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# Existence and uniqueness of positive solutions for boundary value problems of a fractional differential equation with a parameter

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**Abstract**

In this paper, we are concerned with the existence and uniqueness of positive solutions for the following nonlinear fractional two-point boundary value problem

$$\begin{cases} D_{0+}^{\alpha}u(t) + \lambda f(t, u(t), u(t)) = 0, & 0 < t < 1,\ 2 < \alpha \leq 3, \\ u(0) = u'(0) = u'(1) = 0, \end{cases}$$

where $D_{0+}^{\alpha}$ is the standard Riemann-Liouville fractional derivative, and $\lambda$ is a positive parameter. Our analysis relies on a fixed point theorem and some properties of eigenvalue problems for a class of general mixed monotone operators. Our results can not only guarantee the existence of a unique positive solution, but also be applied to construct an iterative scheme for approximating it. An example is given to illustrate the main results.

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**Keywords:** Riemann-Liouville fractional derivative; Fractional differential equation; Positive solution; Existence and uniqueness; Mixed monotone operator.

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## 1. Introduction

Fractional differential equations arise in many fields, such as physics, mechanics, chemistry, economics, engineering and biological sciences,etc;see[1-15] for example. In recent years, the study of positive solutions for fractional differential equation boundary value problems has attracted considerable attention, and fruits from research into it emerge continuously. For a small sample of such work, we refer the reader to [16-28] and the references therein. In these papers, many authors have investigated the existence of positive solutions for nonlinear fractional differential equation boundary value problems. Their results are based on Schauder fixed point theorem, Leggett-Williams theorem, fixed point index theorems in cones, Krasnosel'skii fixed point theorem, the method of upper-lower solutions, fixed point theorems in cones and so on. On the other hand, the uniqueness of positive solutions for nonlinear fractional differential equation boundary value problems has been studied by some authors, see [20-22,24,27] for example. The methods used in these papers are fixed point theorems for mixed monotone operators, $u_0$-concave operators and monotone operators in partially ordered sets.

In [26], by means of Krasnosel'skii fixed point theorem, El-Shahed considered the existence and nonexistence of positive solutions for the nonlinear fractional boundary value problem

$$\begin{cases} D_{0+}^{\alpha}u(t)+\lambda a(t)f(u(t))=0, \quad 0<t<1,\ 2<\alpha<3, \\ u(0)=u'(0)=u'(1)=0, \end{cases}$$

where $D_{0+}^{\alpha}$ is the standard Riemann-Liouville fractional derivative, $a:(0,1)\to[0,+\infty)$ is continuous with $\int_0^1 a(t)dt>0, f\in C([0,+\infty),[0,+\infty))$ and $\lambda$ is a positive parameter.

In [28], by using the properties of the Green function, the method of upper-lower solutions and fixed point theorem, Zhao et al. studied the existence of multiple positive solutions for the nonlinear fractional differential equation boundary value problem

$$\begin{cases} D_{0+}^{\alpha}u(t)+f(t,u(t))=0, \quad 0<t<1,\ 2<\alpha\le 3, \\ u(0)=u'(0)=u'(1)=0. \end{cases}$$

The purpose of this paper is to establish the existence and uniqueness of positive solutions for the following nonlinear fractional two-point boundary value problem

$$\begin{cases} D_{0+}^{\alpha}u(t)+\lambda f(t,u(t),u(t))=0, \quad 0<t<1,\ 2<\alpha\le 3, \\ u(0)=u'(0)=u'(1)=0, \end{cases} \tag{1.1}$$

where $D_{0+}^{\alpha}$ is the standard Riemann-Liouville fractional derivative, $\lambda$ is a positive parameter and $f:[0,1]\times[0,+\infty)\times[0,+\infty)\to[0,+\infty)$ is continuous.

Different from the above works mentioned, we will use a fixed point theorem and some properties of eigenvalue problems for a class of general mixed monotone operators to show the existence and uniqueness of positive solutions for the problem (1.1). Moreover, we can construct two sequences for approximating the unique solution and we show that the positive solution with respect to $\lambda$ has some pleasant properties.

