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Morita theory for group corings

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Abstract

Using the theory of group corings, we study (graded) Morita contexts associated to a comodule over a group coring, which generalize and unify some classical morita contexts. Some applications of our theory are also discussed.

Keywords: group coring, Morita context, cofree.

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1. Introduction

An A-coring is a coalgebra in the monoidal category of A-bimodules over an arbitrary ring A. The concept was introduced by M. Sweedler [14]. In 2000, Takeuchi pointed out that to each entwining structure (A, C, ψ) over a commutative ring k, which was introduced by T. Brzezinski and S. Majid [2], there corresponds an A-coring structure on $\mathcal{C} := A \otimes_k C$. This motivated the revival of the theory of corings and comodules and Brzezinski's paper [3] was the engine behind the revival of the theory of corings and comodules over corings. Many examples of classical categories in noncommutative algebra are special cases of comodules over corings. Let us mention a few of them: the category of a descent datum of a ring extension, graded modules, Hopf modules, Long dimodules, Yetter-Drinfeld modules, Doi-Koppinen modules or entwined modules, and several other categories studied earlier by Hopf algebraists.

One of the important observations is that coring theory provides an elegant approach to descent theory and Galois theory. A systematic study of coring has been carried out in [1, 3, 6, 7, 15]. As the generalization of coring, Caenepeel, Janssen and Wang [5] introduced the group coring and developed Galois theory for group corings.

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It is well-known that the Morita context plays a important role in the theory of Hopf algebras. The first Morita context was constructed by Chase and Sweedler [9], which was generalized by Doi [12]. Morita contexts similar to the one of Doi were studied by Cohen, Fischman and Montgomery in [11]. As the generalization of both contexts, Caenepeel, Vercruysse and Wang associate different types of Morita contexts to a coring with a fixed grouplike element, which was generalized by Caenepeel, Janssen and Wang to group coring with a grouplike family [5]. Without the assumption of a coring with a fixed grouplike element, Caenepeel, De Groot and Vercruysse associated a Morita context to a comodule over a coring in [8]. Morita theory for group corings with fixed grouplike family is a remarkable tool to discuss Hopf-Galois extensions. In order to further discuss coalgebra-Galois extensions, we need to generalize the Morita context for group corings. Naturally, it occurs to us to how to develop (graded) Morita context associated to a comodule over a group coring. This is the motivation of this paper.

The paper is organized as follows.

In Section 2, we recall some basic definitions such as group corings, comodules over a group coring and graded Morita contexts. In Section 3, we associate a Morita context to a comodule over a group coring. In Section 4, we will discuss the graded Morita contexts and their relationship. Some applications of our theory are discussed in Section 5.

2. Preliminaries

Throughout this paper, let G be a group with unit e, and A a ring with unit 1_A, and M an A-module. We will often need collections of A-modules isomorphic to M and indexed by G. We will consider these modules as isomorphic, but distinct. Let $M \times \{\alpha\}$ be the module with index α. We then have isomorphisms

$$\mu_\alpha : M \to M \times \{\alpha\},\ \mu_\alpha(m) = (m, \alpha).$$

We can then write $M \times \{\alpha\} = \mu_\alpha(M)$. μ can be considered as a dummy variable, and we will also use the symbols $\nu, \kappa, \cdots$. We will identify M and $M \times \{e\}$ using μ_e.

2.1. Group Corings. Let A be an algebra. Recall from [5] that a G-group A-coring (or shortly a G-A-coring) $\underline{C}$ is a family $\{C_\alpha\}_{\alpha\in G}$ of A-bimodules together with a family of A-bimodule maps

$$\Delta_{\alpha,\beta} : C_{\alpha\beta} \to C_\alpha \otimes_A C_\beta,\ \varepsilon : C_e \to A$$

such that the following conditions hold:

$$(\Delta_{\alpha,\beta} \otimes_A id) \circ \Delta_{\alpha\beta,\gamma} = (id \otimes_A \Delta_{\beta,\gamma}) \circ \Delta_{\alpha,\beta\gamma},$$

$$(id \otimes_A \varepsilon) \circ \Delta_{\alpha,e} = id = (\varepsilon \otimes_A id) \circ \Delta_{e,\alpha}$$

for all $\alpha, \beta, \gamma \in G$.

For a G-A-coring $\underline{C}$, we also use the following Sweedler-type notation for the comultiplication maps $\Delta_{\alpha,\beta}$:

$$\Delta_{\alpha,\beta}(c) = c_{(1,\alpha)} \otimes_A c_{(2,\beta)}$$

for all $c \in C_{\alpha\beta}$.

A morphism between two G-A-corings $\underline{C}$ and $\underline{D}$ consists of a family of A-bimodule maps $f = \{f_\alpha : C_\alpha \to D_\alpha\}_{\alpha\in G}$ such that

$$(f_\alpha \otimes_A f_\beta) \circ \Delta_{\alpha,\beta} = \Delta_{\alpha,\beta} \circ f_{\alpha\beta}, \quad \varepsilon \circ f_e = \varepsilon.$$

Over a G-A-coring $\underline{C}$, we can define two different types of comodules. A right $\underline{C}$-comodule is a right A-module M together with a family of right A-linear maps $\rho^M = \{\rho^M_\alpha : M \to M \otimes_A C_\alpha\}_{\alpha\in G}$ such that

$$(id \otimes_A \Delta_{\alpha,\beta}) \circ \rho^M_{\alpha\beta} = (\rho^M_\alpha \otimes_A id) \circ \rho_\beta,\ (id \otimes_A \varepsilon) \circ \rho^M_e = id.$$

We use the following Sweedler-type notation:

$$\rho_\alpha^M(m) = m_{[0,\alpha]} \otimes_A m_{[1,\alpha]}$$

for all $m \in M_\alpha$.

A morphism of right $\underline{C}$-comodules is a right A-linear map $f : M \to N$ satisfying the condition

$$(f \otimes_A id) \circ \rho_\alpha^M = \rho_\alpha^N \circ f$$

for all $\alpha \in G$. Let $\mathcal{M}^{\underline{C}}$ denote the category of right $\underline{C}$-comodules.

Similarly, we can define the left $\underline{C}$-comodule and the category ${}^{\underline{C}}\mathcal{M}$ of all left $\underline{C}$-comodules. We use the following Sweedler-type notation for the left $\underline{C}$-comodule structure maps ${}^M\rho_\alpha$:

$${}^M\rho_\alpha(m) = m_{[-1,\alpha]} \otimes_A m_{[0,\alpha]}$$

for all $m \in M_\alpha$.

A right G-$\underline{C}$-comodule $\underline{M}$ is a family of right A-modules $\{M_\alpha\}_{\alpha\in G}$(meaning that each M_α is right A-module), together with a family of right A-linear maps $\rho = \{\rho_{\alpha,\beta}\}_{\alpha,\beta\in G}$, where $\rho_{\alpha,\beta} : M_{\alpha\beta} \to M_\alpha \otimes_A C_\beta$, such that the following conditions hold:

$$(id \otimes_A \Delta_{\beta,\gamma}) \circ \rho_{\alpha,\beta\gamma} = (\rho_{\alpha,\beta} \otimes_A id) \circ \rho_{\alpha\beta,\gamma},\ (id \otimes_A \varepsilon) \circ \rho_{\alpha,e} = id$$

for all $\alpha, \beta, \gamma \in G$.

We use the following standard notation:

$$\rho_{\alpha,\beta}(m) = m_{[0,\alpha]} \otimes_A m_{[1,\beta]}$$

for $m \in M_{\alpha\beta}$.

A morphism between two right G-$\underline{C}$-comodules $\underline{M} = \{M_\alpha\}_{\alpha\in G}$ and $\underline{N} = \{N_\alpha\}_{\alpha\in G}$ is a family of right A-linear maps $f = \{f_\alpha : M_\alpha \to N_\alpha\}_{\alpha\in G}$ such that

$$(f_\alpha \otimes_A id) \circ \rho_{\alpha,\beta} = \rho_{\alpha,\beta} \circ f_{\alpha\beta}.$$

The category of right G-$\underline{C}$-comodules will be denoted by $\mathcal{M}^{G,\underline{C}}$.

Let $\underline{C}$ be a G-A-coring. A family $g = (g_\alpha)_{\alpha\in G} \in \prod_{\alpha\in G} C_\alpha$ is called grouplike, if $\Delta_{\alpha,\beta}(g_{\alpha\beta}) = g_\alpha \otimes_A g_\beta$ and $\varepsilon(g_e) = 1$ for all $\alpha, \beta \in G$.

Let $\underline{C}$ be a G-A-coring with a fixed grouplike family $g = (g_\alpha)_{\alpha\in G}$. Then A can be endowed with a structure of right $\underline{C}$-comodule via the coaction maps

$$\rho_\alpha : A \to A \otimes_A C_\alpha,\ \rho_\alpha(a) = 1_A \otimes_A g_\alpha \cdot a.$$

For $M \in \mathcal{M}^{\underline{C}}$, we define

$$M^{co\underline{C}} = \{m \in M | \rho_\alpha(m) = m \otimes_A g_\alpha, \forall \alpha \in G\}.$$

In particular,

$$A^{co\underline{C}} = \{a \in A | a \cdot g_\alpha = g_\alpha \cdot a, \forall \alpha \in G\}.$$

Let $A \otimes_B A$ be the canonical Sweedler coring associated to the ring morphism $B \to A$ with its comultiplication and counit given by the formulas

$$\Delta(a \otimes_B b) = (a \otimes_B 1_A) \otimes_A (1_A \otimes_B b),\ \varepsilon(a \otimes_B b) = ab.$$

2.2. Graded Rings and Modules. Let A be a ring and $\mathcal{R} = \bigoplus_{\alpha\in G} \mathcal{R}_\alpha$ a G-graded ring. Suppose that we have a ring morphism $i : A \to \mathcal{R}_e$. Then we call $\mathcal{R}$ a G-graded A-ring. Every $\mathcal{R}_\alpha$ is then an A-bimodule and the decomposition of $\mathcal{R}$ is a decomposition of A-bimodules. The category of G-graded right $\mathcal{R}$-modules will be denoted by $\mathcal{M}_{\mathcal{R}}^G$.

