Hacettepe Journal of Mathematics and Statistics
Volume 44 (6) (2015), 1351 – 1359

On strongly and nicely almost ω_1-$p^{\omega+n}$-projective Abelian p-groups

Peter Danchev*

Abstract

We define the classes of *strongly almost* ω_1-$p^{\omega+n}$*-projective* abelian p-groups and *nicely almost* ω_1-$p^{\omega+n}$*-projective* abelian p-groups as well as we study their crucial properties. Our results support those obtained by us in Hacettepe J. Math. Stat. (2014) and Korean J. Math. (2014).

Keywords: almost Σ-cyclic groups, almost $p^{\omega+n}$-projective groups, almost ω_1-$p^{\omega+n}$-projective groups, strongly almost ω_1-$p^{\omega+n}$-projective groups, nicely almost ω_1-$p^{\omega+n}$-projective groups.

2000 AMS Classification: 20K10.

Received : 17.09.2014 *Accepted* : 13.11.2014 *Doi* : 10.15672/HJMS.2015449676

1. Introduction and Terminology

Let all groups into consideration be p-primary abelian, where p is a fixed prime integer, written additively as it is customary. As usual, for some ordinal $\alpha \geq 0$ and a group G, we state the α*-th Ulm subgroup* $p^\alpha G$, consisting of all elements of G with height $\geq \alpha$, inductively as follows: $p^0 G = G$, $pG = \{pg \mid g \in G\}$, $p^\alpha G = p(p^{\alpha-1}G)$ if $\alpha - 1$ exists (so α is non-limit) and $p^\alpha G = \cap_{\beta<\alpha} p^\beta G$ if $\alpha - 1$ does not exist (so α is limit). The group G is called p^α*-bounded* if $p^\alpha G = \{0\}$; note that these groups are necessarily reduced. We shall say that G is *separable* if it is p^ω-bounded. Most of the important unexplained here notations and notions will follow mainly those from [9].

In their seminal work [12], Hill and Ullery have given the following critical concept.

• The reduced group G is called *almost totally projective* if it has a collection $\mathcal{C}$ consisting of nice subgroups of G satisfying the following three conditions:

(1) $\{0\} \in \mathcal{C}$;

(2) $\mathcal{C}$ is closed with respect to ascending unions, i.e., if $H_i \in \mathcal{C}$ with $H_i \subseteq H_j$ whenever $i \leq j$ $(i, j \in I)$ then $\cup_{i\in I} H_i \in \mathcal{C}$;

---

*Department of Mathematics, University of Plovdiv, Plovdiv 4000, Bulgaria
Email : `pvdanchev@yahoo.com, peter.danchev@yahoo.com`

(3) If K is a countable subgroup of G, then there is $L \in \mathcal{C}$ (that is, a nice subgroup L of G) such that $K \subseteq L$ and L is countable.

This concept generalizes the notion of *an almost direct sum of cyclic groups*, defined in [11], hereafter abbreviated as *almost* Σ*-cyclic*. Actually separable almost totally projective groups are almost Σ-cyclic. Moreover, the direct sum of a divisible group and an almost totally projective group is called *almost simply presented*. It readily follows that a group is almost simply presented if and only if its reduced part is almost simply presented as well as that the direct sum of almost simply presented groups is again an almost simply presented group.

Extending the meaning of almost Σ-cyclic groups, the current author defines in [2] (see also [4], [5] and [7]) the following:

• The group G is said to be *almost* $p^{\omega+n}$*-projective* if there is $B \leq G[p^n]$ such that G/B is almost Σ-cyclic.

Observe that when $n = 0$ we obtain almost Σ-cyclic groups, i.e., the almost p^{ω}-projective groups. Moreover, note that P is of necessity nice in G because G/P is separable.

• If there exists a countable subgroup $C \leq G$ of a group G with the property that G/C is almost $p^{\omega+n}$-projective, then we will say that G is *almost* ω_1*-*$p^{\omega+n}$*-projective* – see [7]. Note that by Theorem 2.15 of [7] the subgroup C can be taken to be nice in G.

The following two notions were stated in [4].

• A group G is said to be *almost weak* $p^{\omega\cdot 2+n}$*-projective* if there is an almost $p^{\omega+n}$-projective subgroup $H \leq G$ such that G/H is almost Σ-cyclic.

