Is homotopy perturbation method the traditional Taylor series expansion

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Abstract

The present paper deals with the homotopy perturbation method. The question of whether the homotopy perturbation method is simply the conventional Taylor series expansion is examined. It is proven that under particular choices of the auxiliary parameters the homotopy perturbation method is indeed the Taylor series expansion of the sought solution of nonlinear equations.

1. Introduction

Most of the real-life phenomena is governed by nonlinear equations whose solutions are difficult to find. Therefore, friendly tools have been the focus of past two decade's research.

Recently, investigators have proposed plenty of techniques to find approximate solutions. One of the most recent popular technique is the homotopy perturbation method based on the concept of topology. This method is quite distinct from the classical perturbation technique and does not require a small parameter or a linear term in a differential equation. Essentially, a homotopy with an embedding parameter $p \in [0, 1]$ is constructed. The basic details of homotopy perturbation method for solving nonlinear differential equations were outlined in [1], see also [2, 3, 4]. A numerous nonlinear problems were recently treated by the method, see for instance [5]. The recent works highlight clearly the fact that there is a close relationship between the Adomian decomposition and Taylor series methods as well as the homotopy and Taylor series methods [6, 7].

The investigation of current paper focuses on the homotopy perturbation technique. The prime motivation is to examine the method mathematically and to prove that under certain constraints, by particular choice of auxiliary linear operator and initial approximation, the homotopy perturbation method simply collapses onto the classical Taylor series expansion.

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2. The Homotopy Perturbation Method

The essential idea of this method is to introduce a homotopy parameter, say p, which varies from 0 to 1. Consider the nonlinear initial value problem

$$(2.1) \qquad N(u) = 0, \quad B(u, \frac{du}{dn}) = 0,$$

where u is the function to be solved under the boundary constraints in B. He's homotopy perturbation technique [1, 10] defines a homotopy $u(r, p) : R \times [0, 1] \to R$ so that

$$(2.2) \qquad H(u, p) \quad = \quad (1 - p)[L(u) - L(u_0)] + pN(u),$$

where L is a suitable auxiliary linear operator, u_0 is an initial approximation of equation (2.1) satisfying exactly the boundary conditions, see also [2] for the rest. It is obvious from equation (2.2) that

$$(2.3) \qquad H(u, 0) \quad = \quad L(u) - L(u_0), \quad H(u, 1) \quad = \quad N(u).$$

As p moves from 0 to 1, $u(t, p)$ moves from $u_0(t)$ to $u(t)$. Our basic assumption is that the solution of equation (2.2) when equated to zero can be expressed as a power series in p

$$(2.4) \qquad u(t, p) \quad = \quad u_0(t) + pu_1(t) + p^2 u_2(t) + \cdots = \sum_{k=0}^{\infty} u_k(t)p^k.$$

The approximate solution of equation (2.1), therefore, can be readily obtained as

$$(2.5) \qquad u(t) = \lim_{p \to 1} u(t, p) = \sum_{k=0}^{\infty} u_k(t).$$

3. Homotopy perturbation and Taylor expansion

To answer the question raised in the title of the paper, let's take into account the first-order initial value problem version of (2.1)

$$(3.1) \qquad u'(t) = F(u), \quad u(0) = \alpha,$$

where α is a constant. A straightforward Taylor series representation for the solution $u(t)$ at point $t = 0$ can be given in the form

$$(3.2) \qquad u(t) \quad = \quad u(0) + u'(0)t + \frac{u''(0)}{2!}t^2 + \cdots = \sum_{k=0}^{\infty} a_k t^k,$$

where $a_n = \frac{u^{(n)}(0)}{n!}$ can be immediately found from differentiating (3.1) successively and substituting $t = 0$. A few of the coefficients follow

$$(3.3) \qquad a_1 = u'(0) = F(\alpha),$$

$$a_2 = \frac{u''(0)}{2!} = \frac{1}{2!}F_u(\alpha)a_1,$$

$$a_3 = \frac{u'''(0)}{3!} = \frac{1}{3!}[F_{uu}(\alpha)a_1^2 + 2F_u(\alpha)a_2],$$