Let $\underline{C}$ be a G-A-coring. For every $\alpha \in G$, $\mathcal{R}_\alpha = {}_A\mathrm{HOM}(C_{\alpha^{-1}}, A)$ is an A-bimodule via

$$(a \cdot f_\alpha \cdot b)(c) = f_\alpha(c \cdot a)b$$

for all $f_\alpha \in \mathcal{R}_\alpha$, $a, b \in A$ and $c \in C_{\alpha^{-1}}$. Take $f_\alpha \in \mathcal{R}_\alpha, g_\beta \in \mathcal{R}_\beta$ and define $f_\alpha \star g_\beta \in \mathcal{R}_{\alpha\beta}$ by the following formula:

$$(f_\alpha \star g_\beta)(c) = g_\beta(c_{(1,\beta^{-1})} \cdot f_\alpha(c_{(2,\alpha^{-1})}))$$

for all $c \in C_{(\alpha\beta)^{-1}}$. This defines maps $m_{\alpha,\beta} : \mathcal{R}_\alpha \otimes_A \mathcal{R}_\beta \to \mathcal{R}_{\alpha\beta}$, which make $\mathcal{R} = \oplus_{\alpha\in G}\mathcal{R}_\alpha$ into a G-graded ring with the unit ε. Define $i : A \to \mathcal{R}_e, i(a)(c) = \varepsilon(c)a$ is a ring homomorphism, which make $\mathcal{R} = \bigoplus_{\alpha\in G}\mathcal{R}_\alpha$ be a G-graded A-ring, called the (left) dual (graded) ring of the group coring $\underline{C}$. We will also write $\mathcal{R} = {}^*\underline{C}$.

2.3. Graded Morita Contexts. Let $\mathcal{R}$ be a G-graded ring, and $M, N \in \mathcal{M}^G_{\mathcal{R}}$. A right $\mathcal{R}$-linear map $f : M \to N$ is called homogeneous of degree σ, if $f(M_\alpha) \subset M_{\sigma\alpha}$ for all $\alpha \in G$. $\mathrm{HOM}_{\mathcal{R}}(M, N)_\sigma$ denotes the additive group of all right $\mathcal{R}$-module maps of degree σ.

Let S and $\mathcal{R}$ be G-graded rings. A G-graded Morita context connecting S and $\mathcal{R}$ is a Morita context $(S, \mathcal{R}, P, Q, \varphi, \psi)$ with the following additional structure: P and Q are graded bimodules, and the maps

$$\varphi : P \otimes_{\mathcal{R}} Q \to S,\ \psi : Q \otimes_S P \to \mathcal{R}$$

are homogeneous of degree e.

Given two graded Morita contexts $(S, \mathcal{R}, P, Q, \varphi, \psi)$ and $(\tilde{S}.\tilde{\mathcal{R}}, \tilde{P}, \tilde{Q}, \tilde{\varphi}, \tilde{\psi})$, if there exist two graded ring morphism $\Phi : S \to \tilde{S}$, $\Psi : \mathcal{R} \to \tilde{\mathcal{R}}$ and two graded bimodule morphism $\Theta : Q \to \tilde{Q}$, $\Xi : P \to \tilde{P}$ such that the following two diagrams

$$\begin{tikzcd}
P \otimes_{\mathcal{R}} Q \arrow[r, "\varphi"] \arrow[d, "\Xi\otimes\Theta"'] & S \arrow[d, "\Phi"] & Q \otimes_S P \arrow[r, "\psi"] \arrow[d, "\Theta\otimes\Xi"'] & \mathcal{R} \arrow[d, "\Psi"] \\
\tilde{P} \otimes_{\tilde{\mathcal{R}}} \tilde{Q} \arrow[r, "\tilde{\varphi}"] & \tilde{S} & \tilde{Q} \otimes_{\tilde{S}} \tilde{P} \arrow[r, "\tilde{\psi}"] & \tilde{\mathcal{R}}
\end{tikzcd}$$

are communicative, then we say a quadruple $\widetilde{\Upsilon} = (\Phi, \Psi, \Theta, \Xi)$ a morphism from $(S, \mathcal{R}, P, Q, \varphi, \psi)$ to $(\tilde{S}, \tilde{\mathcal{R}}, \tilde{P}, \tilde{Q}, \tilde{\varphi}, \tilde{\psi})$

Let P be a G-graded right $\mathcal{R}$-module. Then $S = \mathrm{END}_{\mathcal{R}}(P)$ is a G-graded ring, and $Q = \mathrm{HOM}_{\mathcal{R}}(P, R) \in {}_R\mathcal{M}^G_S$ with structure

$$(r \cdot q \cdot s)(p) = rq(s(p))$$

for all $r \in \mathcal{R}$, $s \in S$, $q \in Q$ and $p \in P$. The connecting maps are the following

$$\varphi : P \otimes_{\mathcal{R}} Q \to S,\ \varphi(p \otimes_{\mathcal{R}} q)(p') = pq(p'),$$

$$\psi : Q \otimes_S P \to R,\ \psi(q \otimes_S p) = q(p).$$

Then $(S, \mathcal{R}, P, Q, \varphi, \psi)$ is a graded Morita context.

2.4. Cofree Group Corings. A G-A-coring $\underline{C} = \{C_\alpha\}_{\alpha\in G}$ is called cofree, if there exist A-bimodule isomorphisms $\gamma_\alpha : C_e \to C_\alpha$ such that

$$\Delta_{\alpha,\beta}(\gamma_{\alpha\beta}(c)) = \gamma_\alpha(c_{(1,e)}) \otimes_A \gamma_\beta(c_{(2,e)})$$

for all $c \in C_e$. If $\underline{C}$ is a cofree group coring, then, for every $\alpha \in G$, we have A-bimodule isomorphisms

$$\gamma_{\alpha^{-1}} : C_e \to C_{\alpha^{-1}},\ {}^*\gamma_{\alpha^{-1}} : \mathcal{R}_\alpha \to \mathcal{R}_e,$$

and

$$\chi_\alpha = ({}^*\gamma_{\alpha^{-1}})^{-1} : \mathcal{R}_e \to \mathcal{R}_\alpha.$$

From Proposition 4.6 in [5], the left dual $\mathcal{R} = {}^*\underline{C}$ is the group ring $\mathcal{R}_e[G]$.

Let $\underline{C} = C_e\langle G\rangle$ be a cofree G-A-coring and M be a right $\underline{C}$-comodule. Recall from [10] that we call that M is a cofree $\underline{C}$-comodule, if $(id \otimes_A \gamma_\alpha) \circ \rho^M_e = \rho^M_\alpha$.

2.1. Example. If $\underline{C} = C_e\langle G\rangle$ is a cofree G-A-coring and $g = (g_\alpha)_{\alpha\in G}$ a grouplike family of $\underline{C}$ such that $g_\alpha = \gamma_\alpha(g_e)$. Then A can be endowed with a structure of right $\underline{C}$-comodule via the coaction maps

$$\rho_\alpha^A : A \to A \otimes_A C_\alpha,\ \rho_\alpha^A(a) = 1_A \otimes_A g_\alpha \cdot a.$$

For all $a \in A$, we have

$$(id \otimes_A \gamma_\alpha) \circ \rho_e^A(a) = 1_A \otimes_A \gamma_\alpha(g_e \cdot a) = 1_A \otimes_A g_\alpha \cdot a = \rho_\alpha^A(a),$$

this shows that A is a cofree $\underline{C}$-module.

2.5. Group Entwining Structures. Let $\underline{C} = \{C_\alpha\}_{\alpha\in G}$ be a G-coalgebra and A an algebra. We say that the G-coalgebra $\underline{C}$ and the algebra A are G-entwined, if there is a family of linear maps $\psi = \{\psi_\alpha : C_\alpha \otimes A \to A \otimes C_\alpha\}_{\alpha\in G}$ such that

- $(ab)_{\psi_\alpha} \otimes c^{\psi_\alpha} = a_{\psi_\alpha} b_{\psi'_\alpha} \otimes c^{\psi_\alpha\psi'_\alpha}$,
- $1_{A\psi_\alpha} \otimes c^{\psi_\alpha} = 1_A \otimes c$, for any $c \in C_\alpha$,
- $a_{\psi_{\alpha\beta}} \otimes c^{\psi_{\alpha\beta}}{}_{(1,\alpha)} \otimes c^{\psi_{\alpha\beta}}{}_{(2,\beta)} = a_{\psi_\beta\psi_\alpha} \otimes c_{(1,\alpha)}{}^{\psi_\alpha} \otimes c_{(2,\beta)}{}^{\psi_\beta}$,
- $a_{\psi_e}\varepsilon(c^{\psi_e}) = a\varepsilon(c)$, for any $c \in C_e$ and $a \in A$.

where, we set $\psi_\alpha(c\otimes a) = a_{\psi_\alpha} \otimes c^{\psi_\alpha} = a_{\psi'_\alpha} \otimes c^{\psi'_\alpha} = \cdots$, for $a \in A$ and $c \in C_\alpha$. The triple $(A, \underline{C}, \psi)$ is called a right and right G-entwining structure and is denoted by $(A, \underline{C})_{G-\psi}$.

Given a right-right G-entwining structure $(A, \underline{C})_{G-\psi}$, then $\mathcal{U}_A^{\underline{C}}(\psi)$ is the category of right $(A, \underline{C})_\psi$. The object of $\mathcal{U}_A^{\underline{C}}(\psi)$ are right $\underline{C}$-comodules (M, ρ_α^M) which is also A-module such that

$$\rho_\alpha^M(m \cdot a) = m_{[0,\alpha]} \cdot a_{\psi_\alpha} \otimes m_{[1,\alpha]}{}^{\psi_\alpha}$$

for all $m \in M$ and $a \in A$. Morphisms in $\mathcal{U}_A^{\underline{C}}(\psi)$ are right $\underline{C}$-comodule and right A-module maps and let $\mathcal{U}_A^{G,\underline{C}}(\psi)$ be the category of right $(A, \underline{C})_{G-\psi}$ of which the objects are right G-$\underline{C}$-comodules $(\underline{M}, \rho_{\alpha,\beta}^M)$ which is also right A-module, i.e., each M_α is right A-module, such that

$$\rho_{\alpha,\beta}^M(m \cdot a) = m_{[0,\alpha]} \cdot a_{\psi_\beta} \otimes m_{[1,\beta]}{}^{\psi_\beta}$$

for all $m \in M_{\alpha\beta}$ and $a \in A$. Morphisms in $\mathcal{U}_A^{G,\underline{C}}(\psi)$ are right G-$\underline{C}$-comodule and right A-module maps.

