.

If ρ is a modular, the corresponding modular space is the vector space $\mathcal{X}_\rho$ given by

$$\mathcal{X}_\rho = \{x \in \mathcal{X} : \quad \rho(\lambda x) \to 0 \text{ as } \lambda \to 0\}.$$

Let ρ be a convex modular, the modular space $\mathcal{X}_\rho$ can be equipped with a norm called the Luxemburg norm, defined by

$$\|x\|_\rho = \inf\left\{\lambda > 0 : \quad \rho\left(\frac{x}{\lambda}\right) \leq 1\right\}.$$

A function modular is said to be satisfy the Δ_2–condition if there exists $\kappa > 0$ such that $\rho(2x) \leq \kappa\rho(x)$ for all $x \in \mathcal{X}_\rho$.

1.2. Definition. Let $\{x_n\}$ and x be in $\mathcal{X}_\rho$. Then
(i) the sequence $\{x_n\}$, with $x_n \in \mathcal{X}_\rho$, is ρ–convergent to x and we write $x_n \xrightarrow{\rho} x$ if $\rho(x_n - x) \to 0$ as $n \to \infty$.
(ii) The sequence $\{x_n\}$, with $x_n \in \mathcal{X}_\rho$, is called ρ–Cauchy if $\rho(x_n - x_m) \to 0$ as $n, m \to \infty$.
(iii) A subset $\mathcal{S}$ of $\mathcal{X}_\rho$ is called ρ–complete if and only if any ρ–Cauchy sequence is ρ–convergent to an element of $\mathcal{S}$.

We call the modular ρ has the Fatou property if $\rho(x) \leq \liminf_{n\to\infty} \rho(x_n)$ whenever the sequence $\{x_n\}$ is ρ–convergent to x.

1.3. Remark. Note that $\rho(.x)$ is an increasing function for each $x \in \mathcal{X}$. Suppose $0 < a < b$, and put $y = 0$ in property (iii) of Definition 1.1, then $\rho(ax) = \rho\left(\frac{a}{b}bx\right) \leq \rho(bx)$ for all $x \in \mathcal{X}$. Moreover, if ρ is a convex modular on $\mathcal{X}$ and $|\alpha| \leq 1$, then $\rho(\alpha x) \leq \alpha\rho(x)$ and also $\rho(x) \leq \frac{1}{2}\rho(2x)$ for all $x \in \mathcal{X}$.

1.4. Example. An example of a modular space with Δ_2–condition is the Orlicz space. Let τ be a function defined on the interval $[0, \infty)$ such that $\tau(0) = 0, \tau(\alpha) > 0$ for $\alpha > 0$ and $\tau(\alpha) \to \infty$ as $\alpha \to \infty$. Also assume that τ is convex, nondecreasing and continuous. The function τ is called an Orlicz function. The Orlicz function τ satisfies the Δ_2–condition if there exists $\kappa > 0$ such that $\tau(2\alpha) \leq \kappa\tau(\alpha)$ for all $\alpha > 0$. Let $(\Omega, \mathfrak{M}, \mu)$ be a measure space. Let $L^0(\mu)$ be the space of all measurable real–valued (or complex–valued) functions on Ω. For every $f \in L^0(\mu)$, we define the Orlicz modular $\rho_\tau(f)$ as

$$\rho_\tau(f) = \int_\Omega \tau(|f|)d\mu.$$

The associated modular function space with respect to this modular is called an Orlicz space, and will be denoted by $L^\tau(\Omega,\mu)$ or briefly L^τ. In other words,

$$L^\tau = \{f \in L^0(\mu)| \quad \rho_\tau(\lambda f) \to 0 \text{ as } \lambda \to 0\}$$

or equivalently as

$$L^\tau = \{f \in L^0(\mu)| \quad \rho_\tau(\lambda f) < \infty \text{ for some } \lambda > 0\}.$$

It is known that the Orlicz space L^τ is ρ_τ–complete. Moreover, $(L^\tau, \|.\|_{\rho_\tau})$ is a Banach space, where the Luxemburg norm $\|.\|_{\rho_\tau}$ is defined as follows

$$\|f\|_{\rho_\tau} = \inf\left\{\lambda > 0 : \quad \int_\Omega \tau\left(\frac{|f|}{\lambda}\right) d\mu \le 1\right\}.$$

2. Main results

Throughout this paper, $\mathcal{A}$ and $\mathcal{X}$ denote a Banach algebra with unit and a unital $\mathcal{A}$-module respectively. Also $\mathcal{X}_\rho$ denotes a ρ–complete modular space where ρ is a convex modular on $\mathcal{X}$ with the Fatou property such that satisfies the Δ_2–condition with $0 < \kappa \le 2$. In this section, we present the superstability of generalized left derivations from a Banach algebra into a complete modular space.