$$a_4 = \frac{u^{(4)}(0)}{4!} = \frac{1}{4!}[F_{uuu}(\alpha)a_1^3 + 6F_{uu}(\alpha)a_1 a_2 + 6F_u(\alpha)a_3],$$

$$\vdots$$

Theorem. If the auxiliary linear operator L and the initial approximation $u_0(t)$ to the solution $u(t)$ of equation (3.1) is taken in the homotopy procedure (2.2) as

$$(3.4) \qquad L \;=\; \frac{\partial}{\partial t}, \quad u_0(t) = \alpha,$$

then the homotopy series solution (2.4) converges to the Taylor series expansion (3.2) whose coefficients are evaluated in the order given by (3.3).

Proof. Expanding the homotopy solution $u(t, p)$ from (2.2) into Taylor series according to the parameter p at $p = 0$, it reads

$$(3.5) \qquad u(t, p) \;=\; \sum_{k=0}^{\infty} u_k(t) p^k.$$

When (3.5) is substituted into the homotopy equations (2.2) or equivalently differentiating (2.2) successively with respect to p and replacing $p = 0$ at the end yields a system of linear ordinary differential equations for the coefficients $u_k(t)$ of (3.5)

$$(3.6) \qquad L(u_k - \chi_k u_{k-1}) \;=\; -u'_{k-1} + \frac{1}{(k-1)!}\Big[\frac{\partial^{k-1} F}{\partial p^{k-1}}\Big]\Big|_{p=0},$$
$$u_k(0, p) \;=\; \alpha,$$

where $\chi_k = 0$ for $k = 1$ and $\chi_k = 1$ for $k > 1$. Having solved the equations (3.6) iteratively, the followings result for $u_k(t)$

$$(3.7) \qquad u_1(t) \;=\; F(\alpha)t = a_1 t,$$
$$u_2(t) \;=\; \frac{1}{2!} F_u(\alpha) a_1 t^2 = a_2 t^2,$$
$$u_3(t) \;=\; \frac{1}{3!}[F_{uu}(\alpha) a_1^2 + 2F_u(\alpha) a_2] t^3 = a_3 t^3,$$
$$u_4(t) \;=\; \frac{1}{4!}[F_{uuu}(\alpha) a_1^3 + 6F_{uu}(\alpha) a_1 a_2 + 6F_u(\alpha) a_3] t^4 = a_4 t^4,$$
$$\vdots$$

which generates the homotopy series

$$(3.8) \qquad u(t, p) \;=\; \sum_{k=0}^{\infty} u_k(t) p^k = \sum_{k=0}^{\infty} a_k t^k p^k.$$

The convergence assumption of (3.8) at $p = 1$ yields the homotopy series solution (2.5) which turns out to be the Taylor series expansion (3.2-3.3) to the solution.

Remark 1. Since the homotopy series (3.8) at $p = 1$ is the traditional Taylor series, then the convergence issue of the homotopy series (3.5) is guaranteed for those values t, $|t| < R$ such that $R = \lim_{n \to \infty} |\frac{a_n}{a_{n+1}}|$.

Remark 2. If $u = u(t, r)$ with r denoting space variables and the initial-value problem consists of a partial differential equation of the form

$$(3.9) \qquad u_t \;=\; F(u, u_r),$$
$$u(t = 0, r) \;=\; f(r),$$

then by a similar argument to Theorem 1, the homotopy solution to (3.9) will be again the traditional Taylor series expansion at $t = 0$, provided that the auxiliary linear operator

L and the initial approximation $u_0(t, r)$ are selected as

$$L = \frac{\partial}{\partial t}, \quad u_0(t, r) = f(r).$$

Remark 3. If higher-order ordinary or partial differential initial-value problems (or systems) are considered, by a particular choice of linear differential operator and initial guess, it can be shown that the homotopy perturbation series solution and the Taylor series solution are the same.

4. Illustrative Examples

To justify the presented analysis, the following examples are given, as also stated in reference [2].