2.6. Group Coalgebra Galois Extensions. Let $\underline{C}$ be a G-coalgebra and A an algebra. Let A be a right $\underline{C}$-comodule. Let

$$B = A^{co\underline{C}} = \{a \in A | \rho_\alpha^A(ab) = a\rho_\alpha^A(b), \forall b \in A, \alpha \in G\}.$$

We say that A is a right G-$\underline{C}$-Galois extension of B, if the canonical left A-module right G-$\underline{C}$-comodule map $can = \{can_\alpha : A \otimes_B A \to A \otimes C_\alpha\}$, by $a \otimes_B b \mapsto ab_{[0,\alpha]} \otimes b_{[1,\alpha]}$ for all $a, b \in A$ is bijective, i.e., every map can_α is bijective for al $\alpha \in G$.

3. Morita Context associated to a Comodule over a Group Coring

Let $\underline{C}$ be a G-A-coring, and $M \in {}^{\underline{C}}\mathcal{M}$. We can associate a Morita context to M. The context will connect $T = {}^{\underline{C}}\mathrm{END}(M)^{op}$ and ${}^*\underline{C} = \mathcal{R}$.

For every $\alpha \in G$, $Q_\alpha = {}_A\mathrm{HOM}(C_{\alpha^{-1}}, M) \in {}_R\mathcal{M}_T$ is a left A-module with

$$(a \cdot f_\alpha)(c) = f_\alpha(c \cdot a)$$

for all $f_\alpha \in Q_\alpha$, $a \in A$ and $c \in C_{\alpha^{-1}}$. Let

$$Q = \{q \in \bigoplus_{\alpha\in G} Q_\alpha | q_{\alpha\beta}(c)_{[-1,\beta^{-1}]} \otimes_A q_{\alpha\beta}(c)_{[0,\beta^{-1}]}$$
$$= c_{(1,\beta^{-1})} \otimes_A q_\alpha(c_{(2,\alpha^{-1})}), \forall c \in C_{\beta^{-1}\alpha^{-1}}\}.$$

3.1. Lemma. *With the notation as above,* $^*M = {}_A\mathrm{HOM}(M, A) \in {}_T\mathcal{M}_{\mathcal{R}}$ *and* $Q \in {}_{\mathcal{R}}\mathcal{M}_T$*.*

Proof. Let $\zeta \in {}^*M$, $f_\alpha \in \mathcal{R}_\alpha$, $t \in T$, $q_\beta \in Q_\beta$ and $m \in M$. We define the bimodule structure on $^*M$ as follows:

$$(\zeta \cdot f_\alpha)(m) = f_\alpha(m_{[-1,\alpha^{-1}]} \cdot \zeta(m_{[0,\alpha^{-1}]})) \text{ and } t \cdot \zeta = \zeta \circ t.$$

For all $g_\beta \in \mathcal{R}_\beta$, we have

$$\begin{aligned}
&((\zeta \cdot f_\alpha) \cdot g_\beta)(m)\\
&= g_\beta(m_{[-1,\beta^{-1}]} \cdot f_\alpha(m_{[0,\beta^{-1}][-1,\alpha^{-1}]} \cdot \zeta(m_{[0,\beta^{-1}][0,\alpha^{-1}]})))\\
&= g_\beta(m_{[-1,\beta^{-1}\alpha^{-1}](1,\beta^{-1})} \cdot f_\alpha(m_{[-1,\beta^{-1}\alpha^{-1}](2,\alpha^{-1})} \cdot \zeta(m_{[0,\beta^{-1}\alpha^{-1}]})))\\
&= (f_\alpha \star g_\beta)(m_{[-1,\beta^{-1}\alpha^{-1}]} \cdot \zeta(m_{[0,\beta^{-1}\alpha^{-1}]}))\\
&= (\zeta \cdot (f_\alpha \star g_\beta))(m).
\end{aligned}$$

This shows that $^*M$ is a G-graded right $\mathcal{R}$-module. Let us show that the two actions commute. Indeed, we compute

$$\begin{aligned}
(t \cdot (\zeta \cdot f_\alpha))(m) =&(\zeta \cdot f_\alpha)(t(m))\\
=&f_\alpha(t(m)_{[-1,\alpha^{-1}]} \cdot \zeta(t(m)_{[0,\alpha^{-1}]}))\\
=&f_\alpha(m_{[-1,\alpha^{-1}]} \cdot \zeta(t(m_{[0,\alpha^{-1}]})))\\
=&(t \cdot \zeta) \cdot f_\alpha.
\end{aligned}$$

The bimodule structure on Q is defined by

$$(f_\alpha \cdot q_\beta)(c) = q_\beta(c_{(1,\beta^{-1})} \cdot f_\alpha(c_{(2,\alpha^{-1})}))$$

for all $c \in C_{\beta^{-1}\alpha^{-1}}$ and $q_\beta \cdot t = t \circ q_\beta$. □

3.2. Lemma. *With the notation as above, we have well-defined bimodule maps*

$$\mu : Q \otimes_T {}^*M \to \mathcal{R},\ \ \mu((q \otimes_T \zeta) = \sum_{\alpha \in G} \zeta \circ q_\alpha,$$

$$\tau : {}^*M \otimes_{\mathcal{R}} Q \to T,\ \ \tau(\zeta \otimes_{\mathcal{R}} q)(m) = \sum_{\alpha \in G} q_\alpha(m_{[-1,\alpha^{-1}]} \cdot \zeta(m_{[0,\alpha^{-1}]})).$$

3.3. Theorem. *With the notation as above, we have a Morita context* $(T, \mathcal{R}, {}^*M, Q, \tau, \mu)$*.*

Proof. Here we only check that, for $\zeta, \zeta' \in {}^*M$, $q, q' \in Q$ and $m \in M$,

$$q' \cdot \tau(\zeta \otimes_{\mathcal{R}} q) = \mu(q' \otimes_T \zeta) \cdot q,\ \ \zeta \cdot \mu(q \otimes_T \zeta') = \tau(\zeta \otimes_{\mathcal{R}} q) \cdot \zeta' \tag{3.1}$$

hold. Indeed, for all $c \in C_{(\alpha\gamma)^{-1}}$, we compute

$$\begin{aligned}
(q'_{\alpha\gamma} \cdot \tau(\zeta \otimes_{\mathcal{R}} q))(c) &= \tau(\zeta \otimes_{\mathcal{R}} q)(q'_{\alpha\gamma}(c))\\
&= \sum_{\beta \in G} q_\beta(q'_{\alpha\gamma}(c)_{[-1,\beta^{-1}]} \cdot \zeta(q'_{\alpha\gamma}(c)_{[0,\beta^{-1}]}))\\
&= \sum_{\beta \in G} q_\beta(c_{(1,\beta^{-1})} \cdot \zeta(q'_{\alpha\gamma\beta^{-1}}(c_{(2,\beta\gamma^{-1}\alpha^{-1})})))\\
&= \sum_{\beta \in G} ((\zeta \circ q'_{\alpha\gamma\beta^{-1}}) \cdot q_\beta)(c)\\
&= (\mu(q'_{\alpha\gamma} \otimes_T \zeta) \cdot q)(c).
\end{aligned}$$

Thus we show that the first identity in (3.1) holds. The other identity can be checked similarly. □

Next, we want to make an application of Theorem 3.3 in order to get a new Morita context.

Let M be a right $\underline{C}$-comodule. Assume that M is finitely generated and projective with the finite dual basis $\{e_i, e_i^*\}$ or $\{e_i', e_i'^*\}$. M^* =Hom$_A(M, A)$ can be viewed as a left A-module via $(a \cdot f)(m) = af(m)$. Then M^* is a left $\underline{C}$-comodule with the coaction maps

$$^{M^*}\rho_\alpha : M^* \to C_\alpha \otimes_A M^*, \ ^{M^*}\rho_\alpha(f) = \sum_i f(e_{i[0,\alpha]}) \cdot e_{i[1,\alpha]} \otimes_A e_i^*$$

Indeed, we compute

$$\begin{aligned}(id \otimes^{M^*} \rho_\beta) \circ^{M^*} \rho_\alpha(f) &= (id \otimes^{M^*} \rho_\beta)(\sum_i f(e_{i[0,\alpha]}) \cdot e_{i[1,\alpha]} \otimes_A e_i^*) \\ &= \sum_{i,j} f(e_{i[0,\alpha]}) \cdot e_{i[1,\alpha]} \otimes_A e_i^*(e'_{j[0,\beta]}) \cdot e'_{j[1,\beta]} \otimes_A e_j'^* \\ &= \sum_{i,j} f(e_{i[0,\alpha]}) \cdot e_{i[1,\alpha]} \cdot e_i^*(e'_{j[0,\beta]}) \otimes_A e'_{j[1,\beta]} \otimes_A e_j'^* \\ &= \sum_i f(e_{i[0,\beta][0,\alpha]]}) \cdot e_{i[0,\beta][1,\alpha]} \otimes_A e_{i[1,\beta]} \otimes_A e_i^* \\ &= \sum_i f(e_{i[0,\alpha\beta]}) \cdot e_{i[1,\alpha\beta](1,\alpha)} \otimes_A e_{i[1,\alpha\beta](2,\beta)} \otimes_A e_i^*.\end{aligned}$$

This shows that $^{M^*}\rho = \{^{M^*}\rho_\alpha\}_{\alpha \in G}$ is $\underline{C}$-colinear.

3.4. Lemma. *Let M be a right $\underline{C}$-comodule. Assume that M is finitely generated and projective. Then*

$$^{\underline{C}}\mathrm{END}(M^*)^{op} \cong \mathrm{END}^{\underline{C}}(M).$$

Proof. Let $\{e_i, e_i^*\}$ be the dual basis of M. We construct the desired maps as follows:

$$\Phi :^{\underline{C}} \mathrm{END}(M^*)^{op} \to \mathrm{END}^{\underline{C}}(M), \ \Phi(f)(m) = \sum_i e_i \cdot f(e_i^*)(m)$$

and

$$\Psi : \mathrm{END}^{\underline{C}}(M) \to^{\underline{C}} \mathrm{END}(M^*)^{op}, \ \Psi(f)(g)(m) = g(f(m)).$$

The other verifications are straightforward. □

From Lemma 3.4 and Theorem 3.3, we have the following result.