2.1. Theorem. *Let $d : \mathcal{A} \to \mathcal{X}_\rho$ be a mapping with $d(0) = 0$ such that*

$$\rho\left(d(x+y) - d(x) - d(y)\right) \le \varphi(x,y) \tag{2.1}$$

for all $x, y \in \mathcal{A}$, where $\varphi : \mathcal{A}\times\mathcal{A} \to [0,\infty)$ is a given mapping that

$$\varphi(2x,2x) \le 2L\varphi(x,x)$$

and

$$\lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y)}{2^n} = 0 \tag{2.2}$$

for all $x, y \in \mathcal{A}$ and a constant $0 < L < 1$. Then there exist a unique additive mapping $D : \mathcal{A} \to \mathcal{X}_\rho$ and a convex modular function $\widetilde{\rho}$ such that

$$\widetilde{\rho}(D - d) \le \frac{1}{2(1-L)}. \tag{2.3}$$

Proof. Consider the set

$$\mathfrak{B} = \{\delta : \mathcal{A} \to \mathcal{X}_\rho, \quad \delta(0) = 0\}$$

we define the function $\widetilde{\rho}$ on $\mathfrak{B}$ as follows,

$$\widetilde{\rho}(\delta) = \inf\{c > 0 : \rho(\delta(x)) \le c\varphi(x,x)\}. \tag{2.4}$$

Then $\widetilde{\rho}$ is convex modular. It is enough to show that $\widetilde{\rho}$ satisfies the following condition

$$\widetilde{\rho}(\alpha\delta + \beta\gamma) \le \alpha\widetilde{\rho}(\delta) + \beta\widetilde{\rho}(\gamma) \quad (\alpha,\beta \ge 0,\ \alpha+\beta = 1).$$

Given $\varepsilon > 0$, then there exist $c_1 > 0$ and $c_2 > 0$ such that

$$c_1 \le \widetilde{\rho}(\delta) + \varepsilon, \quad \rho(\delta(x)) \le c_1\varphi(x,x)$$

and

$$c_2 \le \widetilde{\rho}(\gamma) + \varepsilon, \quad \rho(\gamma(x)) \le c_2\varphi(x,x).$$

For $\alpha, \beta \ge 0$ such that $\alpha + \beta = 1$, we get

$$\rho(\alpha\delta(x) + \beta\gamma(x)) \le \alpha\rho(\delta(x)) + \beta\rho(\gamma(x)) \le (\alpha c_1 + \beta c_2)\varphi(x,x),$$

hence

$$\widetilde{\rho}(\alpha\delta + \beta\gamma) \le \alpha\widetilde{\rho}(\delta) + \beta\widetilde{\rho}(\gamma) + (\alpha+\beta)\varepsilon.$$

Consequently $\widetilde{\rho}(\alpha\delta+\beta\gamma)\leq\alpha\widetilde{\rho}(\delta)+\beta\widetilde{\rho}(\gamma)$. Moreover, $\widetilde{\rho}$ satisfies the Δ_2–condition with $0<\kappa<2$. For this, let $\{\delta_n\}$ be a $\widetilde{\rho}$–Cauchy sequence in $\mathcal{E}_{\widetilde{\rho}}$ and given $\varepsilon>0$. There exists a positive integer $n_0\in\mathbb{N}$ such that $\widetilde{\rho}(\delta_n-\delta_m)\leq\varepsilon$ for all $n,m\geq n_0$. Then by definition of the modular $\widetilde{\rho}$, we have

$$\rho\left(\delta_n(x)-\delta_m(x)\right)\leq\varepsilon\varphi(x,x) \tag{2.5}$$