## 2. Preliminaries and previous results

For the convenience of the reader, we present here some definitions, lemmas and basic results that will be used in the proof of our theorem.

**Definition 2.1**([4, Definition 2.1]). The integral

$$I_{0+}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_0^x \frac{f(t)}{(x-t)^{1-\alpha}}dt, \quad x>0$$

is called the Riemann-Liouville fractional integral of order $\alpha$, where $\alpha > 0$ and $\Gamma(\alpha)$ denotes the gamma function.

**Definition 2.2**([4, page 36-37]). For a function $f(x)$ given in the interval $[0, \infty)$, the expression

$$D_{0+}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_0^x \frac{f(t)}{(x-t)^{\alpha-n+1}} dt$$

where $n = [\alpha] + 1, [\alpha]$ denotes the integer part of number $\alpha$, is called the Riemann-Liouville fractional derivative of order $\alpha$.

**Lemma 2.3.**([26]). Given $y \in C[0, 1]$ and $2 < \alpha \le 3$, the following boundary value problem

$$\begin{cases} D_{0+}^{\alpha} u(t) + y(t) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = u'(1) = 0, \end{cases} \tag{2.1}$$

has a unique solution

$$u(t) = \int_0^1 G(t, s) y(s) ds, \ t \in [0, 1],$$

where

$$G(t, s) = \frac{1}{\Gamma(\alpha)} \begin{cases} (1-s)^{\alpha-2} t^{\alpha-1}, & 0 \le t \le s \le 1, \\ (1-s)^{\alpha-2} t^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \le s \le t \le 1. \end{cases}$$

Here $G(t, s)$ is called the Green function of boundary value problem (2.1). Evidently, $G(t, s) \ge 0$ for $t, s \in [0, 1]$.
The following property of the Green function plays important roles in this paper.

**Lemma 2.4.** Let $2 < \alpha \le 3$. Then the Green function $G(t, s)$ in Lemma 2.3 has the following property:

$$\frac{1}{\Gamma(\alpha)} s(1-s)^{\alpha-2} t^{\alpha-1} \le G(t, s) \le \frac{1}{\Gamma(\alpha)} (1-s)^{\alpha-2} t^{\alpha-1} \text{ for } \ t, s \in [0, 1].$$

**Proof.** Evidently, the right inequality holds. So we only need to prove the left inequality. If $0 \le s \le t \le 1$, then we have $0 \le t - s \le t - ts = (1 - s)t$, and thus

$$(t-s)^{\alpha-1} \le (1-s)^{\alpha-1} t^{\alpha-1}.$$

Hence,

$$\begin{aligned} G(t, s) &= \frac{1}{\Gamma(\alpha)} [(1-s)^{\alpha-2} t^{\alpha-1} - (t-s)^{\alpha-1}] \\ &\ge \frac{1}{\Gamma(\alpha)} [(1-s)^{\alpha-2} t^{\alpha-1} - (1-s)^{\alpha-1} t^{\alpha-1}] \\ &= \frac{1}{\Gamma(\alpha)} s(1-s)^{\alpha-2} t^{\alpha-1}. \end{aligned}$$

If $0 \le t \le s \le 1$, then we have

$$G(t, s) = \frac{1}{\Gamma(\alpha)} (1-s)^{\alpha-2} t^{\alpha-1} \ge \frac{1}{\Gamma(\alpha)} s(1-s)^{\alpha-2} t^{\alpha-1}.$$

So the left inequality also holds.□

In the sequel, we present some basic concepts in ordered Banach spaces for completeness and fixed point theorems which we will be used later. For convenience of readers, we suggest that one refer to [29,30] for details.