3.5. Corollary. *Let M be a right $\underline{C}$-comodule. Assume that M is finitely generated and projective. We obtain a Morita context*

$$(\mathrm{END}^{\underline{C}}(M), \mathcal{R}, M, Q =^{\underline{C}} \mathrm{HOM}(\underline{C}, M^*), \tau, \mu)$$

with $M \in_T \mathcal{M}_\mathcal{R}$ by

$$m \cdot f_\alpha = m_{[0,\alpha^{-1}]} \cdot f_\alpha(m_{[1,\alpha^{-1}]}) \ \textit{and} \ t \cdot m = t(m)$$

for all $m \in M$, $f \in \mathcal{R}_\alpha$, $t \in T$, and $Q \in_R \mathcal{M}_T$ by

$$(f_\alpha \cdot q_\beta)(c) = q_\beta(c_{(1,\beta^{-1})} \cdot f_\alpha(c_{(2,\alpha^{-1})}))$$

for all $c \in C_{\beta^{-1}\alpha^{-1}}$ and $q_\beta \in Q_\beta$ and $(q_\beta \cdot t)(c') = q_\beta(c') \circ t$ for all $c' \in C_{\beta^{-1}}$, and

$$\mu : Q \otimes_T M \to R, \ \mu(q \otimes_T m)_\alpha(c) = q_\alpha(c)(m), \forall c \in C_{\alpha^{-1}}$$

$$\tau : M \otimes_R Q \to T, \ \tau(m \otimes_R q)(m') = \sum_{\alpha \in G} m_{[0,\alpha^{-1}]} \cdot (q_\alpha(m_{[0,\alpha^{-1}]})(m')).$$

3.6. Example. Let $\underline{C}$ be a G-A-coring with a grouplike family $g = (g_\alpha)_{\alpha \in G}$. Then A is a right $\underline{C}$-comodule via

$$\rho_\alpha^A : A \to A \otimes_A C_\alpha,\ \rho_\alpha^A(a) = 1_A \otimes_A g_\alpha \cdot a.$$

By [10], $T = \mathrm{END}^{\underline{C}}(A)$ is nothing but the $A^{co\underline{C}}$. Since $A^* \cong A$, we have

$$Q = \{q \in \mathcal{R} | q_{\alpha\beta}(c) g_{\beta^{-1}} = c_{(1,\beta^{-1})} \cdot q_\alpha(c_{(2,\alpha^{-1})}), \forall c \in C_{\beta^{-1}\alpha^{-1}}\}.$$

Applying Corollary 3.5, we have a Morita context as in [5].

4. Graded Morita Context associated to a Comodule over a Group Coring

In this section, we assume that $M \in \mathcal{M}^{\underline{C}}$ is finitely generated and projective with the dual basis $\{e_i, e_i^*\}$. We say that a G-A-coring $\underline{C}$ is left homogeneously finite, if each C_α is finitely generated and projective as a left A-module. For $M \in \mathcal{M}^{\underline{C}}$, it follows that $\{\mu_\alpha(M)\}_{\alpha \in G} \in \mathcal{M}^{G,\underline{C}}$ with the coaction maps

$$\rho_{\alpha,\beta} : \mu_{\alpha\beta}(M) \to \mu_\alpha(M) \otimes_A C_\beta,\ \rho_{\alpha,\beta}(\mu_{\alpha\beta}(m)) = \mu_\alpha(m_{[0,\beta]}) \otimes_A m_{[1,\beta]}.$$

From Proposition 4.1 in [5], we then obtain that

$$M\{G\} = \bigoplus_{\alpha \in G} \mu_\alpha(M) \in \mathcal{M}_{\mathcal{R}}^G.$$

The right $\mathcal{R}$-action is defined by the following formula,

$$\mu_\alpha(m) \cdot f_\beta = \mu_{\alpha\beta}(m_{[0,\beta^{-1}]}) \cdot f_\beta(m_{[1,\beta^{-1}]})$$

for all $f_\beta \in \mathcal{R}_\beta$ and $m \in M$.

Next, we will compute the graded Morita context associated to the graded right $\mathcal{R}$-module $M\{G\}$. Consider the ring

$$S = \{\underline{f} = (f_\alpha)_{\alpha \in G} \in \prod_{\alpha \in G} \mathrm{END}_A(M) | f_\alpha(m)_{[0,\beta^{-1}]} \otimes_A f_\alpha(m)_{[1,\beta^{-1}]} = f_{\alpha\beta}(m_{[0,\beta^{-1}]}) \otimes_A m_{[1,\beta^{-1}]}\}.$$

Observe that we have a ring monomorphism

$$i : T \to S,\ i(f) = \underline{f} = (f)_\alpha.$$

On S, we have the following right G-action:

$$\underline{f}^\sigma = \underline{f} \cdot \sigma = (f_{\sigma\alpha})_{\alpha \in G}.$$

Indeed, if $\underline{f} \in S$, we have $\underline{f} \cdot \sigma \in S$, since

$$f_{\sigma\alpha}(m)_{[0,\beta^{-1}]} \otimes_A f_{\sigma\alpha}(m)_{[1,\beta^{-1}]} = f_{\sigma\alpha\beta}(m_{[0,\beta^{-1}]}) \otimes_A m_{[1,\beta^{-1}]},$$

Now, we consider the twisted group ring $G * S = \bigoplus_{\alpha \in G} \mu_\alpha S$ with multiplication

$$\mu_\alpha \underline{f} \mu_\beta \underline{g} = \mu_{\alpha\beta}((\underline{f} \cdot \beta)\underline{g}).$$

4.1. Proposition. *If G-A-coring $\underline{C}$ is left homogeneously finite, We then have a graded ring isomorphism*

$$\Omega : \mathrm{END}_{\mathcal{R}}(M\{G\}) \to G * S.$$

Proof. For each $\sigma \in G$, we construct a map by

$$\Omega_\sigma : \mathrm{END}_{\mathcal{R}}(M\{G\})_\alpha \to \mu_\sigma S,\ \Omega_\sigma(h) = \mu_\sigma \underline{f}$$

with $f_\alpha(m) = \mu_{\sigma\alpha}^{-1}(h(\mu_\alpha(m)))$. Since h is right $\mathcal{R}$-linear, we have, for all $m \in M$ and $g \in \mathcal{R}_\beta$ that

$$\begin{aligned} h(\mu_\alpha(m) \cdot g_\beta) &= h(\mu_{\alpha\beta}(m_{[0,\beta^{-1}]}) \cdot g_\beta(m_{[1,\beta^{-1}]})) \\ &= \mu_{\sigma\alpha\beta}(f_{\alpha\beta}(m_{[0,\beta^{-1}]})) \cdot g_\beta(m_{[1,\beta^{-1}]})). \end{aligned}$$

Since

$$\begin{aligned} h(\mu_\alpha(m) \cdot g_\beta) &= h(\mu_\alpha(m)) \cdot g_\beta \\ &= \mu_{\sigma\alpha}(f_\alpha(m)) \cdot g_\beta \\ &= \mu_{\sigma\alpha\beta}((f_\alpha(m)_{[0,\beta^{-1}]}) \cdot g_\beta(f_\alpha(m)_{[1,\beta^{-1}]})), \end{aligned}$$

it follows that

$$(f_\alpha(m)_{[0,\beta^{-1}]}) \cdot g_\beta(f_\alpha(m)_{[1,\beta^{-1}]}) = f_{\alpha\beta}(m_{[0,\beta^{-1}]}) \cdot g_\beta(m_{[1,\beta^{-1}]}).$$

Since $\underline{C}$ is left homogeneously finite (also see Lemma 4.2 in [5]), we have

$$(f_\alpha(m)_{[0,\beta^{-1}]}) \otimes_A f_\alpha(m)_{[1,\beta^{-1}]}) = f_{\alpha\beta}(m_{[0,\beta^{-1}]}) \otimes_A m_{[1,\beta^{-1}]}.$$

This means $\underline{f} \in S$. Next, we define a map

$$\Upsilon_\sigma : \mu_\sigma S \to \mathrm{END}_{\mathcal{R}}(M\{G\})_\alpha,\ \Upsilon_\sigma(\underline{f}) = h$$

where h satisfies $h(\mu_\alpha(m)) = \mu_{\sigma\alpha}(f_\alpha(m))$. It is straightforward to check that Υ_σ and Ω_σ are mutually inverses. It is routine to check that

$$\Omega = \bigoplus_{\alpha \in G} \Omega_\alpha : \mathrm{END}_{\mathcal{R}}(M\{G\}) \to G * S$$

preserves the multiplication and the unit. □

Our next aim is to describe $\mathrm{HOM}_{\mathcal{R}}(M\{G\}, \mathcal{R})$. Consider

$$\begin{aligned} Q = \{\underline{q} = (q_\alpha)_{\alpha\in G} \in \prod_{\alpha \in G} {}_A\mathrm{HOM}(C_{\alpha^{-1}}, M^*) | \\ c_{(1,\beta^{-1})} \otimes_A q_\alpha(c_{(2,\alpha^{-1})}) = q_{\alpha\beta}(c)(e_{i[0,\beta^{-1}]}) \cdot e_{i[1,\beta^{-1}]} \otimes_A e_i^*,\ c \in C_{(\alpha\beta)^{-1}}\}. \end{aligned}$$

4.2. Lemma. *If $f_\gamma \in \mathcal{R}_\gamma$ and $\underline{q} \in Q$, then*

$$f_\gamma \cdot \underline{q} = (f_\gamma \cdot q_{\gamma^{-1}\alpha})_{\alpha \in G} \in Q.$$

Proof. For all $c \in C_{(\alpha\beta)^{-1}}$ and $m \in M$, we have

$$\begin{aligned} & c_{(1,\beta^{-1})} \otimes_A (f_\gamma \cdot q_{\gamma^{-1}\alpha})(c_{(2,\alpha^{-1})}) \\ &= c_{(1,\beta^{-1})} \otimes_A q_{\gamma^{-1}\alpha}(c_{(2,\alpha^{-1}\gamma)} \cdot f_\gamma(c_{(3,\gamma^{-1})})) \\ &= (c_{(1,\beta^{-1}\alpha^{-1}\gamma)} \cdot f_\gamma(c_{(2,\gamma^{-1})}))_{(1,\beta^{-1})} \otimes_A q_{\gamma^{-1}\alpha}((c_{(1,\beta^{-1}\alpha^{-1}\gamma)} \cdot f_\gamma(c_{(2,\gamma^{-1})}))_{(2,\alpha^{-1}\gamma)}) \\ &= \sum_i q_{\gamma^{-1}\alpha\beta}(c_{(1,\beta^{-1}\alpha^{-1}\gamma)} \cdot f_\gamma(c_{(2,\gamma^{-1})}))(e_{i[0,\beta^{-1}]}) \cdot e_{i[1,\beta^{-1}]} \otimes_A e_i^* \\ &= (f_\gamma \cdot q_{\gamma^{-1}\alpha\beta})(c)(e_{i[0,\beta^{-1}]}) \cdot e_{i[1,\beta^{-1}]} \otimes_A e_i^*. \end{aligned}$$

□

4.3. Lemma. *If $\underline{q} \in Q$ and $\underline{f} \in S$, then $\underline{q} \cdot \underline{f} = (q_\alpha \cdot f_\alpha)_{\alpha \in G} \in Q$, where*

$$(q_\alpha \cdot f_\alpha)(c) = q_\alpha(c) \circ f_\alpha$$

for all $c \in C_{\alpha^{-1}}$.

