Exact moments of generalized order statistics from type II exponentiated log-logistic distribution

Devendra Kumar [*]

Abstract

In this paper some new simple expressions for single and product moments of generalized order statistics from type II exponentiated log-logistic distribution have been obtained. The results for order statistics and record values are deduced from the relations derived and some ratio and inverse moments of generalized order statistics are also carried out. Further, a characterization result of this distribution by using the conditional expectation of generalized order statistics is discussed.

1. Introduction

A random variable X is said to have type II exponentiated log-logistic distribution if its probability density function (*pdf*) is given by

$$f(x) = \frac{\alpha\beta(x/\sigma)^{\beta-1}}{\sigma[1+(x/\sigma)^\beta]^{\alpha+1}}, \quad x \geq 0, \ \alpha, \ \sigma > 0, \ \beta > 1 \tag{1.1}$$

and the corresponding survival function is

$$\bar{F}(x) = \left(1 + \left\{\frac{x}{\sigma}\right\}^\beta\right)^{-\alpha}, \quad x \geq 0, \ \alpha, \ \sigma > 0, \ \beta > 1. \tag{1.2}$$

It is easy to see that

$$\alpha\beta\bar{F}(x) = \sigma[1+(x/\sigma)^\beta]xf(x). \tag{1.3}$$

Log-logistic distribution is considered as a special case of type II exponentiated log-logistic distribution when $\alpha = 1$. It is used in survival analysis as a parametric model where in the mortality rate first increases then decreases, for example in cancer diagnosis

─────────
[*]Department of Statistics, Amity Institute of Applied Sciences Amity University, Noida-201 303, India
Email: devendrastats@gmail.com

or any other type of treatment. It has also been used in hydrology to model stream flow and precipitation, and in economics to model the distribution of wealth or income.

Kamps [24] introduced the concept of generalized order statistics (gos) as follows: Let $X_1, X_2 \ldots$ be a sequence of independent and identically distributed (iid) random variables (rv) with absolutely continuous cumulative distribution function (cdf) $F(x)$ and pdf, $f(x)$, $x \in (\alpha, \beta)$. Let $n \in N$, $n \geq 2$, $k > 0$, $m \in \Re$, be the parameters such that

$$\gamma_r = k + (n - r)(m + 1) > 0, \quad \text{for all} \quad r \in \{1, 2, \ldots, n - 1\},$$

where $M_r = \sum_{j=r}^{n-1} mj$. Then $X(1, n, m, k), \ldots, X(n, n, m, k)$, $r = 1, 2, \ldots n$ are called gos if their joint pdf is given by

$$k \left(\prod_{j=1}^{n-1} \gamma_j \right) \left(\prod_{i=1}^{n-1} [1 - F(x_i)]^m f(x_i) \right) [1 - F(x_n)]^{k-1} f(x_n) \tag{1.4}$$

on the cone $F^{-1}(0) \leq x_1 \leq x_2 \leq \ldots \leq x_n \leq F^{-1}(1)$.

The model of gos contains as special cases, order statistics, record values, sequential order statistics.

Choosing the parameters appropriately (Cramer, [18]), we get the variant of the gos given in Table 1.

Table 1: Variants of the generalized order statistics

	$\gamma_n = k$	γ_r	m_r
i) Sequential order statistics	α_n	$(n - r + 1)\alpha_r$	$\gamma_r - \gamma_{r+1} - 1$
ii) Ordinary order statistics	1	$n - r + 1$	0
ii) Record values	1	1	-1
iv) Progressively type II censored order statistics	$R_n + 1$	$n - r + 1 + \sum_{j=r}^{n} R_j$	R_r
v) Pfeifer's record values	β_n	β_r	$\beta_r - \beta_{r+1} - 1$

For simplicity we shall assume $m_1 = m_2 = \ldots = m_{n-1} = m$.