Example 1. The steady free convection flow over a vertical semi-infinite flat plate, see [12] and [13] is given by

$$(4.1) \qquad\qquad y' + y^2 \quad = \quad 1, \quad y(0) = 0,$$

To comply with the Theorem, $u_0(t) = 0$ and $L = \frac{d}{dt}$ are chosen so that the homotopy (2.2) becomes

$$(4.2) \qquad\qquad \frac{\partial u(t, p)}{\partial t} + p\, u(x, p)^2 - p \quad = \quad 0, \quad u(0, p) = 0.$$

A few approximate homotopy solutions via the homotopy perturbation (4.2) can be calculated as

$$u_1(t) = t, \ u_2(t) = 0, \ u_3(t) = -\frac{t^3}{3}, \ u_4(t) = 0, \ u_5(t) = \frac{2}{15} t^5,$$

which are the same as those generated from the classical Taylor series expansion of (4.1) at $t = 0$, the validity region is determined to be $-\frac{\pi}{2} < t < \frac{\pi}{2}$.

Example 2. The steady mixed convection flow [14] is given by

$$(4.3) \qquad\qquad 2y'' + y - y^2 \quad = \quad 0, \quad y(0) = 0, \quad y'(0) = \alpha = 1/\sqrt{6}.$$

The Taylor series expansion of (4.3) at point $t = 0$ yields

$$(4.4) \ y(t) = t\alpha - \frac{t^3 \alpha}{12} + \frac{t^4 \alpha^2}{24} + \frac{t^5 \alpha}{480} - \frac{t^6 \alpha^2}{288} + \frac{t^7 \alpha \left(-1 + 40\alpha^2\right)}{40320} + \frac{t^8 \alpha^2}{7680} + \cdots .$$

which totally corresponds to the homotopy perturbation series solution provided that we choose the auxiliary parameters as $u_0(t) = \alpha t$ and $L = 2\frac{\partial^2}{\partial t^2}$, see [2].

Example 3. The approximate theory of the flow through a shock wave traveling in a viscous fluid [15] is given by

$$(4.5) \qquad\qquad u_t + uu_x \quad = \quad u_{xx}, \quad u(x, 0) = 2x, \ (x, t) \in R \times [0, 1/2),$$

which receives an exact solution given by (see [2])

$$(4.6) \qquad\qquad u(x, t) \quad = \quad \frac{2x}{1 + 2t}.$$

Exact solution (4.6) is approximated by the auxiliary parameters $u_0(x, t) = 2x$ and $L = \frac{\partial}{\partial t}$. Then, the homotopy (2.2) turns out to be

$$(4.7) \quad u_t(x, t, p) + p(u(x, t, p)u_x(x, p, t) - u_{xx}(x, t, p) = 0, \ u(x, 0, p) = 2x.$$

Equation (4.7) produces the below homotopy series for the solution of (4.5)

$$(4.8) \quad u(x,t) \quad = \quad 2x - 4xt + 8xt^2 - 16xt^3 + 32xt^4 + ... + (-1)^n 2^{n+1} xt^n + \cdots,$$

which is the same as the classical Taylor series expansion of (4.6) around $t = 0$. The interval of convergence is easy to identify as $0 \le t < 1/2$.

Example 4. Consider now the well-known KdV-Burger's equation involving both dispersion and dissipation terms

$$(4.9) \qquad u_t + 2(u^3)_x - u_{xxx} + u_{xx} \quad = \quad 0, \quad u(x,0) = \frac{1}{6}\left(1 + \tanh\left[\frac{x}{6}\right]\right),$$

whose exact travelling-wave solution is given by

$$(4.10) \qquad\qquad u(x,t) \quad = \quad \frac{1}{6}\left(1 + \tanh\left[\frac{1}{6}\left(x - \frac{2}{9}t\right)\right]\right).$$

To approximate the exact solution (4.10), if we choose the auxiliary parameters $u_0(x,t) = \frac{1}{6}\left(1 + \tanh\left[\frac{x}{6}\right]\right)$ and $L = \frac{\partial}{\partial t}$, the homotopy (2.2) turns out to be

$$(4.11) \qquad u_t(x,t,p) + p(2(u(x,t,p)^3)_x - u_{xxx}(x,p,t) + u_{xx}(x,t,p) \quad = \quad 0,$$
$$u(x,0,p) = \frac{1}{6}\left(1 + \tanh\left[\frac{x}{6}\right]\right).$$

It is no hard to deduce that the Taylor series and homotopy perturbation series completely coincide again for this specific problem.