Suppose that $(E, \|\cdot\|)$ is a real Banach space which is partially ordered by a cone $P \subset E$, *i.e.*, $x \leq y$ if and only if $y - x \in P$. If $x \leq y$ and $x \neq y$, then we denote $x < y$ or $y > x$. By $\theta$ we denote the zero element of $E$. Recall that a non-empty closed convex set $P \subset E$ is a cone if it satisfies (i) $x \in P, \lambda \geq 0 \Rightarrow \lambda x \in P$; (ii) $x \in P, -x \in P \Rightarrow x = \theta$.

$P$ is called normal if there exists a constant $M > 0$ such that, for all $x, y \in E$, $\theta \leq x \leq y$ implies $\|x\| \leq M\|y\|$; in this case $M$ is called the normality constant of $P$. If $x_1, x_2 \in E$, the set $[x_1, x_2] = \{x \in E | x_1 \leq x \leq x_2\}$ is called the order interval between $x_1$ and $x_2$.

For all $x, y \in E$, the notation $x \sim y$ means that there exist $\lambda > 0$ and $\mu > 0$ such that $\lambda x \leq y \leq \mu x$. Clearly, $\sim$ is an equivalence relation. Given $h > \theta$($i.e.$, $h \geq \theta$ and $h \neq \theta$), we denote by $P_h$ the set $P_h = \{x \in E | \ x \sim h\}$. It is easy to see that $P_h \subset P$ is convex and $\lambda P_h = P_h$ for all $\lambda > 0$.

**Definition 2.5**(see [29,30]). $A : P \times P \to P$ is said to be a mixed monotone operator if $A(x, y)$ is increasing in $x$ and decreasing in $y$, i.e., $u_i, v_i (i = 1, 2) \in P, u_1 \leq u_2, v_1 \geq v_2$ implies $A(u_1, v_1) \leq A(u_2, v_2)$. Element $x \in P$ is called a fixed point of $A$ if $A(x, x) = x$.

In a recent paper [30], Zhai and Zhang considered the following operator equations

$$A(x, x) = x \text{ and } A(x, x) = \lambda x,$$

where $A : P \times P \to P$ is a mixed monotone operator which satisfy the following conditions:

$(A_1)$ there exists $h \in P$ with $h \neq \theta$ such that $A(h, h) \in P_h$.
$(A_2)$ for any $u, v \in P$ and $t \in (0, 1)$, there exists $\varphi(t) \in (t, 1)$ such that $A(tu, t^{-1}v) \geq \varphi(t)A(u, v)$.
They established the existence and uniqueness of positive solutions for the above equations and they present the following interesting results.

**Theorem 2.6.** Suppose that $P$ is a normal cone of $E$, and $(A_1), (A_2)$ hold. Then operator $A$ has a unique fixed point $x^*$ in $P_h$. Moreover, for any initial $x_0, y_0 \in P_h$, constructing successively the sequences

$$x_n = A(x_{n-1}, y_{n-1}), y_n = A(y_{n-1}, x_{n-1}), n = 1, 2, \ldots,$$

we have $\|x_n - x^*\| \to 0$ and $\|y_n - x^*\| \to 0$ as $n \to \infty$.

**Theorem 2.7.** Suppose that $P$ is a normal cone of $E$, and $(A_1), (A_2)$ hold. Let $x_\lambda (\lambda > 0)$ denote the unique solution of nonlinear eigenvalue equation $A(x, x) = \lambda x$ in $P_h$. Then we have the following conclusions:

$(R_1)$ If $\varphi(t) > t^{\frac{1}{2}}$ for $t \in (0, 1)$, then $x_\lambda$ is strictly decreasing in $\lambda$, that is, $0 < \lambda_1 < \lambda_2$ implies $x_{\lambda_1} > x_{\lambda_2}$;
$(R_2)$ If there exists $\beta \in (0, 1)$ such that $\varphi(t) \geq t^\beta$ for $t \in (0, 1)$, then $x_\lambda$ is continuous in $\lambda$, that is, $\lambda \to \lambda_0 (\lambda_0 > 0)$ implies $\|x_\lambda - x_{\lambda_0}\| \to 0$;
$(R_3)$ If there exists $\beta \in (0, \frac{1}{2})$ such that $\varphi(t) \geq t^\beta$ for $t \in (0, 1)$, then $\lim\limits_{\lambda \to \infty} \|x_\lambda\| = 0$, $\lim\limits_{\lambda \to 0^+} \|x_\lambda\| = \infty$.