for all $x\in\mathcal{A}$ and $n,m\geq n_0$. Let x be a point of $\mathcal{A}$, (2.5) implies that $\{\delta_n(x)\}$ is a ρ–Cauchy sequence in $\mathcal{X}_\rho$. Since $\mathcal{X}_\rho$ is ρ–complete, so $\{\delta_n(x)\}$ is ρ–convergent in $\mathcal{X}_\rho$, for each $x\in\mathcal{A}$. Therefore we can define a function $\delta:\mathcal{A}\to\mathcal{X}_\rho$ by

$$\delta(x)=\lim_{n\to\infty}\delta_n(x)$$

for any $x\in\mathcal{A}$. Letting $m\to\infty$, then (2.5) implies that

$$\widetilde{\rho}(\delta_n-\delta)\leq\varepsilon$$

for all $n\geq n_0$. Since ρ has the Fatou property, thus $\{\delta_n\}$ is $\widetilde{\rho}$–convergent sequence in $\mathfrak{B}_{\widetilde{\rho}}$. Therefore $\mathcal{E}_{\widetilde{\rho}}$ is $\widetilde{\rho}$–complete.

Now, we define the function $\mathcal{T}:\mathcal{E}_{\widetilde{\rho}}\to\mathfrak{B}_{\widetilde{\rho}}$ as follows

$$\mathcal{T}\delta(x):=\frac{1}{2}\delta(2x)$$

for all $\delta\in\mathfrak{B}_{\widetilde{\rho}}$. Let $\delta,\gamma\in\mathfrak{B}_{\widetilde{\rho}}$ and let $c\in[0,\infty]$ be an arbitrary constant with $\widetilde{\rho}(\delta-\gamma)\leq c$. We have

$$\rho(\delta(x)-\gamma(x))\leq c\varphi(x,x)$$

for all $x\in\mathcal{A}$. The last inequality implies that

$$\rho\left(\frac{\delta(2x)}{2}-\frac{\gamma(2x)}{2}\right)\leq\frac{1}{2}\rho(\delta(2x)-\gamma(2x))\leq\frac{1}{2}c\varphi(2x,2x)\leq Lc\varphi(x,x)$$

for all $x\in\mathcal{A}$. Hence, $\widetilde{\rho}(\mathcal{T}\delta-\mathcal{T}\gamma)\leq L\widetilde{\rho}(\delta-\gamma)$, for all $\delta,\gamma\in\mathfrak{B}_{\widetilde{\rho}}$. Therefore $\mathcal{T}$ is a $\widetilde{\rho}$–strict contraction. We show that the $\widetilde{\rho}$–strict mapping $\mathcal{T}$ satisfies the conditions of Theorem 3.4 of [6]. Letting $x=y$ in (2.12), we get

$$\rho(d(2x)-2d(x))\leq\varphi(x,x) \tag{2.6}$$

for all $x\in\mathcal{A}$. Replacing x by $2x$ in (2.6) we get

$$\rho(d(4x)-2d(2x))\leq\varphi(2x,2x)$$

for all $x\in\mathcal{A}$. Since ρ is convex modular and satisfies the Δ_2–condition, for all $x\in\mathcal{A}$ we have

$$\begin{aligned}\rho\left(\frac{d(4x)}{2}-2d(x)\right)&\leq\frac{1}{2}\rho(d(4x)-2d(2x))+\frac{1}{2}\rho(2d(2x)-4d(x))\\&\leq\frac{1}{2}\varphi(2x,2x)+\frac{\kappa}{2}\varphi(x,x).\end{aligned}$$

Moreover,

$$\rho\left(\frac{d(2^2x)}{2^2}-d(x)\right)\leq\frac{1}{2}\rho\left(2\frac{d(4x)}{2^2}-2d(x)\right)\leq\frac{1}{2^2}\varphi(2x,2x)+\frac{\kappa}{2^2}\varphi(x,x).$$

for all $x\in\mathcal{A}$. By induction we obtain

$$\rho\left(\frac{d(2^nx)}{2^n}-d(x)\right)\leq\frac{1}{2^n}\sum_{i=1}^{n}\kappa^{n-i}\varphi(2^{i-1}x,2^{i-1}x)\leq\frac{1}{2(1-L)}\varphi(x,x) \tag{2.7}$$

for all $x \in \mathcal{A}$. Now we claim that $\delta_{\widetilde{\rho}}(d) = \sup\{\widetilde{\rho}\,(\mathfrak{T}^n(d) - \mathfrak{T}^m(d))\,; n, m \in \mathbb{N})\} < \infty$. It follows from (2.7) that