• A group G is said to be *almost* ω_1*-weak* $p^{\omega\cdot 2+n}$*-projective* if there is a countable subgroup $K \leq G$ such that G/K is almost weak $p^{\omega\cdot 2+n}$-projective.

On the other hand, in [2] it was formulated the following:

• The group G is said to be *almost* n*-simply presented* if there is $H \leq G[p^n]$ such that G/H is almost simply presented.

If G/H is almost totally projective, then we will say that G is *almost* n*-totally projective*.

In case that H is nice in G, we give

• The group G is called *nicely almost* n*-simply presented* if there exists a p^n-bounded nice subgroup $N \leq G$ with G/N almost simply presented.

On the other hand these groups could be termed as *strongly almost* n*-simply presented* and *strongly almost* n*-totally projective*, respectively.

Apparently almost $p^{\omega+n}$-projective groups are nicely almost n-totally projective.

We will now state our new machinery like this:

1.1 Definition. A group G is said to be *strongly almost* ω_1*-*$p^{\omega+n}$*-projective* if it contains a p^n-bounded nice subgroup P such that G/P is the sum of a countable group and an almost Σ-cyclic group.

1.2 Definition. A group G is said to be *nicely almost* ω_1*-*$p^{\omega+n}$*-projective* if it contains a nice subgroup X such that X is almost $p^{\omega+n}$-projective and G/X is countable.

The goal of the present paper is to give a comprehensive study of these two concept. The work is organized as follows: In the next section, we establish our basic results which are stated in two different subsections. In the final section, we list some interesting left-open questions.

And so, we come to

2. Main Results

We distribute the chief results into two subsections. We start with

2.1. Strongly Almost ω_1-$p^{\omega+n}$-Projective p-Groups. We begin here with two useful necessary and sufficient conditions when a group is strongly almost ω_1-$p^{\omega+n}$-projective. First, we need two more preliminaries.

The following can be seen in [2].

2.1. Lemma. *If C is a countable subgroup of a group A such that A/C is almost Σ-cyclic, then A is the sum of a countable group and an almost Σ-cyclic group.*

The following somewhat extends the corresponding result from [13] (see [8] and [2], too).

2.2. Proposition. *The group A is almost simply presented with countable $p^\omega A$ if and only if A is the sum of a countable group and an almost Σ-cyclic group.*

Proof. **"Necessity"**. In conjunction with [12], one may write that $A/p^\omega A$ is almost Σ-cyclic. We furthermore appeal to Lemma 2.1 to get the desired decomposition of the group A.

"Sufficiency". Write $A = C+S$, where C is countable and S is almost Σ-cyclic. Since $C \cap S \subseteq S$ is countable, there is a nice countable subgroup K of S such that $C \cap S \subseteq K$. Therefore, $A/K = [(C+K)/K] \oplus [S/K]$. But $p^\omega(S/K) = (p^\omega S + K)/K = \{0\}$, so that $p^\omega(A/K) = p^\omega((C+K)/K)$ is countable because it is obvious that the same is $(C+K)/K$. Thus $p^\omega A/(p^\omega A \cap K) \cong (p^\omega A + K)/K \subseteq p^\omega(A/K)$ is countable, whence so is $p^\omega A$ as asserted, since $p^\omega A \cap K$ is countable. ∎

The last can be slightly extended to the following one:

2.3. Lemma. *Suppose G is a group. Then the following are equivalent:*

(1) G is almost simply presented with countable $p^\omega G$;

(2) $G/p^\omega G$ is almost Σ-cyclic such that $p^\omega G$ is countable;

(3) G is the sum of a countable group and an almost Σ-cyclic group.

Proof. The implication (1) $\Rightarrow$ (2) follows from [12]. The implication (2) $\Rightarrow$ (3) follows from Lemma 2.1. The implication (3) $\Rightarrow$ (1) follows from Proposition 2.2. ∎

As a helpful consequence we derive:

2.4. Corollary. *(a) A subgroup of the sum of a countable group and an almost Σ-cyclic group is again the sum of a countable group and an almost Σ-cyclic group.*

(b) If G is the sum of a countable group and an almost Σ-cyclic group, then for each $\alpha \geq \omega$ the quotient $G/p^\alpha G$ is also the sum of a countable group and an almost Σ-cyclic group.