4.4. Lemma.

$$QG = \bigoplus_{\alpha \in G} \omega_\alpha(Q) \in {}_{\mathcal{R}}\mathcal{M}^{G}_{G*S}$$

with bimodule structures defined as follows: for all $f \in \mathcal{R}_\beta$, $\underline{q} \in Q$ *and* $\underline{b} \in S$,

$$f_\beta \cdot \omega_\alpha(\underline{q}) = \omega_{\beta\alpha}(f_\beta \cdot \underline{q}),\ \omega_\alpha(\underline{q}) \cdot \mu_\tau \underline{b} = \omega_{\alpha\tau}(\underline{q} \cdot (\underline{b} \cdot (\alpha\tau)^{-1})).$$

4.5. Proposition. *If* G*-*A *coring* $\underline{C}$ *is left homogeneously finite, and* $M \in \mathcal{M}^{\underline{C}}$ *is finitely generated and projective as a right* A*-module. We then have an isomorphism of graded bimodules*

$$\Psi : \mathrm{HOM}_{\mathcal{R}}(M\{G\}, \mathcal{R}) \to QG$$

Proof. For each $\sigma \in G$, we construct a map by

$$\Psi_\sigma : \mathrm{HOM}_{\mathcal{R}}(M\{G\}, \mathcal{R})_\sigma \to \omega_\sigma(Q),\ \Psi_\sigma(g) = \omega_\sigma(\underline{q})$$

with $q_\alpha(c)(m) = g(\mu_{\sigma^{-1}\alpha}(m))(c)$ for all $c \in C_{\alpha^{-1}}$ and $m \in M$. Take $\beta \in G$ and $f_\beta \in \mathcal{R}_\beta$. Since g is right $\mathcal{R}$-linear, we have, for all $m \in M$ that

$$\begin{aligned} g(\mu_{\sigma^{-1}\alpha}(m) \cdot f_\beta) &= g(\mu_{\sigma^{-1}\alpha\beta}(m_{[0,\beta^{-1}]}) \cdot f_\beta(m_{[1,\beta^{-1}]})) \\ &= g(\mu_{\sigma^{-1}\alpha\beta}(m_{[0,\beta^{-1}]})) \cdot f_\beta(m_{[1,\beta^{-1}]}). \end{aligned}$$

Notice that

$$g(\mu_{\sigma^{-1}\alpha}(m) \cdot f_\beta) = g(\mu_{\sigma^{-1}\alpha}(m)) \cdot f_\beta.$$

Thus, for all $c \in C_{(\alpha\beta)^{-1}}$, we have

$$\begin{aligned} &(g(\mu_{\sigma^{-1}\alpha\beta}(m_{[0,\beta^{-1}]})) \cdot f_\beta(m_{[1,\beta^{-1}]}))(c) \\ &= (g(\mu_{\sigma^{-1}\alpha\beta}(m_{[0,\beta^{-1}]})))(c) f_\beta(m_{[1,\beta^{-1}]}) \\ &= q_{\alpha\beta}(c)(m_{[0,\beta^{-1}]}) f_\beta(m_{[1,\beta^{-1}]}) \end{aligned}$$

and

$$(g(\mu_{\sigma^{-1}\alpha}(m)) \cdot f_\beta)(c) = f_\beta(c_{(1,\beta^{-1})} \cdot (q_\alpha(c_{(2,\alpha^{-1})}))(m)),$$

it follows that

$$f(q_{\alpha\beta}(c)(m_{[0,\beta^{-1}]}) \cdot m_{[1,\beta^{-1}]}) = f(c_{(1,\beta^{-1})} \cdot g(\mu_{\sigma^{-1}\alpha}(m))(c_{(2,\alpha^{-1})})).$$

Since $\underline{C}$ is left homogeneously finite, we have

$$q_{\alpha\beta}(c)(m_{[0,\beta^{-1}]}) \cdot m_{[1,\beta^{-1}]} = c_{(1,\beta^{-1})} \cdot (q_\alpha(c_{(2,\alpha^{-1})}))(m).$$

Using the above equation and by M being finitely generated and projective (also see Lemma 4.2 in [5]), we have

$$c_{(1,\beta^{-1})} \otimes_A q_\alpha(c_{(2,\alpha^{-1})}) = q_{\alpha\beta}(c)(e_{i[0,\beta^{-1}]}) \cdot e_{i[1,\beta^{-1}]} \otimes_A e_i^*.$$

This means $\underline{q} \in Q$. Next, we define a map

$$\Phi_\sigma : \omega_\sigma(Q) \to \mathrm{HOM}_{\mathcal{R}}(M\{G\}, \mathcal{R})_\sigma,\ \Phi_\sigma(\omega_\sigma(\underline{q})) = g$$

where g satisfies $g(\mu_{\sigma^{-1}\alpha}(m))(c) = q_\alpha(c)(m)$ for all $c \in C_{\alpha^{-1}}$ with $\alpha \in G$. It is straightforward to check that Ψ_σ and Φ_σ are mutually inverse. It is routine to check that the bijection

$$\Psi = \bigoplus_{\alpha \in G} \Psi_\alpha : \mathrm{HOM}_{\mathcal{R}}(M\{G\}, \mathcal{R}) \to QG$$

preserves the bimodule structure. □

Now, we will achieve the main goal in this section.

4.6. Theorem. *If G-A coring $\underline{C}$ is left homogeneously finite, and $M \in \mathcal{M}^{\underline{C}}$ is finitely generated and projective as a right A-module. Consider the graded Morita context* $(\mathrm{END}_{\mathcal{R}}(M\{G\}), \mathcal{R}, M\{G\}, \mathrm{HOM}_{\mathcal{R}}(M\{G\}, \mathcal{R}), \mathcal{R}, \varphi, \psi)$ *associated to the graded $\mathcal{R}$-module $M\{G\}$. Using the isomorphism Ω and Ψ from Proposition 4.1 and 4.5, we find an isomorphic graded Morita context $\mathbb{GM} = (G * S, \mathcal{R}, M\{G\}, QG, \omega', \nu')$ with connecting map ω' and ν' given by the formulas*

$$\begin{gathered}
\omega' : M\{G\} \otimes_{\mathcal{R}} QG \to G * S,\\
\omega'(\mu_\alpha(m) \otimes_{\mathcal{R}} \omega_\sigma(\underline{q})) = \mu_{\alpha\sigma}(\{f_{\sigma\beta}\}_{\beta \in G}),\\
f_{\sigma\beta}(m') = m_{[0,(\sigma\beta)^{-1}]} \cdot q_{\sigma\beta}(m_{[1,(\sigma\beta)^{-1}]})(m')\\
\nu' : QG \otimes_{G*S} M\{G\} \to \mathcal{R},\\
\nu'(\omega_\sigma(\underline{q}) \otimes_{G*S} \mu_\alpha(m))(c) = q_{\sigma\alpha}(c)(m),\ \forall c \in C_{(\sigma\alpha)^{-1}}.
\end{gathered}$$

Proof. It is routine to check that the following two diagrams are commutative

$$\begin{tikzcd}
M\{G\} \otimes_{\mathcal{R}} \mathrm{HOM}_{\mathcal{R}}(M\{G\}, \mathcal{R}) \arrow[r] \arrow[d, "id \otimes_{\mathcal{R}} \Psi"'] & \mathrm{END}_{\mathcal{R}}(M\{G\}) \arrow[d, "\Omega"] \\
M\{G\} \otimes_{\mathcal{R}} QG \arrow[r, "\omega'"] & G * S
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{HOM}_{\mathcal{R}}(M\{G\}, \mathcal{R}) \otimes_{\mathrm{END}_{\mathcal{R}}(M\{G\})} M\{G\} \arrow[r] \arrow[d, "\Psi \otimes_{\mathcal{R}} id"'] & \mathcal{R} \arrow[d, "="] \\
QG \otimes_{G*S} M\{G\} \arrow[r, "\nu'"] & \mathcal{R}
\end{tikzcd}$$

This ends the proof. □

4.7. Remark. Let $(\underline{C}, \underline{x})$ be a G-A-coring with a fixed grouplike family $\underline{x} = (x_\alpha)_{\alpha \in G}$. The Morita context in Theorem 4.6 is just the Morita context studied in [5].