The pdf of the $r-$th gos, $X(r, n, m, k)$, $1 \leq r \leq n$, is

$$f_{X(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!} [\bar{F}(x)]^{\gamma_r - 1} f(x) g_m^{r-1}(F(x)) \tag{1.5}$$

and the joint pdf of $X(r, n, m, k)$ and $X(s, n, m, k)$, $1 \leq r < s \leq n$, is

$$f_{X(r,n,m,k),X(s,n,m,k)}(x, y) = \frac{C_{s-1}}{(r-1)!(s-r-1)!} [\bar{F}(x)]^m f(x) g_m^{r-1}(F(x))$$

$$\times [h_m(F(y)) - h_m(F(x))]^{s-r-1} [\bar{F}(y)]^{\gamma_s - 1} f(y), \quad x < y, \tag{1.6}$$

where

$$\bar{F}(x) = 1 - F(x), \quad C_{r-1} = \prod_{i=1}^{r} \gamma_i, \quad \gamma_i = k + (n-i)(m+1),$$

$$h_m(x) = \begin{cases} -\frac{1}{m+1}(1-x)^{m+1}, & m \neq -1 \\ -ln(1-x), & m = -1 \end{cases}$$

and

$$g_m(x) = h_m(x) - h_m(1), \ \ x \in [0, 1).$$

Theory of record values and its distributional properties have been extensively studied in the literature, Ahsanullah [4], Balakrishnan *et al*. [12], Nevzorov [33], Glick [21] and Arnold *et al*. [8, 9]. Resnick [35] discussed the asymptotic theory of records. Sequential order statistics have been studies by Arnold and Balakrishnan [7], Kamps [24], Cramer and Kamps [19] and Schenk [37], among others.

Aggarawala and Balakrishnan [1] established recurrence relations for single and product moments of progressive type II right censored order statistics from exponential and truncated exponential distributions. Balasooriya and saw [14] develop reliability sampling plans for the two parameter exponential distribution under progressive censoring. Balakrishnan *et al*. [13] obtained bounds for the mean and variance of progressive type II censored order statistics. Ordinary via truncated distributions and censoring schemes and particularly progressive type II censored order statistics have been discuss by Kamps [24] and Balakrishnan and Aggarwala [10], among others.

Kamps [24] investigated recurrence relations for moments of *gos* based on non-identically distributed random variables, which contains order statistics and record values as special cases. Cramer and Kamps [20] derived relations for expectations of functions of *gos* within a class of distributions including a variety of identities for single and product moments of ordinary order statistics and record values as particular cases. Various developments on *gos* and related topics have been studied by Kamps and Gather [23], Ahsanullah [5], Pawlas and Szynal [34], Kamps and Cramer [22], Ahmad and Fawzy [2], Ahmad [3], Kumar [27, 28, 29] among others. Characterizations based on *gos* have been studied by some authors, Keseling [25] characterized some continuous distributions based on conditional distributions of *gos* . Bieniek and Szynal [15] characterized some distributions via linearity of regression of *gos*. Cramer *et al*. [17] gave a unifying approach on characterization via linear regression of ordered random variables. Khan *et al*. [26] characterized some continuous distributions through conditional expectation of functions of *gos*.

The aim of the present study is to give some explicit expressions and recurrence relations for single and product moments of *gos* from type II exponentiated log-logistic distribution. In Section 2, we give the explicit expressions and recurrence relations for single moments of type II exponentiated log-logistic distribution and some inverse moments of *gos* are also worked out. Then we show that results for order statistics and record values are deduced as special cases. In Section 3, we present the explicit expressions and recurrence relations for product moments of type II exponentiated log-logistic distribution and we show that results for order statistics and record values are deduced as special cases and ratio moments of *gos* are also established. Section 4, provides a characterization result on type II exponentiated log-logistic distribution based on conditional moment of *gos*. Two applications are performed in Section 5. Some concluding remarks are given in Section 6.

2. Relations for single Moments

In this Section, the explicit expressions, recurrence relations for single moments of *gos* and inverse moments of *gos* are considered. First we need the basic result to prove the main Theorem.