Proof. (a) By the usage of Lemma 2.3 (2), let $A \leq G$ where $G/p^\omega G$ is almost Σ-cyclic and $p^\omega G$ is countable. Thus

$$A/(A \cap p^\omega G) \cong (A + p^\omega G)/p^\omega G \subseteq G/p^\omega G$$

is almost Σ-cyclic with the aid of [1]. But $A \cap p^\omega G \leq p^\omega G$ is countable. Hence we employ Lemma 2.1 to deduce the desired claim.

(b) By virtue of Lemma 2.3 (2) we have that $G/p^\omega G$ is almost Σ-cyclic and $p^\omega G$ is countable. Consequently,

$$G/p^\omega G \cong (G/p^\alpha G)/(p^\omega G/p^\alpha G) = (G/p^\alpha G)/(p^\omega(G/p^\alpha G))$$

is almost Σ-cyclic with $p^\omega(G/p^\alpha G) = p^\omega G/p^\alpha G$ being countable. Again an application of point (2) in Lemma 2.3 gives the wanted claim. ∎

The next assertion gives two new necessary and sufficient conditions when a group is strongly almost ω_1-$p^{\omega+n}$-projective.

2.5. Proposition. *(a) A group G is strongly almost ω_1-$p^{\omega+n}$-projective if and only if there exists a p^n-bounded nice subgroup $N \leq G$ such that $p^\omega(G/N)$ is countable and $G/(N + p^\omega G)$ is almost Σ-cyclic.*

(b) A group G is strongly almost ω_1-$p^{\omega+n}$-projective if and only if there exists a countable subgroup K and a p^n-bounded nice subgroup N such that $G/(K + N)$ is almost Σ-cyclic.

Proof. (a) "$\Rightarrow$". By definition G/P is the sum of a countable group and an almost Σ-cyclic group for some nice subgroup P of G which is bounded by p^n. Since G/P is almost simply presented in conjunction with Proposition 2.2 (see [2] as well), we deduce that

$$(G/P)/p^\omega(G/P) = (G/P)/((p^\omega G + P)/P) \cong G/(p^\omega G + P)$$

is almost Σ-cyclic, as stated. That $p^\omega(G/P)$ is countable again follows directly from Proposition 2.2.

"$\Leftarrow$". Since $G/(N + p^\omega G) \cong [G/N]/[(N + p^\omega G)/N]$ is almost Σ-cyclic with countable quotient $(N + p^\omega G)/N = p^\omega(G/N)$, Lemma 2.1 leads us to this that G/N is the sum of a countable group and an almost Σ-cyclic subgroup, as expected.

(b) "$\Rightarrow$". Write $G/P = (A/P) + (B/P)$, where the first term A/P is countable and the second term B/P is almost Σ-cyclic for some $A, B \leq G$ and some nice subgroup P of G with $p^n P = \{0\}$. Since $(A/P) \cap (B/P) \subseteq B/P$ is countable, there is a countable nice subgroup C/P of B/P for some $C \leq B$ such that $(A/P) \cap (B/P) \subseteq C/P$. In accordance to [7], the factor-group $(B/P)/(C/P) \cong B/C$ is always almost Σ-cyclic. We also may write twice $A = K_1 + P$ and $C = K_2 + P$, where both K_1 and K_2 are countable groups. Furthermore, one can decompose

$$(G/P)/(C/P) = [((A/P) + (C/P))/(C/P)] \oplus [(B/P)/(C/P)].$$

Since $(G/P)/(C/P) \cong G/C$ as the first term of the above decomposition is isomorphic to $(A+C)/C$ while the second one is isomorphic to B/C, it is routinely seen that $G/(K+P) = G/(A+C) \cong B/C$ is almost Σ-cyclic for some countable subgroup $K = K_1 + K_2$, as required.

"$\Leftarrow$". Suppose that $G/(K+N)$ is almost Σ-cyclic, for some countable subgroup $K \leq G$ and some bounded by p^n nice subgroup $N \leq G$. Observing that $G/(K+N) \cong (G/N)/((K+N)/N)$, where $(K+N)/N \cong K/(K \cap N)$ is obviously countable, Lemma 2.1 allows us to conclude that G/N is the sum of a countable group and an almost Σ-cyclic group, as required in Definition 1.1. ∎

2.6. Corollary. *If G is strongly almost ω_1-$p^{\omega+n}$-projective, then so are both $p^\alpha G$ and $G/p^\alpha G$ for every ordinal α.*

Proof. Assume that $G/P = (L/P) + (S/P)$, where $L, S \leq G$ and L/P is countable while S/P is an almost Σ-cyclic group, for some nice subgroup $P \leq G$ with $p^n P = \{0\}$. Thus, with Corollary 2.4 at hand, all of

$$G/P \supseteq (p^\alpha G + P)/P \cong p^\alpha G/(p^\alpha G \cap P)$$

are also sums of countable groups and almost Σ-cyclic groups, where $p^\alpha G \cap P$ is nice in $p^\alpha G$, as needed.