## 3. Existence and uniqueness of positive solutions for the problem (1.1)

In this section, we apply Theorem 2.6 and Theorem 2.7 to study the problem (1.1), and we obtain a new result on the existence and uniqueness of positive solutions. Moreover, we show that the positive solution with respect to $\lambda$ has some pleasant properties. The method used here is new to the literature and so is the existence and uniqueness result to the fractional differential equations.

In our considerations we will work in the Banach space $C[0,1] = \{x : [0,1] \to \mathbf{R} \text{ is continuous}\}$ with the standard norm $\|x\| = \sup\{|x(t)| : t \in [0,1]\}$. Notice that this space can be equipped with a partial order given by

$$x, y \in C[0,1], x \leq y \Leftrightarrow x(t) \leq y(t) \text{ for } t \in [0,1].$$

Set $P = \{x \in C[0,1] | x(t) \geq 0,\ t \in [0,1]\}$, the standard cone. It is clear that $P$ is a normal cone in $C[0,1]$ and the normality constant is 1. Our main result is summarized in the following theorem.

**Theorem 3.1.** Assume that

$(H_1)$ $f(t,x,y)$ is nondecreasing in $x$ for each $t \in [0,1]$ and $y \in [0,+\infty)$, nonincreasing in $y$ for each $t \in [0,1]$ and $x \in [0,+\infty)$ with $f(t,0,1) \not\equiv 0$;
$(H_2)$ for any $\gamma \in (0,1)$, there exist constants $\varphi_1(\gamma), \varphi_2(\gamma) \in (0,1)$ with $\varphi_1(\gamma)\varphi_2(\gamma) > \gamma$ such that

$$f(t,\gamma x,y) \geq \varphi_1(\gamma) f(t,x,y), f(t,x,\gamma y) \leq \frac{1}{\varphi_2(\gamma)} f(t,x,y) \text{ for any } \ x,y \in [0,+\infty).$$

Then: (1) the problem (1.1) has a unique positive solution $u_\lambda^*$ in $P_h$, where $h(t) = t^{\alpha-1}, t \in [0,1]$. Moreover, for any initial values $u_0, v_0 \in P_h$, constructing successively the sequences

$$u_{n+1}(t) = \lambda \int_0^1 G(t,s) f(s,u_n(s),v_n(s))ds,\ v_{n+1}(t) = \lambda \int_0^1 G(t,s) f(s,v_n(s),u_n(s))ds,\ n = 0,1,2,\ldots,$$

we have $u_n(t) \to u_\lambda^*(t), v_n(t) \to u_\lambda^*(t)$ as $n \to \infty$, where $G(t,s)$ is given as in Lemma 2.3;
(2) if $\varphi_1(t)\varphi_2(t) > t^{\frac{1}{2}}$ for $t \in (0,1)$, then $u_\lambda^*$ is strictly increasing in $\lambda$, that is, $0 < \lambda_1 < \lambda_2$ implies $u_{\lambda_1}^* \leq u_{\lambda_2}^*, u_{\lambda_1}^* \neq u_{\lambda_2}^*$. If there exists $\beta \in (0,1)$ such that $\varphi_1(t)\varphi_2(t) \geq t^\beta$ for $t \in (0,1)$, then $u_\lambda^*$ is continuous in $\lambda$, that is, $\lambda \to \lambda_0 (\lambda_0 > 0)$ implies $\|u_\lambda^* - u_{\lambda_0}^*\| \to 0$. If there exists $\beta \in (0,\frac{1}{2})$ such that $\varphi_1(t)\varphi_2(t) \geq t^\beta$ for $t \in (0,1)$, then $\lim_{\lambda \to 0^+} \|u_\lambda^*\| = 0$, $\lim_{\lambda \to \infty} \|u_\lambda^*\| = \infty$.