$$\begin{aligned}\rho\left(\frac{d(2^n x)}{2^n} - \frac{d(2^m x)}{2^m}\right) &\le \frac{1}{2}\rho\left(2\frac{d(2^n x)}{2^n} - 2d(x)\right) + \frac{1}{2}\rho\left(2\frac{d(2^m x)}{2^m} - 2d(x)\right)\\ &\le \frac{\kappa}{2}\rho\left(\frac{d(2^n x)}{2^n} - d(x)\right) + \frac{\kappa}{2}\rho\left(\frac{d(2^m x)}{2^m} - d(x)\right)\\ &\le \frac{1}{1-L}\varphi(x,x),\end{aligned}$$

for every $x \in \mathcal{A}$ and $n, m \in \mathbb{N}$, which implies that

$$\widetilde{\rho}\,(\mathfrak{T}^n(d) - \mathfrak{T}^m(d)) \le \frac{1}{1-L},$$

for all $n, m \in \mathbb{N}$. Therefore $\delta_{\widetilde{\rho}}(d) < \infty$. [6, Lemma 3.3] shows that $\{\mathfrak{T}^n(d)\}$ is $\widetilde{\rho}$–convergent to $D \in \mathfrak{B}_{\widetilde{\rho}}$. Since ρ has the Fatou property, (2.7) gives $\widetilde{\rho}(\mathfrak{T}D - d) < \infty$.

If we replace x by $2^n x$ in (2.6), then

$$\widetilde{\rho}\left(d(2^{n+1}x) - 2d(2^n x)\right) \le \varphi(2^n x, 2^n x),$$

for all $x \in \mathcal{A}$. Hence

$$\begin{aligned}\rho\left(\frac{d(2^{n+1}x)}{2^{n+1}} - \frac{d(2^n x)}{2^n}\right) &\le \frac{1}{2^{n+1}}\rho\left(d(2^{n+1}x) - 2d(2^n x)\right) \le \frac{1}{2^{n+1}}\varphi(2^n, 2^n x)\\ &\le \frac{1}{2^{n+1}}2^n L^n \varphi(x,x) \le \frac{L^n}{2}\varphi(x,x) \le \varphi(x,x)\end{aligned}$$

for all $x \in \mathcal{A}$, therefore $\widetilde{\rho}(\mathfrak{T}(D) - D) < \infty$. It follows from [6, Theorem 3.4] that $\widetilde{\rho}$–limit D of $\{\mathfrak{T}^n(d)\}$ is fixed point of map $\mathfrak{T}$. If we replace x by $2^n x$ and y by $2^n y$ in (2.12), then we obtain

$$\rho\left(d(2^n(x+y)) - d(2^n x) - d(2^n y)\right) \le \varphi(2^n x, 2^n y)$$

for all $x, y \in \mathcal{A}$. Hence,

$$\begin{aligned}\rho\left(\frac{d(2^n(x+y))}{2^n} - \frac{d(2^n x)}{2^n} - \frac{d(2^n y)}{2^n}\right) &\le \frac{1}{2^n}\rho\left(d(2^n(x+y)) - d(2^n x) - d(2^n y)\right)\\ &\le \frac{\varphi(2^n x, 2^n y)}{2^n}\end{aligned}$$

for all $x, y \in \mathcal{A}$. Taking the limit, we deduce that $D(x+y) = D(x) + D(y)$ for all $x, y \in \mathcal{A}$, that is, D is additive. Now, let D^* be another fixed point of $\mathfrak{T}$, then

$$\begin{aligned}\widetilde{\rho}(D - D^*) &\le \frac{1}{2}\widetilde{\rho}(2\mathfrak{T}(D) - 2d) + \frac{1}{2}\widetilde{\rho}(2\mathfrak{T}(D^*) - 2d)\\ &\le \frac{\kappa}{2}\widetilde{\rho}(\mathfrak{T}(D) - d) + \frac{\kappa}{2}\widetilde{\rho}(\mathfrak{T}(D^*) - d) \le \frac{\kappa}{2(1-L)} < \infty.\end{aligned}$$

Since $\mathfrak{T}$ is $\widetilde{\rho}$–strict contraction, we get

$$\widetilde{\rho}(D - D^*) = \widetilde{\rho}(\mathfrak{T}(D) - \mathfrak{T}(D^*)) \le L\widetilde{\rho}(D - D^*),$$

which implies that $\widetilde{\rho}(D - D^*) = 0$ or $D = D^*$, since $\widetilde{\rho}(D - D^*) < \infty$. This proves the uniqueness of D. Also it follows from inequality (2.7) that

$$\widetilde{\rho}(D - d) \le \frac{1}{2(1-L)}.$$

This completes the proof. □

We now investigate the superstability of a generalized left derivation from a unital algebra into a modular space.