Concerning the second half-part, it follows directly from Corollary 2.4, because the isomorphism sequence

$$(G/p^\alpha G)/((P + p^\alpha G)/p^\alpha G) \cong G/(P + p^\alpha G) \cong (G/P)/((P + p^\alpha G)/P).$$

holds. ∎

2.7. Corollary. *If G is a group such that $p^{\omega+n} G = \{0\}$, then G is strongly almost ω_1-$p^{\omega+n}$-projective if and only if G is almost $p^{\omega+n}$-projective.*

Proof. In accordance with Proposition 2.5, the quotient $G/(N + p^\omega G)$ is almost Σ-cyclic for some $N \leq G[p^n]$. Thus $p^n(N + p^\omega G) = \{0\}$ and the claim follows at once by definition. ∎

We are now ready to proceed by proving one of our basic results, which reduces the investigation of strongly almost ω_1-$p^{\omega+n}$-projective groups to groups of lengths not exceeding $\omega + n$.

2.8. Theorem. *For every $n \geq 1$ the group G is strongly almost ω_1-$p^{\omega+n}$-projective if and only if*

(1) $p^{\omega+n} G$ is countable;

(2) $G/p^{\omega+n} G$ is almost $p^{\omega+n}$-projective.

Proof. "$\Rightarrow$". According to Proposition 2.5, one may write that $p^\omega G/(p^\omega G \cap N)$ is countable for some p^n-bounded nice subgroup N of G. Thus $p^\omega G = p^\omega G \cap N + C$ where $C \leq p^\omega G$ is countable. Furthermore, $p^{\omega+n} G = p^n C$ is countable, so that clause (1) follows.

Next, point (2) follows directly from Corollary 2.6.

"$\Leftarrow$". Suppose that $P \leq G$ such that $p^{\omega+n} G \subseteq P$, $p^n P \subseteq p^{\omega+n} G$ (thereby $P/p^{\omega+n} G$ is p^n-bounded) and G/P is Σ-cyclic. Let Y be a maximal p^n-bounded summand of $p^\omega G$; so there is a decomposition $p^\omega G = X \oplus Y$ and thus the inclusions $X \subseteq p^\omega G \subseteq P$ hold. We may assume without loss of generality that X is countable; in fact, $p^{\omega+n} G = p^n X$

is countable and so we can decompose $X = K \oplus T$ where K is countable and T is p^n-bounded (whence T is a p^n-bounded summand of $p^\omega G$ and thereby $T \subseteq Y$; then even $T = T \cap Y \subseteq X \cap Y = \{0\}$ and $X = K$ - in any case $p^\omega G = K \oplus (T \oplus Y)$ where $T \oplus Y$ is p^n-bounded). That is why $p^\omega G = K \oplus Y$ with a countable summand K, as desired. An other verification of this fact is like this: Note that $X[p] = (p^{\omega+n}G)[p] = (p^n X)[p]$, and hence $X[p]$ is countable. So X will be countable, provided that it is reduced.