2.1. Lemma. *For type II exponentiated log-logistic distribution as given in (1.2) and any non-negative and finite integers a and b with m $\neq$ −1*

$$J_j(a,0) = \alpha\sigma^j \sum_{p=0}^{\infty} \frac{(-1)^p(j/\beta)_{(p)}}{[\alpha(a+1)+p-(j/\beta)]}, \quad \beta > j \ \ and \ \ j = 0,1,2,\ldots, \tag{2.1}$$

where

$$(\alpha)_{(i)} = \begin{cases} \alpha(\alpha+1)\ldots(\alpha+i-1), & i>0 \\ 1, & i=0 \end{cases}.$$

and

$$J_j(a,b) = \int_0^{\infty} x^j[\bar{F}(x)]^a f(x)g_m^b(F(x))dx. \tag{2.2}$$

Proof From (2.2), we have

$$J_j(a,0) = \int_0^{\infty} x^j[\bar{F}(x)]^a f(x)dx. \tag{2.3}$$

By making the substitution $z = [\bar{F}(x)]^{1/\alpha}$ in (2.3), we get

$$J_j(a,0) = \alpha\sigma^j \int_0^{\infty} (1-z)^{j/\beta} z^{\alpha(a+1)-(j/\beta)-1}dz.$$

$$= \alpha\sigma^j \sum_{p=0}^{\infty} (-1)^p(j/\beta)_{(p)} \int_0^1 z^{\alpha(a+1)-(j/\beta)+p-1}dz$$

and hence the result given in (2.1).

2.2. Lemma. *For type II exponentiated log-logistic distribution as given in (1.2) and any non-negative and finite integers a and b*

$$J_j(a,b) = \frac{1}{(m+1)^b} \sum_{u=0}^{b} (-1)^u \begin{pmatrix} b \\ u \end{pmatrix} J_j(a+u(m+1),0) \tag{2.4}$$

$$= \frac{\alpha\sigma^j}{(m+1)^b} \sum_{p=0}^{\infty} \sum_{u=0}^{b} (-1)^{p+u} \begin{pmatrix} b \\ u \end{pmatrix} \frac{(j/\beta)_{(p)}}{[\alpha\{a+(m+1)u+1\}+p-(j/\beta)]},$$

$$m \neq -1 \tag{2.5}$$

$$= \alpha^{b+1}\sigma^j b! \sum_{p=0}^{\infty} \frac{(j/\beta)_{(p)}}{[\alpha(a+1)+p-(j/\beta)]^{b+1}}, \quad m = -1, \tag{2.6}$$

where $J_j(a,b)$ is as given in (2.2).

Proof: On expanding $g_m^b(F(x)) = \left[\frac{1}{m+1}(1-(F(x))^{m+1})\right]^b$ binomially in (2.2), we get when $m \neq -1$

$$J_j(a,b) = \frac{1}{(m+1)^b} \sum_{u=0}^{b} (-1)^u \begin{pmatrix} b \\ u \end{pmatrix} \int_0^{\infty} x^j[F(x)]^{a+u(m+1)} f(x)dx$$

$$= \frac{1}{(m+1)^b} \sum_{u=0}^{b} (-1)^u \begin{pmatrix} b \\ u \end{pmatrix} J_j(a+u(m+1),0).$$

Making use of Lemma 2.1, we establish the result given in (2.5)
and when $m = -1$ that
$J_j(a,b) = \frac{0}{0}$ as $\sum_{u=0}^{b} (-1)^u \begin{pmatrix} b \\ u \end{pmatrix} = 0$.
Since (2.5) is of the form $\frac{0}{0}$ at $m = -1$, therefore, we have

$$J_j(a,b) = A\sum_{u=0}^{b}(-1)^u \begin{pmatrix} b \\ u \end{pmatrix} \frac{[\alpha\{a+u(m+1)+1\}+p-(j/\beta)]^{-1}}{(m+1)^b}, \qquad (2.7)$$

where

$$A = \alpha\sigma^j \sum_{p=0}^{\infty}(-1)^p(j/\beta)_{(p)}.$$

Differentiating numerator and denominator of (2.7) b times with respect to m, we get

$$J_j(a,b) = A\alpha^b \sum_{u=0}^{b}(-1)^{u+b} \begin{pmatrix} b \\ u \end{pmatrix} \frac{u^b}{[\alpha\{a+u(m+1)+1\}+p-(j/\beta)]^{b+1}}.$$