**Proof.** To begin with, from [26] the problem (1.1) has an integral formulation given by

$$u(t) = \lambda \int_0^1 G(t,s) f(s,u(s),u(s))ds,$$

where $G(t,s)$ is given as in Lemma 2.3. For any $u, v \in P$, we define

$$A(u,v)(t) = \int_0^1 G(t,s) f(s,u(s),v(s))ds.$$

Noting that $f(t,x,y) \geq 0$ and $G(t,s) \geq 0$, it is easy to check that $A : P \times P \to P$. In the sequel we check that $A$ satisfies all assumptions of Theorem 2.6.

Firstly, we prove that $A$ is a mixed monotone operator. In fact, for $u_i, v_i \in P, i = 1,2$ with $u_1 \geq u_2, v_1 \leq v_2$, we know that $u_1(t) \geq u_2(t), v_1(t) \leq v_2(t), t \in [0,1]$ and by $(H_1)$

and Lemma 2.3,

$$A(u_1,v_1)(t)=\int_0^1 G(t,s)f(s,u_1(s),v_1(s))ds \geq \int_0^1 G(t,s)f(s,u_2(s),v_2(s))ds = A(u_2,v_2)(t).$$

That is, $A(u_1,v_1)\geq A(u_2,v_2)$.

Next we show that $A$ satisfies the condition $(A_2)$. From $(H_2)$, for $\gamma\in(0,1)$ we can get $f(t,x,\gamma^{-1}y)\geq \varphi_2(\gamma)f(t,x,y)$ for any $x,y\in[0,+\infty)$. Then for any $\gamma\in(0,1)$ and $u,v\in P$, we obtain

$$\begin{aligned}
A(\gamma u,\gamma^{-1}v)(t) &= \int_0^1 G(t,s)f(s,\gamma u(s),\gamma^{-1}v(s))ds\\
&\geq \int_0^1 G(t,s)\varphi_1(\gamma)f(s,u(s),\gamma^{-1}v(s))ds\\
&\geq \int_0^1 G(t,s)\varphi_1(\gamma)\varphi_2(\gamma)f(s,u(s),v(s))ds\\
&= \varphi_1(\gamma)\varphi_2(\gamma)A(u,v)(t),\ t\in[0,1].
\end{aligned}$$

Let $\varphi(t)=\varphi_1(t)\varphi_2(t), t\in(0,1)$. Then $\varphi(t)\in(t,1)$ for $t\in(0,1)$. Hence, $A(\gamma u,\gamma^{-1}v)\geq \varphi(\gamma)A(u,v),\ \forall\ u,v\in P,\ \gamma\in(0,1)$. So the condition $(A_2)$ in Theorem 2.6 is satisfied. Now we show that $A(h,h)\in P_h$. On one hand, it follows from $(H_1),(H_2)$ and Lemma 2.4 that

$$\begin{aligned}
A(h,h)(t) &= \int_0^1 G(t,s)f(s,h(s),h(s))ds\\
&= \int_0^1 G(t,s)f(s,s^{\alpha-1},s^{\alpha-1})ds\\
&\geq \int_0^1 \frac{1}{\Gamma(\alpha)}s(1-s)^{\alpha-2}t^{\alpha-1}f(s,0,1)ds\\
&= \frac{1}{\Gamma(\alpha)}h(t)\int_0^1 s(1-s)^{\alpha-2}f(s,0,1)ds,\ t\in[0,1].
\end{aligned}$$