Let us now H be a $p^{\omega+n}$-high subgroup of G containing Y (thus H is maximal with respect to $H \cap p^{\omega+n}G = \{0\}$). We next assert that $(G/p^{\omega+n}G)[p^n] = (X \oplus H[p^n])/p^{\omega+n}G$. To this aim, given $v \in G$ with $p^n v \in p^{\omega+n}G$, it suffices to prove that $v \in X \oplus H[p^n]$. If $x \in X$ is chosen such that $p^n x = p^n v$, then replacing v by $v - x$, we may assume that $p^n v = 0$. Since $G[p] = (p^{\omega+n}G)[p] \oplus H[p] = X[p] \oplus H[p]$ and H is pure in G, it easily follows that $G[p^n] = X[p^n] \oplus H[p^n]$. Therefore, $v = x' + h$ where $x' \in X[p^n]$ and $h \in H[p^n]$ as required. Moreover, $X \cap H = \{0\}$ because as noted above $X[p] = (p^{\omega+n}G)[p]$, which substantiates our assertion. Furthermore, by what we have just shown above, $P/p^{\omega+n}G \subseteq (G/p^{\omega+n}G)[p^n]$ implies that $P \subseteq X \oplus H[p^n]$. Note also the fact from above that $X \leq P$. Let $L = P \cap H[p^n] \subseteq H[p^n] \subseteq G[p^n]$; so $p^n L = \{0\}$. Clearly, the inclusion $L \subseteq H$ forces that $L \cap p^{\omega+n}G = \{0\}$. Likewise, $P \subseteq X \oplus H[p^n]$ yields that $P = X + (P \cap H[p^n]) = X + L$; indeed the modular law applies to get that $P = (X \oplus H[p^n]) \cap P = X + P \cap H[p^n]$ as stated. Consequently, we conclude that $P = p^\omega G + P = p^\omega G + L$. Thus $G/P = G/(p^\omega G + L)$ is Σ-cyclic.

We next will show that L is nice in G. Since $L \cap p^{\omega+n}G = \{0\}$, it readily follows via some technical efforts that $L \cap p^\omega G$ is nice in $p^\omega G$ and so nice in G. But $L + p^\omega G = P$ is also nice in G because $G/(p^\omega G + L)$ is separable, and these two conditions together imply that L is nice in G, as wanted (see, e.g., Section 79, Exercise 10 of [9]).

Furthermore, we claim that $p^\omega(G/L) = (p^\omega G + L)/L = P/L$ is countable. In fact, $P/L = P/(P \cap H[p^n]) \cong (P + H[p^n])/H[p^n] = (p^\omega G + H[p^n])/H[p^n] \cong p^\omega G/(p^\omega G \cap H[p^n])$. But $p^\omega G = X \oplus Y$ and since $Y \subseteq H$, one may have in view of the modular law that $p^\omega G \cap H = (X \oplus Y) \cap H = (X \cap H) \oplus Y = Y$. We therefore establish that $P/L \cong (X \oplus Y)/Y[p^n] \cong X \oplus (Y/Y[p^n]) \cong X \oplus p^n Y = X$, because $p^n Y = \{0\}$. As noticed above, X is countable, so that $p^\omega(G/L)$ is really countable as claimed. Finally, Proposition 2.5 (a) allows us to infer that G is strongly ω_1-$p^{\omega+n}$-projective, as required. ∎

As a direct consequence, we obtain the following:

2.9. Corollary. *The group G is strongly almost n-simply presented with countable $p^{\omega+n}G$ if and only if G is strongly almost ω_1-$p^{\omega+n}$-projective.*

Proof. Concerning the necessity, in conjunction with [2], the quotient $G/p^{\omega+n}G$ is almost $p^{\omega+n}$-projective. We next apply Theorem 2.8 to get the desired assertion.

As for the sufficiency, it follows immediately from either Proposition 2.2 or Lemma 2.3 accomplished with Theorem 2.8. ∎

Another immediate consequence is the following one:

2.10. Corollary. *Strongly almost n-simply presented groups are almost ω_1-$p^{\omega+n}$-projective if and only if they are strongly almost ω_1-$p^{\omega+n}$-projective.*

Proof. The "if" part being elementary, we concentrate on the "and only if" part. To this aim, owing to [7], the Ulm subgroup $p^{\omega+n}G$ has to be countable. On the other hand, according to [2], the factor-group $G/p^{\omega+n}G$ must be almost $p^{\omega+n}$-projective. We therefore with Theorem 2.8 at hand deduce that G is strongly almost ω_1-$p^{\omega+n}$-projective groups, as claimed. ∎

2.11. Proposition. *The countable direct sum of strongly almost ω_1-$p^{\omega+n}$-projective groups is again a strongly almost ω_1-$p^{\omega+n}$-projective group.*

Proof. Write $G = \oplus_{i\in I} G_i$, where all summands G_i are strongly almost ω_1-$p^{\omega+n}$-projective groups, and $|I| = \aleph_0$. Thus, in view of Theorem 2.8, $p^{\omega+n}G = \oplus_{i\in I} p^{\omega+n}G_i$ remains countable. On the other vein, in virtue of [7] along with Theorem 2.8, the quotient $G/p^{\omega+n}G \cong \oplus_{i\in I} G_i/p^{\omega+n}G_i$ remains almost $p^{\omega+n}$-projective. We finally again take into account Theorem 2.8 to get the wanted assertion that G is a strongly almost ω_1-$p^{\omega+n}$-projective group. ■

2.12. Proposition. *Let $G = H \oplus K$, where K is a countable subgroup of a group G. Then G is a strongly almost ω_1-$p^{\omega+n}$-projective group if and only if H is a strongly almost ω_1-$p^{\omega+n}$-projective group.*

Proof. The "if" part follows directly from Proposition 2.11.