On applying the L' Hospital rule, we have

$$lim_{m\to-1}J_j(a,b) = A\alpha^b \sum_{u=0}^{b}(-1)^{u+b} \begin{pmatrix} b \\ u \end{pmatrix} \frac{u^b}{[\alpha(a+1)+p-(j/\beta)]^{b+1}}. \qquad (2.8)$$

But for all integers $n \geq 0$ and for all real numbers x, we have Ruiz [36]

$$\sum_{i=0}^{n}(-1)^i \begin{pmatrix} n \\ i \end{pmatrix} (x-i)^n = n!. \qquad (2.9)$$

Therefore,

$$\sum_{u=0}^{b}(-1)^{u+b} \begin{pmatrix} b \\ u \end{pmatrix} u^b = b!. \qquad (2.10)$$

Now on substituting (2.10) in (2.8), we have the result given in (2.6).

2.3. Theorem. *For type II exponentiated log-logistic distribution as given in (1.2) and* $1 \leq r \leq n,\ k = 1, 2, \ldots$ *and* $m \neq -1$

$$E[X^j(r,n,m,k)] = \frac{C_{r-1}}{(r-1)!}J_j(\gamma_r-1,r-1) \qquad (2.11)$$

$$= \frac{\alpha\sigma^j C_{r-1}}{(r-1)!(m+1)^{r-1}}\sum_{p=0}^{\infty}\sum_{u=0}^{r-1}(-1)^{p+u} \begin{pmatrix} r-1 \\ u \end{pmatrix}$$

$$\times \frac{(j/\beta)_{(p)}}{[\alpha\gamma_{r-u}+p-(j/\beta)]}, \quad \beta > j \ \ and \ \ j = 0, 1, 2, \ldots \qquad (2.12)$$

where $J_j(\gamma_r-1,r-1)$ is as defined in (2.2).
Proof. From (1.5) and (2.2), we have

$$E[X^j(r,n,m,k)] = \frac{C_{r-1}}{(r-1)!}J_j(\gamma_r-1,r-1)$$

Making use of Lemma 2.2, we establish the result given in (2.12).
Identity 2.1. For $\gamma_r \geq 1,\ k \geq 1,\ 1 \leq r \leq n$ and $m \neq -1$

$$\sum_{u=0}^{r-1}(-1)^u \begin{pmatrix} r-1 \\ u \end{pmatrix} \frac{1}{\gamma_{r-u}} = \frac{(r-1)!(m+1)^{r-1}}{\prod_{t=1}^{r}\gamma_t}. \qquad (2.13)$$

Proof. (2.13) can be proved by setting $j=0$ in (2.12).
Special Cases

i) Putting $m = 0$, $k = 1$ in (2.12), the explicit formula for the single moments of order statistics of the type II exponentiated log-logistic distribution can be obtained as

$$E[X_{r:n}^j] = C_{r:n} \sum_{p=0}^{\infty} \sum_{u=0}^{r-1} (-1)^{u+p} \left(\begin{array}{c} r-1 \\ u \end{array} \right) \frac{\alpha \sigma^j (j/\beta)_{(p)}}{[\alpha(n-r+u+1)+p-(j/\beta)]},$$

where

$$C_{r:n} = \frac{n!}{(r-1)!(n-r)!},$$

ii) Setting $m = -1$ in (2.12), we deduce the explicit expression for the single moments of upper k record values for type II exponentiated log-logistic distribution in view of (2.11) and (2.6) in the form

$$E[X^j(r, n, -1, k)] = E[(Z_r^{(k)})^j] = (\alpha k)^r \sigma^j \sum_{p=0}^{\infty} \frac{(-1)^p (j/\beta)_{(p)}}{[\alpha k + p - (j/\beta)]^r}$$

and hence for upper records

$$E[(Z_r^{(1)})^j] = E[X_{U(r)}^j] = \alpha^r \sigma^j \sum_{p=0}^{\infty} \frac{(-1)^p (j/\beta)_{(p)}}{[\alpha + p - (j/\beta)]^r}.$$

Recurrence relations for single moments of *gos* from (1.5) can be obtained in the following theorem.