2.2. Theorem. *Let $d : \mathcal{A} \to \mathcal{X}_\rho$ be a mapping with $d(0) = 0$. If there exists a mapping $g : \mathcal{A} \to \mathcal{X}_\rho$ such that*

$$\rho\left(d(x + y + zw) - d(x) - d(y) - zd(w) - wg(z)\right) \leq \varphi(x, y, z, w) \tag{2.8}$$

for all $x, y, z, w \in \mathcal{A}$, where $\varphi : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \times \mathcal{A} \to [0, \infty)$ is a given mapping such that

$$\varphi(2x, 2x, 0, 0) \leq 2L\varphi(x, x, 0, 0)$$

and

$$\lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y, 0, 0)}{2^n} = \lim_{n\to\infty} \frac{\varphi(0, 0, 2^n z, w)}{2^n} = \lim_{n\to\infty} \frac{\varphi(0, 0, z, 2^n w)}{2^n} = 0 \tag{2.9}$$

for all $x, y \in \mathcal{A}$ and a constant $0 < L < 1$, then d is a generalized left derivation and g is a left derivation.

Proof. Letting $z = w = 0$ in (2.8), then d satisfies (2.12) and so the Theorem 2.1 shows that there exists a unique additive mapping $D : \mathcal{A} \to \mathcal{X}_\rho$ for which satisfies

$$\widetilde{\rho}(D - d) \leq \frac{1}{2(1 - L)},$$

where $\widetilde{\rho}$ is the convex modular defined in (2.4). Now, we prove that d is a generalized left derivation and g is a left derivation. Substituting $x = y = 0$ in (2.8), we get

$$\rho\left(d(zw) - zd(w) - wg(z)\right) \leq \varphi(0, 0, z, w), \tag{2.10}$$

for all $z, w \in \mathcal{A}$. Moreover, if we replace z and w with $2^n z$ and $2^n w$ in (2.10), respectively, and then divide both sides by 2^{2n}, we deduced that

$$\rho\left(\frac{d(2^{2n} zw)}{2^{2n}} - z\frac{d(2^n w)}{2^n} - w\frac{g(2^n z)}{2^n}\right) \leq \frac{\varphi(0, 0, 2^n z, 2^n w)}{2^{2n}},$$

for all $z, w \in \mathcal{A}$. Letting $n \to \infty$, we obtain

$$D(zw) - zD(w) = \lim_{n\to\infty} w\frac{g(2^n z)}{2^n},$$

for all $z, w \in \mathcal{A}$. Suppose that $w = e$, hence it follows

$$\lim_{n\to\infty} \frac{g(2^n z)}{2^n} = D(z) - zD(e),$$

for all $z \in \mathcal{A}$. If $\gamma(z) = D(z) - zD(e)$, then by the additivity of D, we get

$$\gamma(z+w) = D(z+w)-(z+w)D(e) = (D(z)-zD(e))+(D(w)-wD(e)) = \gamma(z)+\gamma(w),$$

for all $z, w \in \mathcal{A}$. Therefore γ is additive.

Suppose $\Delta(z, w) = d(zw) - zd(w) - wg(z)$, for all $z, w \in \mathcal{A}$. The inequality given in (2.10) implies that

$$\lim_{n\to\infty} \frac{\Delta(2^n z, w)}{2^n} = 0,$$

for all $z, w \in \mathcal{A}$. Thus we get

$$\begin{aligned} D(zw) = \widetilde{\rho} \lim_{n\to\infty} \frac{d(2^{2n} zw)}{2^n} &= \lim_{n\to\infty} \frac{2^n zd(w) + wg(2^n z) + \Delta(2^n z, w)}{2^n} \\ &= zd(w) + \lim_{n\to\infty} \frac{wg(2^n z)}{2^n} = zd(w) + w\gamma(z), \end{aligned}$$