On the other hand, also from $(H_1),(H_2)$ and Lemma 2.4, we obtain

$$\begin{aligned}
A(h,h)(t) &= \int_0^1 G(t,s)f(s,s^{\alpha-1},s^{\alpha-1})ds\\
&\leq \int_0^1 \frac{1}{\Gamma(\alpha)}(1-s)^{\alpha-2}t^{\alpha-1}f(s,1,0)ds\\
&= \frac{1}{\Gamma(\alpha)}h(t)\int_0^1 f(s,1,0)ds,\ t\in[0,1].
\end{aligned}$$

Let

$$r_1=\int_0^1 s(1-s)^{\alpha-2}f(s,0,1)ds,\ r_2=\int_0^1 f(s,1,0)ds.$$

Since $f$ is continuous and $f(t,0,1)\not\equiv 0$, we can get

$$0<r_1=\int_0^1 s(1-s)^{\alpha-2}f(s,0,1)ds\leq \int_0^1 f(s,1,0)ds=r_2.$$

Consequently,

$$A(h,h)(t)\geq \frac{r_1}{\Gamma(\alpha)}\cdot h(t),\ A(h,h)(t)\leq \frac{r_2}{\Gamma(\alpha)}\cdot h(t),\ t\in[0,1].$$

So we have

$$\frac{r_1}{\Gamma(\alpha)}\cdot h\leq A(h,h)\leq \frac{r_2}{\Gamma(\alpha)}\cdot h.$$

Hence $A(h,h) \in P_h$, the condition $(A_1)$ in Theorem 2.6 is satisfied. Therefore, by Theorem 2.7, there exists a unique $u_\lambda^* \in P_h$ such that $A(u_\lambda^*, u_\lambda^*) = \frac{1}{\lambda} u_\lambda^*$. That is, $u_\lambda^* = \lambda A(u_\lambda^*, u_\lambda^*)$. It is easy to check that $u_\lambda^*$ is a unique positive solution of the problem (1.1) for given $\lambda > 0$. Further, if $\varphi(t) > t^{\frac{1}{2}}$ for $t \in (0,1)$, then Theorem 2.7 $(R_1)$ means that $u_\lambda^*$ is strictly increasing in $\lambda$, that is, $0 < \lambda_1 < \lambda_2$ implies $u_{\lambda_1}^* \leq u_{\lambda_2}^*, u_{\lambda_1}^* \neq u_{\lambda_2}^*$. If there exists $\beta \in (0,1)$ such that $\varphi(t) \geq t^\beta$ for $t \in (0,1)$, then Theorem 2.7 $(R_2)$ means that $u_\lambda^*$ is continuous in $\lambda$, that is, $\lambda \to \lambda_0 (\lambda_0 > 0)$ implies $\|u_\lambda^* - u_{\lambda_0}^*\| \to 0$. If there exists $\beta \in (0, \frac{1}{2})$ such that $\varphi(t) \geq t^\beta$ for $t \in (0,1)$, then Theorem 2.7 $(R_3)$ means $\lim\limits_{\lambda \to 0^+} \|u_\lambda^*\| = 0, \ \lim\limits_{\lambda \to \infty} \|u_\lambda^*\| = \infty$.

Let $A_\lambda = \lambda A$, then $A_\lambda$ also satisfies all the conditions of Theorem 2.6. By Theorem 2.6, for any initial values $u_0, v_0 \in P_h$, constructing successively the sequences $u_{n+1} = A_\lambda(u_n, v_n), \ v_{n+1} = A_\lambda(v_n, u_n), n = 0, 1, 2, \ldots$, we have $u_n \to u_\lambda^*, v_n \to u_\lambda^*$ as $n \to \infty$. That is,

$$u_{n+1}(t) = \lambda \int_0^1 G(t,s) f(s, u_n(s), v_n(s)) ds \to u_\lambda^*(t),$$

$$v_{n+1}(t) = \lambda \int_0^1 G(t,s) f(s, v_n(s), u_n(s)) ds, \ \to u_\lambda^*(t)$$

as $n \to \infty$. □

**Remark 3.1.** Let $f(t,x,y) \equiv C > 0$. Then the conditions $(H_1), (H_2)$ are satisfied and the problem (1.1) has a unique solution $u_\lambda(t) = \lambda C \int_0^1 G(t,s) ds, \ t \in [0,1]$. From Lemma 2.4, the unique solution $u_\lambda$ is a positive solution and satisfies $u_\lambda \in P_h = P_{t^{\alpha-1}}$.