2.4. Theorem. *For the distribution given in (1.2) and $2 \leq r \leq n$, $n \geq 2$ and $k = 1, 2, \ldots$,*

$$\left(1 - \frac{\sigma j}{\alpha \beta \gamma_r} \right) E[X^j(r, n, m, k)] = E[X^j(r-1, n, m, k)]$$

$$+ \frac{j \sigma^{\beta+1}}{\alpha \beta \gamma_r} E[X^{j-\beta}(r, n, m, k)]. \tag{2.14}$$

Proof. From (1.5), we have

$$E[X^j(r, n, m, k)] = \frac{C_{r-1}}{(r-1)!} \int_0^{\infty} x^j [\bar{F}(x)]^{\gamma_r - 1} f(x) g_m^{r-1}(F(x)) dx. \tag{2.15}$$

Integrating by parts treating $[\bar{F}(x)]^{\gamma_r - 1} f(x)$ for integration and rest of the integrand for differentiation, we get

$$E[X^j(r, n, m, k)] = E[X^j(r-1, n, m, k)] + \frac{j C_{r-1}}{\gamma_r (r-1)!} \int_0^{\infty} x^{j-1} [\bar{F}(x)]^{\gamma_r} g_m^{r-1}(F(x)) dx$$

the constant of integration vanishes since the integral considered in (2.15) is a definite integral. On using (1.3), we obtain

$$E[X^j(r, n, m, k)] - E[X^j(r-1, n, m, k)]$$

$$= \frac{\sigma j C_{r-1}}{\alpha \beta \gamma_r (r-1)!} \int_0^{\infty} x^j [\bar{F}(x)]^{\gamma_r - 1} f(x) g_m^{r-1}(F(x)) dx$$

$$+ \frac{\sigma^{\beta+1} j C_{r-1}}{\alpha \beta \gamma_r (r-1)!} \int_0^{\infty} x^{j-\beta} [\bar{F}(x)]^{\gamma_r - 1} f(x) g_m^{r-1}(F(x)) dx$$

and hence the result given in (2.14).

Remark 2.1: Setting $m = 0$, $k = 1$, in (2.14), we obtain a recurrence relation for single moments of order statistics for type II exponentiated log-logistic distribution in the form

$$\left(1 - \frac{\sigma j}{\alpha \beta (n-r+1)} \right) E[X_{r:n}^j] = E[X_{r-1:n}^j] + \frac{j \sigma^{\beta+1}}{\alpha \beta (n-r+1)} E[X_{r-1:n}^{j-\beta}].$$

Remark 2.2: Putting $m = -1$, in Theorem 2.4, we get a recurrence relation for single moments of upper k record values from type II exponentiated log-logistic distribution in the form

$$\left(1 - \frac{\sigma j}{\alpha \beta k}\right) E[(X_{U(r)}^{(k)})^j] = E[(X_{U(r-1)}^{(k)})^j] + \frac{j\sigma^{\beta+1}}{\alpha \beta k} E[(X_{U(r)}^{(k)})^{j-\beta}].$$

Inverse moments of *gos* from type II exponentiated log-logistic distribution can be obtain by the following Theorem.