for all $z, w \in \mathcal{A}$. Since γ is additive, we have

$$2^n zd(w) + 2^n w\gamma(z) = D(2^n z.w) = D(z.2^n w) = zd(2^n w) + 2^n w\gamma(z),$$

for all $z, w \in \mathcal{A}$. Therefore $zd(w) = z\frac{1}{2^n}d(2^n w)$, for all $z, w \in \mathcal{A}$. By letting $n \to \infty$, we obtain $zd(w) = zD(w)$. If $z = e$, we have $d = D$. Consequently we get

$$d(zw) = zd(w) + w\gamma(z), \tag{2.11}$$

for all $z, w \in \mathcal{A}$. Now, we verify that γ is a left derivation. Using the fact that d satisfies (2.11), we have

$$\begin{aligned}\gamma(xy) &= d(xy) - xyd(e) = xd(y) + y\gamma(x) - xyd(e)\\ &= x(d(y) - yd(e)) + y\gamma(x) = x\gamma(y) + y\gamma(x),\end{aligned}$$

for all $x, y \in \mathcal{A}$, which means that γ is a derivation and hence d is a generalized left derivation.

Finally, we show that g is a left derivation. If we replace w by $2^n w$ in (2.10) and then divide both sides by 2^{2n}, we obtain

$$\rho\left(\frac{d(2^n zw)}{2^n} - z\frac{d(2^n w)}{2^n} - 2^n w\frac{g(z)}{2^n}\right) \leq \frac{\varphi(0, 0, 2^n z, w)}{2^n},$$

for all $z, w \in \mathcal{A}$. Passing the limit as $n \to \infty$, we get

$$d(zw) - zd(w) - wg(z) = 0,$$

for all $z, w \in \mathcal{A}$. Therefore $d(zw) = zd(w) + wg(z)$, for all $z, w \in \mathcal{A}$, and hence if $w = e$, then $g(z) = d(z) - zd(e) = \gamma(z)$, for all $z \in \mathcal{A}$. Since γ is a left derivation, hence g is a left derivation and this completes the proof. ☐

The similar way as in the proof of Theorem 2.2, we get the following result for a generalized derivation.

2.3. Theorem. *Let $d : \mathcal{A} \to \mathcal{X}_\rho$ be a mapping with $d(0) = 0$. If there exists a mapping $g : \mathcal{A} \to \mathcal{X}_\rho$ such that*

$$\rho\left(d(x + y + zw) - d(x) - d(y) - zd(w) - g(z)w\right) \leq \varphi(x, y, z, w) \tag{2.12}$$

for all $x, y, z, w \in \mathcal{A}$, where $\varphi : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \times \mathcal{A} \to [0, \infty)$ is a given mapping such that

$$\varphi(2x, 2x, 0, 0) \leq 2L\varphi(x, x, 0, 0)$$

and

$$\lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y, 0, 0)}{2^n} = \lim_{n\to\infty} \frac{\varphi(0, 0, 2^n z, w)}{2^n} = \lim_{n\to\infty} \frac{\varphi(0, 0, z, 2^n w)}{2^n} = 0 \tag{2.13}$$

for all $x, y \in \mathcal{A}$ and a constant $0 < L < 1$, then d is a generalized derivation and g is a derivation.

With the help of Theorem 2.1, the following result can be derived for a linear generalized left derivation.

2.4. Theorem. *Let $\mathcal{A}$ be a unital algebra and let $\mathcal{X}$ be a unital $\mathcal{A}$-module and $\mathcal{X}_\rho$ a ρ–complete modular space. Suppose $d : \mathcal{A} \to \mathcal{X}_\rho$ satisfies the condition $d(0) = 0$ and an inequality of the form*

$$\rho\left(d(\alpha x + \beta y + zw) - \alpha d(x) - \beta d(y) - zd(w) - wg(z)\right) \leq \varphi(x, y, z, w) \tag{2.14}$$

for all $x, y, z, w \in \mathcal{A}$ and all $\alpha, \beta \in \mathbb{U} = \{z \in \mathbb{C} : |z| = 1\}$, where $\varphi : \mathcal{A}\times\mathcal{A}\times\mathcal{A}\times\mathcal{A} \to [0, \infty)$ is a given mapping such that

$$\varphi(2x, 2x, 0, 0) \leq 2L\varphi(x, x, 0, 0)$$

and

$$\lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y, 0, 0)}{2^n} = \lim_{n\to\infty} \frac{\varphi(0, 0, 2^n z, w)}{2^n} = \lim_{n\to\infty} \frac{\varphi(0, 0, z, 2^n w)}{2^n} = 0 \tag{2.15}$$

for all $x, y \in \mathcal{A}$ and a constant $0 < L < 1$. Then d is a linear generalized left derivation and g is a linear left derivation.