Example 5. The transverse vibrations of a uniform flexible beam [16] is given by

$$(4.12) \quad u_{tt} + \left(\frac{y+z}{2\cos x} - 1\right)u_{xxxx} + \left(\frac{z+x}{2\cos y} - 1\right)u_{yyyy} + \left(\frac{x+y}{2\cos z} - 1\right)u_{zzzz} \quad = \quad 0,$$
$$u(x,y,z,0) = -u_t(x,y,z,0) = x + y + z - (\cos x + \cos y + \cos z),$$

admitting an exact solution

$$(4.13) \qquad\qquad u(x,t) = (x + y + z - \cos x - \cos y - \cos z)e^{-t}, see[2].$$

This exact solution (4.13) is approximated by selecting the auxiliary parameters respectively, $u_0(x,t) = (x + y + z - \cos x - \cos y - \cos z)(1 - t)$ and $L = \frac{\partial^2}{\partial t^2}$. As a result, we obtain the homotopy series

$$(4.14) \quad u(x,t) \quad = \quad (x + y + z - \cos x - \cos y - \cos z)(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots)$$

which matches exactly onto the Taylor series expansion of (4.13) around $t = 0$

$$(4.15) \qquad\qquad u(x,t) \quad = \quad \sum_{n=0}^{\infty}(x + y + z - \cos x - \cos y - \cos z)\frac{t^n}{n!},$$

that is obviously convergent for all t.

Example 6. As a final example, we consider the linear partial differential equation

$$(4.16) \qquad\qquad u_t + u_x - 2u_{xxt} \quad = \quad 0, \quad u(x,0) = e^{-x},$$

having the exact solution

$$(4.17) \qquad\qquad u(x,t) \quad = \quad e^{-x-t}, see[2].$$

Choosing the auxiliary parameters $u_0(x,t) = e^{-x}$ and $L = \frac{\partial}{\partial t}$, then the homotopy (2.2) becomes

$$(4.18) \qquad u_t(x,t,p) + p(u_x(x,p,t) - u_{xxt}(x,t,p) = 0, \ u(x,0,p) = e^{-x}.$$

The homotopy series solution of (4.13) from (2.1) can be found as

$$(4.19) \quad u(x,t) \quad = \quad \frac{e^{-x}}{720}(720 + 45360t + 46440t^2 + 13320t^3 + 1470t^4 + 66t^5 + \cdots),$$

whose radius of convergence is zero, so that the homotopy series (4.19) is convergent only at the point $t = 0$. On the other hand, the classical Taylor series expansion applied to (4.16) predicts the exact result (4.17). It should be remarked that this example does not contradict at all with the Theorem, since (4.16) involves mixed partial derivatives. The weakness of the homotopy perturbation method on this example may be overcame by a better choice of auxiliary parameters.

It can be concluded as an answer to the title of the paper that for specific choices of auxiliary homotopy parameters, the homotopy perturbation technique produces exactly the same series as the traditional Taylor series. If this is the case, then there seems no a scientific merit to publish papers regarding the homotopy perturbation technique.

5. Concluding remarks

The homotopy perturbation method is mathematically analyzed in the present work. The theorem presented here proves that under certain special conditions the traditional homotopy perturbation method becomes the well-known Taylor series expansion. An example has also been given to demonstrate the advantage of the Taylor series expansion over the homotopy perturbation method. It can be concluded that a great deal of the papers published under the topic of homotopy perturbation technique is simply the traditional Taylor series expansion, whose contributions to science are questionable.

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