2.5. Theorem. *For type II exponentiated log-logistic distribution as given in (1.2) and* $1 \le r \le n$, $k = 1, 2, \ldots$,

$$E[X^{j-\beta}(r, n, m, k)] = \sum_{p=0}^{\infty} \frac{\sigma^{j-\beta}(-1)^p \Gamma\left(\frac{j}{\beta}\right)}{p! \Gamma\left(\frac{j}{\beta} - p\right) \prod_{i=1}^{r} \left(1 + \frac{p+1-(j/\beta)}{\alpha \gamma_i}\right)}, \quad \beta > j. \tag{2.16}$$

Proof. From (1.5), we have

$$E[X^{j-\beta}(r, n, m, k)] = \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}} \sum_{u=0}^{r-1} (-1)^u \left(\begin{array}{c} r-1 \\ u \end{array} \right)$$

$$\times \int_0^{\infty} x^{j-\beta} [\bar{F}(x)]^{\gamma_{r-u}-1} f(x) dx. \tag{2.17}$$

Now letting $t = [\bar{F}(x)]^{1/\alpha}$ in (2.17), we get

$$E[X^{j-\beta}(r, n, m, k)] = \frac{\sigma^{j-\beta} C_{r-1}}{(r-1)!(m+1)^r} \sum_{u=0}^{r-1} \sum_{p=0}^{\infty} (-1)^{u+p} \left(\begin{array}{c} r-1 \\ u \end{array} \right) \frac{\Gamma\left(\frac{j}{\beta}\right)}{p! \Gamma\left(\frac{j}{\beta} - p\right)}$$

$$\times B\left(\frac{k}{m+1} + n - r + u + \frac{p+1-(j/\beta)}{\alpha(m+1)}, 1\right).$$

Since

$$\sum_{a=0}^{b} (-1)^a \left(\begin{array}{c} b \\ a \end{array} \right) B(a+k, c) = B(k, c+b) \tag{2.18}$$

where $B(a, b)$ is the complete beta function.
Therefore,

$$E[X^{j-\beta}(r, n, m, k)] = \frac{\sigma^{j-\beta} C_{r-1}}{(m+1)^r} \sum_{p=0}^{\infty} (-1)^p \frac{\Gamma\left(\frac{j}{\beta}\right)}{p! \Gamma\left(\frac{j}{\beta} - p\right)}$$

$$\times \frac{\Gamma\left(\frac{\alpha\{k+(n-r)(m+1)\}+p+1-(j/\beta)}{\alpha(m+1)}\right)}{\Gamma\left(\frac{\alpha\{k+n(m+1)\}+p+1-(j/\beta)}{\alpha(m+1)}\right)} \tag{2.19}$$

and hence the result given in (2.16).
Special Cases
iii) Putting $m = 0$, $k = 1$ in (2.19), we get inverse moments of order statistics from type II exponentiated log-logistic distribution as;

$$E[X_{r:n}^{j-\beta}] = \frac{\sigma^{j-\beta} n!}{(n-r)!} \sum_{p=0}^{\infty} \frac{(-1)^p \Gamma\left(\frac{j}{\beta}\right) \Gamma[\alpha(n-r+1)+p+1-(j/\beta)]}{p! \Gamma\left(\frac{j}{\beta} - p\right) \Gamma[\alpha(n+1)+p+1-(j/\beta)]}.$$

iv) Putting $m = -1$ in (2.16), to get inverse moments of k record values from type II exponentiated log-logistic distribution as;

$$E[X_{U(r)}^{j-\beta}] = \sigma^{j-\beta} \sum_{p=0}^{\infty} \frac{(-1)^p \Gamma\left(\frac{j}{\beta}\right)}{p!\Gamma\left(\frac{j}{\beta} - p\right)\left(1 + \frac{p+1-(j/\beta)}{\alpha k}\right)^r}.$$

Recurrence relations for inverse moments of *gos* from (1.2) can be obtained in the following theorem.

2.6. Theorem. *For type II exponentiated log-logistic distribution and for $2 \leq r \leq n$, $n \geq 2$ $k = 1, 2, \ldots,$*

$$\left(1 - \frac{\sigma(j - \beta)}{\alpha\beta\gamma_r}\right) E[X^{j-\beta}(r, n, m, k)] = E[X^{j-\beta}(r-1, n, m, k)]$$

$$+ \frac{(j-\beta)\sigma^{\beta+1}}{\alpha\beta\gamma_r} E[X^{j-2\beta}(r, n, m, k)], \quad \beta > j. \tag{2.20}$$

Proof. The proof is easy.