Proof. We consider $\alpha = \beta = 1 \in \mathbb{U}$ in (2.14) and then d satisfies the inequality (2.8). It follows from Theorem 2.3 that d is a generalized left derivation and g is a left derivation. It is enough to prove that d and g are linear. By the proof of Theorem 2.2 we know that

$$d(x) = \widetilde{\rho} - \lim_{n\to\infty} \mathcal{T}^n(d)(x) = \widetilde{\rho} - \lim_{n\to\infty} \frac{1}{2^n} d(2^n x). \tag{2.16}$$

Letting $w = 0$ in (2.14), we have

$$\rho\left(d(\alpha x + \beta y) - \alpha d(x) - \beta d(y)\right) \le \varphi(x, y, 0, 0), \tag{2.17}$$

for all $x, y \in \mathcal{A}$ and all $\alpha, \beta \in \mathbb{U}$. If we replace x and y with $2^n x$ and $2^n y$ in (2.16), respectively, and then divide both sides by 2^n, we see that

$$\rho\left(\frac{1}{2^n} d(\alpha 2^n x + \beta 2^n y) - \frac{1}{2^n}\alpha d(2^n x) - \frac{1}{2^n}\beta d(2^n y)\right) \le \frac{1}{2^n}\varphi(2^n x, 2^n y, 0, 0) \to 0, \tag{2.18}$$

for all $x, y \in \mathcal{A}$ and all $\alpha, \beta \in \mathbb{U}$, as $n \to \infty$. Hence, we get

$$d(\alpha x + \beta y) = \alpha d(x) + \beta d(y), \tag{2.19}$$

for all $x, y \in \mathcal{A}$ and all $\alpha, \beta \in \mathbb{U}$. Now the proof of [5, Theorem 2.3] implies that

$$d(\alpha x + \beta y) = \alpha d(x) + \beta d(y), \tag{2.20}$$

for all $x, y \in \mathcal{A}$ and all $\alpha, \beta \in \mathbb{C}$. □

Employing the similar way as in the proof of Theorem 2.3 and Theorem 2.4, we get the next corollary for a linear generalized derivation.

2.5. Corollary. *Let $\mathcal{A}$ be a unital algebra and let $\mathcal{X}$ be a unital $\mathcal{A}$-module and $\mathcal{X}_\rho$ a ρ–complete modular space. Suppose $d : \mathcal{A} \to \mathcal{X}_\rho$ satisfies the condition $d(0) = 0$ and an inequality of the form*

$$\rho\left(d(\alpha x + \beta y + zw) - \alpha d(x) - \beta d(y) - zd(w) - g(z)w\right) \le \varphi(x, y, z, w) \tag{2.21}$$

for all $x, y, z, w \in \mathcal{A}$ and all $\alpha, \beta \in \mathbb{U} = \{z \in \mathbb{C} : |z| = 1\}$, where $\varphi : \mathcal{A}\times\mathcal{A}\times\mathcal{A}\times\mathcal{A} \to [0, \infty)$ is a given mapping such that

$$\varphi(2x, 2x, 0, 0) \le 2L\varphi(x, x, 0, 0)$$

and

$$\lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y, 0, 0)}{2^n} = \lim_{n\to\infty} \frac{\varphi(0, 0, 2^n z, w)}{2^n} = \lim_{n\to\infty} \frac{\varphi(0, 0, z, 2^n w)}{2^n} = 0 \tag{2.22}$$

for all $x, y \in \mathcal{A}$ and a constant $0 < L < 1$. Then d is a linear generalized derivation and g is a linear derivation.

2.6. Remark. Let $\mathcal{A}$ be a normed algebra and let $\mathfrak{B}$ be a Banach algebra. It is known that every normed space is modular space with the modular $\rho(x) = \|x\|$ and $\kappa = 2$. A typical example of φ in the above results is $\varphi(x, y) = \varepsilon + \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)$, such that $\varepsilon, \theta \ge 0$ and $p \in [0, 1)$.

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