To treat the "and only if" part, since by Theorem 2.8 the group $p^{\omega+n}G$ is countable, it follows at once that so is its subgroup $p^{\omega+n}H$. Moreover, the direct decomposition $G/p^{\omega+n}G \cong (H/p^{\omega+n}H) \oplus (K/p^{\omega+n}K)$ implies with the aid of [7] that $H/p^{\omega+n}H$ is almost $p^{\omega+n}$-projective, because by virtue of Theorem 2.8 the same is $G/p^{\omega+n}G$. We consequently may now employ once again Theorem 2.8 to obtain that H is strongly almost ω_1-$p^{\omega+n}$-projective, as expected. ■

2.13. Proposition. *(i) Suppose $H \leq G$ with G/H finite. If H is strongly almost ω_1-$p^{\omega+n}$-projective, then G is strongly almost ω_1-$p^{\omega+n}$-projective.*

(ii) Suppose $F \leq G$ is finite. If G is strongly almost ω_1-$p^{\omega+n}$-projective, then G/F is strongly almost ω_1-$p^{\omega+n}$-projective.

Proof. (i) Write $G = H + F$ where $F \leq G$ is finite. By definition, let $H/P = (C/P) + (S/P)$, where the first term is countable while the second one is almost Σ-cyclic, for some p^n-bounded nice subgroup P of H. It follows that $G/P = [(C/P)+((F+P)/P)]+(S/P)$, where the first term remain countable. Owing to [2] or [3] it follows that P is nice in $H + F = G$, as required.

(ii) With Theorem 2.8 in hand, we know that $p^{\omega+n}G$ is countable and $G/p^{\omega+n}G$ is almost Σ-cyclic. But F being finite is nice in G, so that $p^{\omega+n}(G/F) = (p^{\omega+n}G+F)/F \cong p^{\omega+n}G/(F \cap p^{\omega+n}G)$ is also countable. Moreover,

$$(G/F)/p^{\omega+n}(G/F) \cong G/(p^{\omega+n}G + F) \cong (G/p^{\omega+n}G)/((p^{\omega+n}G + F)/p^{\omega+n}G).$$

Since $(p^{\omega+n}G + F)/p^{\omega+n}G) \cong F/(p^{\omega+n}G \cap F)$ is finite, we refer to [2] or to [7] to obtain that $(G/p^{\omega+n}G)/((p^{\omega+n}G + F)/p^{\omega+n}G)$ is almost Σ-cyclic, and hence so is $(G/F)/p^{\omega+n}(G/F)$ thus getting the wanted claim. ■

2.2. Nicely Almost ω_1-$p^{\omega+n}$-Projective p-Groups.

2.14. Proposition. *If G is nicely almost ω_1-$p^{\omega+n}$-projective, then so is $p^\alpha G$ for any ordinal α.*

Proof. Letting G/X be countable for some nice subgroup $X \leq G$ such that X is almost $p^{\omega+n}$-projective, one sees that $p^\alpha(G/X) = (p^\alpha G + X)/X \cong p^\alpha G/(p^\alpha G \cap X)$ remains also countable. Besides, in accordance to [9], $p^\alpha G \cap X$ is nice in $p^\alpha G$ as well as $p^\alpha G \cap X \subseteq X$ is almost $p^{\omega+n}$-projective by application of [7]. Thus Definition 1.2 is satisfied, as required. ■

2.15. Proposition. *The countable direct sum of nicely almost ω_1-$p^{\omega+n}$-projective groups is again a nicely almost ω_1-$p^{\omega+n}$-projective group.*