Let C_e be an A-coring and M a C_e-comodule such that M is finitely generated and projective as right A-module. Recall from [8] that we have a Morita context

$$\mathbb{M}_e = (T = \mathrm{END}^{C_e}(M), \mathcal{R}_e, M, Q_e = {}^{C_e}\mathrm{HOM}(C_e, M^*), \tau_e, \mu_e)$$

with $M \in {}_T\mathcal{M}_{\mathcal{R}_e}$ by

$$m \cdot f_e = m_{[0,e]} \cdot f_e(m_{[1,e]}) \text{ and } t \cdot m = t(m)$$

for all $m \in M$, $f_e \in \mathcal{R}_e$, $t \in T$, and $Q_e \in {}_{\mathcal{R}_e}\mathcal{M}_T$ by

$$(f_e \cdot q_e)(c) = q_e(c_{(1,e)} \cdot f_e(c_{(2,e)}))$$

for all $c \in C_e$ and $q_e \in Q_e$ and $(q_e \cdot t)(c') = q_e(c') \circ t$ for all $c' \in C_e$, and

$$\begin{gathered}
\mu_e : Q_e \otimes_T M \to \mathcal{R}_e,\ \mu_e(q_e \otimes_T m)(c) = q_e(c)(m), \forall c \in C_e\\
\tau_e : M \otimes_{\mathcal{R}_e} Q_e \to T,\ \tau_e(m \otimes_{\mathcal{R}_e} q_e)(m') = m_{[0,e]} \cdot (q_e(m_{[1,e]})(m')).
\end{gathered}$$

4.8. Proposition. *Let $\mathbb{M}_e$ be the Morita context defined as above. Consider the group rings $T[G]$ and $\mathcal{R}_e[G]$. Then $M[G] = \bigoplus_{\sigma \in G} M\mu_\sigma \in_{T[G]} \mathcal{M}^G_{\mathcal{R}_e[G]}$ and $Q_e[G] = \bigoplus_{\sigma \in G} Q_e\mu_\sigma \in_{\mathcal{R}_e[G]} \mathcal{M}^G_{T[G]}$ with*

$$f\mu_\sigma \cdot m\mu_\alpha \cdot r_e\mu_\beta = (f \cdot m \cdot r_e)\mu_{\sigma\alpha\beta},\ r_e\mu_\beta \cdot q_e\mu_\alpha \cdot f\mu_\sigma = (r_e \cdot q_e \cdot f)\mu_{\beta\alpha\sigma}$$

for all $\sigma, \alpha, \beta \in G$, $f \in T$, $r_e \in \mathcal{R}_e$, $m \in M$ and $q_e \in Q_e$. We have well-defined maps

$$\begin{gathered}
\mu : Q_e[G] \otimes_{T[G]} M[G] \to \mathcal{R}_e[G],\ \mu(q_e\mu_\sigma \otimes_{T[G]} m\mu_\alpha) = \mu_e(q_e \otimes_T m)\mu_{\sigma\alpha},\\
\tau : M[G] \otimes_{\mathcal{R}_e[G]} Q_e[G] \to T[G],\ \tau(m\mu_\sigma \otimes_{\mathcal{R}_e[G]} q_e\mu_\alpha) = \tau_e(m \otimes_{\mathcal{R}_e} q_e)\mu_{\sigma\alpha}.
\end{gathered}$$

Then $\mathbb{M}_e[G] = (T[G], \mathcal{R}_e[G], M[G], Q_e[G], \tau, \mu)$ *is a graded Morita context.*

4.9. Lemma. *Let* $\underline{C}$ *be a cofree group coring and* M *a cofree* $\underline{C}$*-comodule such that* M *is finitely generated and projective. Then* $i : T \to S$ *is isomorphism, and* $\mathrm{END}_{\mathcal{R}}(M\{G\}) \cong G * S$ *is isomorphic as a graded ring to the group ring* $T[G]$.

Proof. It suffices to show that i is surjective. For $\underline{f} \in S$, we have that

$$\begin{aligned} f_\alpha(m)_{[0,e]} \otimes_A \gamma_{\beta^{-1}}(f_\alpha(m)_{[1,e]}) &= f_\alpha(m)_{[0,\beta^{-1}]} \otimes_A f_\alpha(m)_{[1,\beta^{-1}]} \\ &= f_{\alpha\beta}(m_{[0,\beta^{-1}]}) \otimes_A m_{[1,\beta^{-1}]} \\ &= f_{\alpha\beta}(m_{[0,e]}) \otimes_A \gamma_{\beta^{-1}}(m_{[1,e]}). \end{aligned}$$

Applying $id \otimes_A \varepsilon \circ \gamma_{\beta^{-1}}^{-1}$, we have $f_\alpha(m) = f_{\alpha\beta}(m)$, hence $f_e = f_\beta$ for all $\beta \in G$, and $\underline{f} = i(f_e)$. □

4.10. Proposition. *Let* $\underline{C}$ *be a cofree group coring and* M *a cofree* $\underline{C}$*-comodule. Then we have an isomorphism of* G*-graded* $(G * S, \mathcal{R})$*-bimodules*

$$\vartheta : M\{G\} \to M[G],\ \vartheta(\mu_\alpha(m)) = m\mu_\alpha.$$

Proof. Straightforward. □

4.11. Lemma. *Let* $\underline{C}$ *be a cofree group coring and* M *a cofree* $\underline{C}$*-comodule such that* M *is finitely generated and projective. Then* $Q \cong Q_e$. *Consequently* $\mathrm{HOM}_{\mathcal{R}}(M\{G\}, \mathcal{R}) \cong Q_e[G]$.

Proof. Let us take $\underline{q} = \{q_\alpha\}_{\alpha \in G} \in Q$. Then for all $\alpha, \beta \in G$ and $c \in C_e$, we have

$$\begin{aligned} &\gamma_{\beta^{-1}}(c_{(1,e)}) \otimes_A q_\alpha(\gamma_{\alpha^{-1}}(c_{(2,e)})) \\ =&\gamma_{(\alpha\beta)^{-1}}(c)_{(1,\beta^{-1})} \otimes_A q_\alpha(\gamma_{(\alpha\beta)^{-1}}(c)_{(2,\alpha^{-1})}) \\ =&q_{\alpha\beta}(\gamma_{(\alpha\beta)^{-1}}(c))(e_{i[0,e]}) \cdot \gamma_{\beta^{-1}}(e_{i[1,e]}) \otimes_A e_i^* \end{aligned}$$

Taking $\alpha = \beta = e$, we find that $q_e \in Q_e$. For all $m \in M$, it follows that

$$\gamma_{\beta^{-1}}(c_{(1,e)}) \cdot q_\alpha(\gamma_{\alpha^{-1}}(c_{(2,e)}))(m) = q_{\alpha\beta}(\gamma_{(\alpha\beta)^{-1}}(c))(m_{[0,e]}) \cdot \gamma_{\beta^{-1}}(m_{[1,e]}).$$

Applying $\gamma_{\beta^{-1}}^{-1}$ to both sides of the equation above, we have

$$c_{(1,e)} \cdot q_\alpha(\gamma_{\alpha^{-1}}(c_{(2,e)}))(m) = q_{\alpha\beta}(\gamma_{(\alpha\beta)^{-1}}(c))(m_{[0,e]}) \cdot m_{[1,e]}.$$

Applying ε to both sides, we find that

$$q_\alpha(\gamma_{\alpha^{-1}}(c))(m) = q_{\alpha\beta}(\gamma_{(\alpha\beta)^{-1}}(c))(m),$$

and

$$q_e(c) = q_\beta(\gamma_{\beta^{-1}}(c)).$$

Hence, we have $q_\beta = q_e \circ \gamma_{\beta^{-1}}^{-1}$. These arguments show that the map

$$j : Q_e \to Q,\ j(q) = (\sigma_\alpha(q))_{\alpha \in \pi}$$

is a well-defined isomorphism. □

4.12. Theorem. *Let* $\underline{C}$ *be a cofree group coring and* M *a cofree* $\underline{C}$*-comodule such that* M *is finitely generated and projective. Then the graded Morita contexts* $\mathbb{GM}$ *and* $\mathbb{M}_e[G]$ *are isomorphic.*

Proof. Let $\Xi : G * S \to T[G]$ be the isomorphism in Lemma 4.9. We will show that the diagram

$$\begin{tikzcd}
M\{G\} \otimes_{\mathcal{R}} QG \arrow[r, "\omega'"] \arrow[d, "\vartheta\otimes j^{-1}G"'] & G * S \arrow[d, "\Xi"] \\
M[G] \otimes_{\mathcal{R}_e[G]} Q_e[G] \arrow[r, "\varphi"] & T[G]
\end{tikzcd}$$

commutes. Indeed, for $\alpha, \sigma \in G$, $a \in A$ and $\underline{q} \in Q$, we have

$$\begin{aligned}(\Xi \circ \omega')(\mu_\alpha(m) \otimes \omega_\sigma(\underline{q})) =& \Xi(\mu_{\alpha\sigma}((f_{\sigma\beta})_{\beta\in G})) \\ =& f_\sigma \mu_{\alpha\sigma}\end{aligned}$$

where $f_{\sigma\beta}(m') = m_{[0,(\sigma\beta)^{-1}]} \cdot q_{\sigma\beta}(m_{[1,(\sigma\beta)^{-1}]})(m')$, and

$$\begin{aligned}&(\varphi \circ (\vartheta \otimes j^{-1}G))(\mu_\alpha(m) \otimes \omega_\sigma(\underline{q})) \\ &= \varphi(m\mu_\alpha \otimes q_e\mu_\sigma) \\ &= \varphi_e(m \otimes q_e)\mu_{\alpha\sigma},\end{aligned}$$

for all $m' \in M$, since

$$\begin{aligned}f_\sigma(m') =& m_{[0,\sigma^{-1}]} \cdot q_\sigma(m_{[1,\sigma^{-1}]})(m') \\ =& m_{[0,e]} \cdot q_\sigma(\gamma_{\sigma^{-1}}(m_{[1,e]}))(m') \\ =& m_{[0,e]} \cdot q_e(m_{[1,e]})(m') \\ =& \varphi_e(m \otimes q_e)(m'),\end{aligned}$$

it follows that $\Xi \circ \omega = \varphi \circ (\vartheta \otimes j^{-1}G)$. Let

$$\Gamma : \mathcal{R}_e[G] \to \mathcal{R},\ \Gamma(f\mu_\alpha) = f \circ \gamma^{-1}_{\alpha^{-1}}$$

be the isomorphism from Proposition 4.6 in [5]. We will show that the diagram

$$\begin{tikzcd}
QG \otimes_{G*S} M\{G\} \arrow[r, "\nu'"] \arrow[d, "j^{-1}G\otimes\vartheta"'] & \mathcal{R} \arrow[d, "\Gamma^{-1}"] \\
Q_e[G] \otimes_{T[G]} M[G] \arrow[r, "\psi"] & R_e[G]
\end{tikzcd}$$

commutes. Take $\sigma, \alpha \in G$, $\underline{q} \in Q$ and $a \in A$,

$$\begin{aligned}&(\Gamma \circ \psi \circ (j^{-1}G \otimes \vartheta))(\omega_\sigma(\underline{q}) \otimes \mu_\alpha(m)) \\ &= (\Gamma \circ \psi)(q_e\mu_\sigma \otimes m\mu_\alpha) \\ &= \Gamma(q_e(-)(m)\mu_{\sigma\alpha}) \\ &= q_e(-)(m) \circ \gamma^{-1}_{(\sigma\alpha)^{-1}}\end{aligned}$$

and

$$\nu'(\omega_\sigma(\underline{q}) \otimes_{G*S} \mu_\alpha(m))(-) = q_{\sigma\alpha}(-)(m).$$