**Example 3.1.** Consider the following problem:

$$\begin{cases} D_{0+}^{\frac{5}{2}} u(t) + \lambda a(t)[u^{\frac{1}{5}}(t) + (u(t)+3)^{-\frac{1}{4}}] = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = u'(1) = 0, \end{cases} \tag{3.1}$$

where $a : [0,1] \to [0, +\infty)$ is continuous with $a \not\equiv 0$.

In this example, we have $\alpha = \frac{5}{2}$. Let $f(t,x,y) = a(t)[x^{\frac{1}{5}} + (y+3)^{-\frac{1}{4}}]$. Evidently, $f(t,x,y)$ is increasing in $x$ for $t \in [0,1], y \geq 0$, decreasing in $y$ for $t \in [0,1], x \geq 0$. Moreover, $f(t,0,1) = a(t) 4^{-\frac{1}{4}} \not\equiv 0$. Set $\varphi_1(\gamma) = \gamma^{\frac{1}{5}}, \varphi_2(\gamma) = \gamma^{\frac{1}{4}}, \gamma \in (0,1)$. Then $\varphi_1(\gamma)\varphi_2(\gamma) = \gamma^{\frac{9}{20}} > \gamma$ and

$$f(t,\gamma x,y) = a(t)[\gamma^{\frac{1}{5}} x^{\frac{1}{5}} + (y+3)^{-\frac{1}{4}}] \geq \varphi_1(\gamma) f(t,x,y), f(t,x,\gamma y) = a(t)[x^{\frac{1}{5}} + \frac{1}{\gamma^{\frac{1}{4}}}(y+3)^{-\frac{1}{4}}] \leq \frac{1}{\varphi_2(\gamma)} f(t,x,y),$$

for $t \in [0,1], x, y \geq 0$. Hence, all the conditions of Theorem 3.1 are satisfied. An application of Theorem 3.1 implies that the problem (3.1) has a unique positive solution $u_\lambda^*$ in $P_h = P_{t^{\alpha-1}}$, and for any initial values $u_0, v_0 \in P_{t^{\alpha-1}}$, constructing successively the sequences

$$u_{n+1}(t) = \lambda \int_0^1 G(t,s) a(s)[u_n^{\frac{1}{5}}(s) + (v_n(s)+3)^{-\frac{1}{4}}] ds, \ v_{n+1}(t) = \lambda \int_0^1 G(t,s) a(s)[v_n^{\frac{1}{5}}(s) + (u_n(s)+3)^{-\frac{1}{4}}] ds,$$

$n = 0, 1, 2, \ldots$, we have $u_n(t) \to u_\lambda^*(t), v_n(t) \to u_\lambda^*(t)$ as $n \to \infty$, where $G(t,s)$ is given as in Lemma 2.3. Moreover, note that $\varphi_1(t)\varphi_2(t) > t^{\frac{1}{2}}$ for $t \in (0,1)$, then from Theorem 3.1, $u_\lambda^*$ is strictly increasing in $\lambda$, that is, $0 < \lambda_1 < \lambda_2$ implies $u_{\lambda_1}^* \leq u_{\lambda_2}^*, u_{\lambda_1}^* \neq u_{\lambda_2}^*$. Take $\beta \in [\frac{9}{20}, \frac{1}{2})$ and applying Theorem 3.1, we know that $u_\lambda^*$ is continuous in $\lambda$ and $\lim\limits_{\lambda \to 0^+} \|u_\lambda^*\| = 0, \ \lim\limits_{\lambda \to \infty} \|u_\lambda^*\| = \infty$.

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