Proof. Write $G = \oplus_{i\in I} G_i$, where all summands G_i are strongly almost ω_1-$p^{\omega+n}$-projective groups, and $|I| = \aleph_0$. By definition, for each index $i \in I$, there is a nice subgroup $X_i \leq G_i$ such that G_i/X_i is countable and X_i is almost $p^{\omega+n}$=projective. Setting $X = \oplus_{i\in I} X_i$, one can see that X is nice in G, and X is almost $p^{\omega+n}$-projective by [7]. Moreover, $G/X \cong \oplus_{i\in I} G_i/X_i$ is countable, so that Definition 1.2 is applicable to obtain that G is a nicely almost ω_1-$p^{\omega+n}$-projective group, as promised. ■

2.16. Proposition. *(i) Suppose $H \leq G$ with G/H finite. If H is nicely almost ω_1-$p^{\omega+n}$-projective, then G is nicely almost ω_1-$p^{\omega+n}$-projective.*

(ii) Suppose $F \leq G$ is finite. If G is nicely almost ω_1-$p^{\omega+n}$-projective, then G/F is nicely almost ω_1-$p^{\omega+n}$-projective.

Proof. (i) Letting H/X be countable for some nice subgroup $X \leq H$ such that X is almost $p^{\omega+n}$-projective and writing $G = H + F$, where F is a finite subgroup of G, we observe that $G/X = (H/X) + ((F + X)/X)$ is countable. By [2] or [3], we have that X is nice in $H + F = G$, as required.

(ii) Let G/X be countable for some nice subgroup $X \leq G$ such that X is almost $p^{\omega+n}$-projective. Thus, as being its epimorphic image, $G/(X+F) \cong (G/F)/((X+F)/F)$ remains countable as well. But, in virtue of [2] or [3], $X + F$ is nice in G whence by [9] the factor-group $(X + F)/F$ is nice in G/F. Finally, [7] enables us that $(X + F)/F \cong X/(F \cap X)$ is almost $p^{\omega+n}$-projective, because $F \cap X$ is finite. Consequently, Definition 1.2 leads us to G/F is nicely almost ω_1-$p^{\omega+n}$-projective, as claimed. ■

3. Open Problems

We state here two problems of interest.

Problem 1. Does it follow that strongly almost ω_1-$p^{\omega+n}$-projective groups are nicely almost ω_1-$p^{\omega+n}$-projective?

Removing off the word "almost" this is true (see [3]). However, the same proof does not work directly in the current case, because the two definitions are almost identical.

Problem 2. Decide whether or not nicely almost ω_1-$p^{\omega+n}$-projective groups are nicely almost n-simply presented.

Acknowledgment. The author would like to thank the referee for the valuable comments on the present work.

References

[1] B. Balof and P. Keef, *Invariants on primary abelian groups and a problem of Nunke,* Note Mat. **29** (2009), 83–114.

[2] P. V. Danchev, *On almost n-simply presented abelian p-groups,* Korean J. Math. **21** (2013), 401–419.

[3] P. V. Danchev, *On strongly and separably ω_1-$p^{\omega+n}$-projective abelian p-groups,* Hacettepe J. Math. Stat. **43** (2014), 51–64.

[4] P. V. Danchev, *On almost ω_1-weak $p^{\omega\cdot 2+n}$-projective abelian p-groups,* Georgian Math. J. **21** (2014).
[5] P. V. Danchev, *Primary abelian almost n-Σ-groups,* Ric. Mat. **63** (2014).
[6] P. V. Danchev, *On nicely and separately ω_1-$p^{\omega+n}$-projective abelian p-groups,* Math. Reports (2015).
[7] P. V. Danchev, *On almost ω_1-$p^{\omega+n}$-projective abelian p-groups,* Korean J. Math. **22** (2014), 501-516.
[8] P. V. Danchev and P. W. Keef, *An application of set theory to $\omega+n$-totally $p^{\omega+n}$-projective primary abelian groups,* Mediterr. J. Math. **8** (2011), 525–542.
[9] L. Fuchs, Infinite Abelian Groups, volumes I and II, Acad. Press, New York and London, 1970 and 1973.
[10] P. Griffith, Infinite Abelian Group Theory, The University of Chicago Press, Chicago-London, 1970.
[11] P. D. Hill, *Almost coproducts of finite cyclic groups,* Commentat. Math. Univ. Carolin. **36** (1995), 795–804.
[12] P. D. Hill and W. D. Ullery, *Almost totally projective groups,* Czechoslovak Math. J. **46** (1996), 249–258.
[13] C. K. Megibben, *On high subgroups*, Pac. J. Math. **14** (1964), 1353–1358.