For $\gamma_{(\sigma\alpha)^{-1}}(c) \in C_{(\sigma\alpha)^{-1}}$, we compute that

$$\begin{aligned}(q_e(-)(m) \circ \gamma^{-1}_{(\sigma\alpha)^{-1}})(\gamma_{(\sigma\alpha)^{-1}}(c)) =& q_e((\gamma^{-1}_{(\sigma\alpha)^{-1}} \circ \gamma_{(\sigma\alpha)^{-1}})(c))(m) \\ =& q_{\sigma\alpha}(\gamma_{(\sigma\alpha)^{-1}}(c))(m) \\ =& \nu'(\omega_\sigma(\underline{q}) \otimes_{G*S} \mu_\alpha(m))(\gamma_{(\sigma\alpha)^{-1}}(c)).\end{aligned}$$

Thus we have $\psi \circ (j^{-1}G \otimes \vartheta) = \Gamma^{-1} \circ \nu'$. $\square$

5. Applications

In this section, we will give the application of our theory to G-entwining structure.

Given a G-entwining structure $(A, \underline{C})_{G-\psi}$, then we have a G-A-coring $\{A \otimes C_\alpha\}_{\alpha \in G}$ arising from $(A, \underline{C})_{G-\psi}$. First observe that

$$\mathcal{R}_\alpha = {}_A\mathrm{HOM}(A \otimes C_{\alpha^{-1}}, A) \cong \mathrm{HOM}(C_{\alpha^{-1}}, A)$$

as spaces. This graded ring structure on $\mathcal{R}$ induces a graded ring structure on $\bigoplus_{\alpha \in G} \mathrm{HOM}(C_{\alpha^{-1}}, A)$, and this graded ring is denoted by $\sharp(\underline{C}, A)$. The product is given by the formula

$$(f_\alpha \sharp g_\beta)(c) = f_\alpha(c_{(2,\alpha^{-1})})_{\psi_{\beta^{-1}}} g_\beta(c_{(1,\beta^{-1})}{}^{\psi_{\beta^{-1}}})$$

for all $f_\alpha \in \mathrm{HOM}(C_{\alpha^{-1}}, A)$, $g_\beta \in \mathrm{HOM}(C_{\beta^{-1}}, A)$ and $c \in C_{(\alpha\beta)^{-1}}$.

Fix a grouplike family $x = \{x_\alpha\}_{\alpha \in G}$ of $\underline{C}$, then we have that $A \in \mathcal{U}_A^{\underline{C}}(\psi)$ with right $\underline{C}$-coaction $\rho_\alpha^A(a) = a_{\psi_\alpha} \otimes x_\alpha{}^{\psi_\alpha}$.

Generally, let A be a right $\underline{C}$-comodule. Suppose that A is an object of $\mathcal{U}_A^{\underline{C}}(\psi)$ with the structure maps m_A and $\rho^A = \{\rho_\alpha^A\}$. Then we have $\rho_\alpha^A(ab) = a_{[0,\alpha]} b_{\psi_\alpha} \otimes a_{[1,\alpha]}{}^{\psi_\alpha}$. Specially, the coaction can be written as $\rho_\alpha^A(b) = 1_{A[0,\alpha]} b_{\psi_\alpha} \otimes 1_{A[1,\alpha]}{}^{\psi_\alpha}$. The ring of coinvariants is

$$B = \{a \in A | 1_{A[0,\alpha]} a_{\psi_\alpha} \otimes 1_{A[1,\alpha]}{}^{\psi_\alpha} = a 1_{A[0,\alpha]} \otimes 1_{A[1,\alpha]}, \forall \alpha \in G\}.$$

Q and S can be described as follows:

$$\begin{aligned} Q = \{&\underline{q} = (q_\alpha)_{\alpha \in G} \in \prod_{\alpha \in G} \sharp(C_{\alpha^{-1}}, A)_\alpha | \\ &q_\alpha(c_{(2,\alpha^{-1})})_{\psi_{\beta^{-1}}} \otimes c_{(1,\beta^{-1})}{}^{\psi_{\beta^{-1}}} = q_{\alpha\beta}(c) 1_{A[0,\beta^{-1}]} \otimes 1_{A[1,\beta^{-1}]}, c \in C_{(\alpha\beta)^{-1}}\} \end{aligned}$$

and

$$\begin{aligned} S = \{&\underline{b} = (b_\alpha)_{\alpha \in G} \in \prod_{\alpha \in G} A | \\ &1_{A[0,\beta^{-1}]} b_{\alpha\psi_{\beta^{-1}}} \otimes 1_{A[0,\beta^{-1}]}{}^{\psi_{\beta^{-1}}} = 1_{A[0,\beta^{-1}]} b_{\alpha\beta} \otimes 1_{A[1,\beta^{-1}]}\}. \end{aligned}$$

Then we have the twisted group ring $G * S = \bigoplus_{\alpha \in G} \mu_\alpha S$ with the multiplication given by $\mu_\alpha \underline{b} \mu_\beta \underline{c} = \mu_{\alpha\beta}(\underline{b}^\beta \underline{c})$, where $\underline{b}^\beta = (b_{\beta\alpha})_{\alpha \in G}$.

From Theorem 4.6, we have the following result.

5.1. Theorem. *With the notation as above, we have a graded Morita context* $\mathbb{GM} = (G * S, \sharp(\underline{C}, A), A\{G\}, QG, \omega', \nu')$ *with connecting map* ω' *and* ν' *given by the formulas*

$$\omega' : A\{G\} \otimes_{\sharp(\underline{C},A)} QG \to G * S,$$

$$\omega'(\mu_\alpha(a) \otimes_{\sharp(\underline{C},A)} \omega_\sigma(\underline{q})) = \mu_{\alpha\sigma}(1_{A[0,(\sigma\beta)^{-1}]} a_{\psi_{(\sigma\beta)^{-1}}} q_{\sigma\beta}(1_{A[1,(\sigma\beta)^{-1}]}{}^{(\sigma\beta)^{-1}})),$$

$$\nu' : QG \otimes_{G*S} A\{G\} \to \sharp(\underline{C}, A),$$

$$\nu'(\omega_\sigma(\underline{q}) \otimes_{G*S} \mu_\alpha(a))(c) = q_{\sigma\alpha}(c) a, \forall c \in C_{(\sigma\alpha)^{-1}}.$$

As was stated above, if we fix a grouplike family $x = \{x_\alpha\}_{\alpha \in G}$ of $\underline{C}$, then we have that $A \in \mathcal{U}_A^{\underline{C}}(\psi)$ with right G-$\underline{C}$-coaction $\rho_\alpha^A(a) = a_{\psi_\alpha} \otimes x_\alpha{}^{\psi_\alpha}$. In particular, it follows that $\rho_\alpha^A(1_A) = 1_A \otimes x_\alpha$. Then B, S and Q have the following forms:

$$B = \{a \in A | a_{\psi_\alpha} \otimes x_\alpha{}^{\psi_\alpha} = a \otimes x_\alpha, \forall \alpha \in G\},$$

$$\begin{aligned} Q = \{&\underline{q} = (q_\alpha)_{\alpha \in G} \in \prod_{\alpha \in G} \sharp(C_{\alpha^{-1}}, A)_\alpha | \\ &q_\alpha(c_{(2,\alpha^{-1})})_{\psi_{\beta^{-1}}} \otimes c_{(1,\beta^{-1})}{}^{\psi_{\beta^{-1}}} = q_{\alpha\beta}(c) \otimes x_{\beta^{-1}}, c \in C_{(\alpha\beta)^{-1}}\} \end{aligned}$$

and

$$S = \{\underline{b} =(b_\alpha)_{\alpha\in G} \in \prod_{\alpha\in G} A | b_{\alpha\psi_{\beta^{-1}}} \otimes {x_{\beta^{-1}}}^{\psi_{\beta^{-1}}} = b_{\alpha\beta} \otimes x_{\beta^{-1}}\}.$$

From Theorem 5.1, we have a graded Morita context $\mathbb{GM} = (G * S, \sharp(\underline{C}, A), A\{G\}, QG, \omega', \nu')$ with connecting map ω' and ν' given by the formulas

$$\omega' : A\{G\} \otimes_{\sharp(\underline{C},A)} QG \to G * S,$$

$$\omega'(\mu_\alpha(a) \otimes_{\sharp(\underline{C},A)} \omega_\sigma(\underline{q})) = \mu_{\alpha\sigma}(a_{\psi_{(\sigma\beta)^{-1}}} q_{\sigma\beta}({x_{(\sigma\beta)^{-1}}}^{(\sigma\beta)^{-1}})),$$

$$\nu' : QG \otimes_{G*S} A\{G\} \to \sharp(\underline{C}, A),$$

$$\nu'(\omega_\sigma(\underline{q}) \otimes_{G*S} \mu_\alpha(a))(c) = q_{\sigma\alpha}(c)a, \forall c \in C_{(\sigma\alpha)^{-1}}.$$

Furthermore, if G is a trivial group, then $B = S$ and the graded Morita context $\mathbb{GM} = (G*S, \sharp(\underline{C}, A), A\{G\}, QG, \omega', \nu')$ recovers to the Morita context in the sense of [7, Section 4].

In order to proceed the further discussion, we need the following result [10].

5.2. Proposition. *Let A and E be rings, and $\underline{C}$ a G-A-coring, and M both a $\underline{C}$-comodule and a (E, A)-bimodule such that the comodule maps ρ_α are left E-linear. Then we have a pair of adjoint functors (F, U):*

$$F : \mathcal{M}_E \to \mathcal{M}^{G,\underline{C}},\ \underline{F(N)} = \{\mu_\alpha(N \otimes_E M)\}_{\alpha\in G}.$$

The coaction maps are

$$\rho_{\alpha,\beta} : \mu_{\alpha\beta}(N \otimes_E M) \to \mu_\alpha(N \otimes_E M) \otimes_A C_\beta,$$

$$\rho_{\alpha,\beta}(\mu_{\alpha\beta}(n \otimes_E m)) = \mu_\alpha(n \otimes_E m_{[0,\beta]}) \otimes_A m_{[1,\beta]}.$$

For $\underline{X} \in \mathcal{M}^{G,\underline{C}}$, define U as follows:

$$U : \mathcal{M}^{G,\underline{C}} \to \mathcal{M}_E,\ U_2(\underline{X}) = \mathrm{HOM}^{G,\underline{C}}(\mu_\alpha(M), \underline{X}).$$

Next, we apply Proposition 5.2 to the particular G-A-coring $\{A \otimes C_\alpha\}_{\alpha\in G}$ arising from $(A, \underline{C})_{G-\psi}$. Under the assumption that A is an object of $\mathcal{U}_A^{\underline{C}}(\psi)$ with the structure maps m_A and $\rho^A = \{\rho^A_\alpha\}$, we have a special pair of adjoint functors $(\widetilde{F}, \widetilde{U})$:

$$\widetilde{F} : \mathcal{M}_B \to \mathcal{U}_A^{G,\underline{C}}(\psi),\ \underline{\widetilde{F}(N)} = \{\mu_\alpha(N \otimes_B A)\}_{\alpha\in G}.$$

The coaction maps are

$$\rho_{\alpha,\beta} : \mu_{\alpha\beta}(N \otimes_B A) \to \mu_\alpha(N \otimes_B A) \otimes_A C_\beta,$$

$$\rho_{\alpha,\beta}(\mu_{\alpha\beta}(n \otimes_B a)) = \mu_\alpha(n \otimes_B 1_{A[0,\beta]} a_{\psi_\beta}) \otimes_A {1_{A[1,\beta]}}^{\psi_\beta}.$$

For $\underline{X} \in \mathcal{U}_A^{G,\underline{C}}(\psi)$, define $\widetilde{U}$ as follows:

$$\widetilde{U} : \mathcal{U}_A^{G,\underline{C}}(\psi) \to \mathcal{M}_B,\ \widetilde{U}(\underline{X}) = \mathrm{HOM}_A^{\underline{C}}(\mu_\alpha(A), \underline{X}).$$

For $\underline{M} \in \mathcal{U}_A^{G,\underline{C}}$, we define

$$\underline{M}^{co} = \{m = (m_\alpha)_{\alpha\in G} \in \prod_{\alpha\in G} M_\alpha | m_{[0,\alpha]} \otimes m_{[1,\beta]} = m \cdot 1_{A[0,\beta]} \otimes 1_{A[1,\beta]}\}.$$

Then we have

5.3. Lemma. *There exists an isomorphism*

$$\mathrm{HOM}_A^{\underline{C}}(\mu_\alpha(A), \underline{M}) \cong \underline{M}^{co}$$

as right B-modules.

Proof. For any $\underline{f} \in \mathrm{HOM}_A^{\underline{C}}(\mu_\alpha(A), \underline{M})$, since f is a right $\underline{C}$-comodule, then we have

$$\rho^{M}_{\alpha\beta}(f_{\alpha\beta}(\mu_{\alpha\beta}(1_A))) = f_\alpha(\mu_\alpha(1_{A[0,\beta]})) \otimes 1_{A[1,\beta]}.$$

Set $m = (m_\alpha)_{\alpha \in G}$, where $m_\alpha = f_\alpha(\mu_\alpha(1_A))$. Straightforward calculation can show that $\underline{m} \in \underline{M}^{co}$. Thus we define a map

$$\widehat{\Phi} : \mathrm{HOM}_A^{\underline{C}}(\mu_\alpha(A), \underline{M}) \to \underline{M}^{co},\ \widehat{\Phi}(\underline{f}) = (f_\alpha(\mu_\alpha(1_A)))_{\alpha \in G}.$$

Take $\underline{m} \in \underline{M}^{co}$, we define a map

$$\widehat{\Psi} : \underline{M}^{co} \to \mathrm{HOM}_A^{\underline{C}}(\mu_\alpha(A), \underline{M}),\ \widehat{\Psi}(m)_\alpha(\mu_\alpha(a)) = m_\alpha \cdot a.$$

It follow easily that $\widehat{\Phi}$ and $\widehat{\Psi}$ are both B-linear and mutually inverses. ☐

From Lemma 5.3 and what was discussed above, we have a pair of adjoint functors $(\widetilde{F}, \widetilde{U})$:

$$\widetilde{F} : \mathcal{M}_B \to \mathcal{U}_A^{G,\underline{C}}(\psi),\ \underline{\widetilde{F}(N)} = \{\mu_\alpha(N \otimes_B A)\}_{\alpha \in G}.$$

$$\widetilde{U} : \mathcal{U}_A^{G,\underline{C}}(\psi) \to \mathcal{M}_B,\ \widetilde{U}(\underline{X}) = \underline{X}^{co}.$$

By the discussion as above, and [7, Theorem 9.2], we can achieve the main goal in this section.

5.4. Theorem. *Let $(A, \underline{C})$ be a G-entwined structure. Suppose that $A \in \mathcal{U}_A^{\underline{C}}(\psi)$. Consider the map*

$$can : (A \otimes_B A)\langle G\rangle \to A \otimes \underline{C},\ can_\alpha(a \otimes_B b) = a1_{A[0,\alpha]}b_{\psi_\beta} \otimes {1_{A[1,\alpha]}}^{\psi_\beta}.$$

Then the following statements are equivalent:

1. *can is an isomorphism of group corings, and A is faithfully flat as a left B-module,*
2. *${}^*can$ is an isomorphism of graded rings and A is a left B-progenerator,*
3. *The graded Morita context $\mathbb{GM} = (G * S, \sharp(\underline{C}, A), A\{G\}, QG, \omega', \nu')$ is strict,*
4. *$(\widetilde{F}, \widetilde{U})$ is an equivalence of categories.*

As the end of this paper, we discuss the $(\underline{H}, A)$-Hopf module for an $\underline{H}$-comodule algebra A over a Hopf G-coalgebra $\underline{H}$.

Let $\underline{H} = (\{H_\alpha, m_\alpha, 1_\alpha, \Delta, \varepsilon\})$ be a Hopf G-coalgebra in the sense of [16] and A an algebra. We recall from [16] that a right $\underline{H}$-comodule algebra is a right $\underline{H}$-comodule $(A, \rho^A = \{\rho^A_\alpha\})$, such that the following conditions are satisfied:

- $\rho^A_\alpha(ab) = a_{[0,\alpha]}b_{[0,\alpha]} \otimes a_{[1,\alpha]}b_{[1,\alpha]}$ for all $a, b \in A$ and $\alpha \in G$,
- $\rho^A_\alpha(1_A) = 1_A \otimes 1_\alpha$ for all $\alpha \in G$.

Given an $\underline{H}$-comodule algebra A, we have a G-entwined structure $\psi_\alpha : H_\alpha \otimes A \to A \otimes H_\alpha,\ \psi_\alpha(h \otimes a) = a_{[0,\alpha]} \otimes ha_{[1,\alpha]}$. We call a special $(A, \underline{C})_\psi$-module a (right-right) $(\underline{H}, A)$-Hopf module and denote the category of $(\underline{H}, A)$-Hopf modules by $\mathcal{U}_A^{\underline{H}}$. It is easy to see that $A \in \mathcal{U}_A^{\underline{H}}$. Let us take the grouplike family $\{1_\alpha\}_{\alpha \in G}$. Then we have a graded Morita context $\mathbb{GM} = (G * S, \sharp(\underline{H}, A), A\{G\},\ QG, \omega', \nu')$ with connecting map ω' and ν' given by the formulas

$$\omega' : A\{G\} \otimes_{\sharp(\underline{H},A)} QG \to G * S,$$

$$\omega'(\mu_\alpha(a) \otimes_{\sharp(\underline{H},A)} \omega_\sigma(\underline{q})) = \mu_{\alpha\sigma}(a_{[0,(\sigma\beta)^{-1}]}q_{\sigma\beta}(a_{[1,(\sigma\beta)^{-1}]})),$$

$$\nu' : QG \otimes_{G*S} A\{G\} \to \sharp(\underline{H}, A),$$

$$\nu'(\omega_\sigma(\underline{q}) \otimes_{G*S} \mu_\alpha(a))(c) = q_{\sigma\alpha}(c)a, \forall c \in C_{(\sigma\alpha)^{-1}},$$

where

$$\begin{aligned} Q =\{&\underline{q} = (q_\alpha)_{\alpha\in G} \in \prod_{\alpha\in G} \sharp(H_{\alpha^{-1}}, A)_\alpha | \\ &q_\alpha(h_{(2,\alpha^{-1})})_{[0,\beta^{-1}]} \otimes h_{(1,\beta^{-1})} q_\alpha(h_{(2,\alpha^{-1})})_{[1,\beta^{-1}]} = q_{\alpha\beta}(h) \otimes 1_{\beta^{-1}}, h \in H_{(\alpha\beta)^{-1}}\} \end{aligned}$$

and

$$S = \{\underline{b} =(b_\alpha)_{\alpha\in G} \in \prod_{\alpha\in G} A | b_{\alpha[0,\beta^{-1}]} \otimes b_{\alpha[1,\beta^{-1}]} = b_{\alpha\beta} \otimes 1_{\beta^{-1}}\}.$$

5.5. Remark. If π is a trivial group, then $S = A^{coH}$ and

$$Q = \{q \in \sharp(H, A) | q(h_{(2)})_{[0]} \otimes h_{(1)} q(h)_{[1]} = q(h) \otimes 1_H, h \in H\}.$$

Hence, the graded Morita context $\mathbb{GM} = (G * S, \sharp(\underline{H}, A), A\{G\}, QG, \omega', \nu')$ is just the Morita context of Doi in [12].

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