Remark 2.3: Setting $m = 0$, $k = 1$ in (2.20), we obtain a recurrence relation for inverse moments of order statistics for type II exponentiated log-logistic distribution in the form

$$\left(1 - \frac{\sigma(j - \beta)}{\alpha\beta(n - r + 1)}\right) E[X_{r:n}^{j-\beta}] = E[X_{r-1:n}^{j-\beta}] + \frac{(j-\beta)\sigma^{\beta+1}}{\alpha\beta(n - r + 1)} E[X_{r:n}^{j-2\beta}].$$

Remark 2.4: Putting $m = -1$, in Theorem 2.6, we get a recurrence relation for inverse moments of upper k record values from type II exponentiated log-logistic distribution in the form

$$\left(1 - \frac{\sigma(j - \beta)}{\alpha\beta k}\right) E[(X_{U(r)}^{(k)})^{j-\beta}] = E[(X_{U(r-1)}^{(k)})^{j-\beta}] + \frac{(j-\beta)\sigma^{\beta+1}}{\alpha\beta k} E[(X_{U(r)}^{(k)})^{j-2\beta}].$$

3. Relations for product moments

In this Section, the explicit expressions and recurrence relations for single moments of *gos* and ratio moments of *gos* are considered. First we need the following Lemmas to prove the main result.

3.1. Lemma. *For type II exponentiated log-logistic distribution as given in (1.2) and any non-negative integers a, b, c with $m \neq -1$*

$$J_{i,j}(a, 0, c) = \alpha^2 \sigma^{i+j} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{(-1)^{p+q}(j/\beta)_{(p)}(j/\beta)_{(q)}}{[\alpha(c+1) + p - (j/\beta)]}$$

$$\times \frac{1}{[\alpha(a + c + 2) + p + q - \{(i + j)/\beta\}]}, \tag{3.1}$$

where

$$J_{i,j}(a, b, c) = \int_0^{\infty} \int_x^{\infty} x^i y^j [\bar{F}(x)]^a f(x)[h_m(F(y)) - h_m(F(x))]^b [\bar{F}(y)]^c f(y) dy dx. \tag{3.2}$$

Proof: From (3.2), we have

$$J_{i,j}(a, 0, c) = \int_0^{\infty} x^i [\bar{F}(x)]^a f(x) G(x) dx, \tag{3.3}$$

where

$$G(x) = \int_x^{\infty} y^j [\bar{F}(y)]^c f(y) dy. \tag{3.4}$$

By setting $z = [\bar{F}(y)]^{1/\alpha}$ in (3.4), we find that

$$G(x) = \alpha\sigma^j \sum_{p=0}^{\infty}(-1)^p \frac{(j/\beta)_p[\bar{F}(x)]^{c+1+\{p-(j/\beta)\}/\alpha}}{[\alpha(c+1)+p-(j/\beta)]}.$$

On substituting the above expression of $G(x)$ in (3.3), we get

$$J_{i,j}(a,0,c) = \alpha\sigma^j \sum_{p=0}^{\infty} \frac{(-1)^p(j/\beta)_p}{[\alpha(c+1)+p-(j/\beta)]}$$

$$\times \int_0^{\infty} x^i[\bar{F}(x)]^{a+c+1+\{p-(j/\beta)\}/\alpha}f(x)dx. \tag{3.5}$$

Again by setting $t = [\bar{F}(x)]^{1/\alpha}$ in (3.5) and simplifying the resulting expression, we derive the relation given in (3.1).

3.2. Lemma. *For the distribution as given in (1.2) and any non-negative integers a, b, c*

$$J_{i,j}(a,b,c) = \frac{1}{(m+1)^b}\sum_{v=0}^{b}(-1)^v \binom{b}{v} J_{i,j}(a+(b-v)(m+1),0,c+v(m+1)) \tag{3.6}$$

$$= \frac{\alpha^2\sigma^{i+j}}{(m+1)^b}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{v=0}^{b}(-1)^{p+q+v}\binom{b}{v}\frac{(j/\beta)_p}{[\alpha\{c+(m+1)v+1\}+p-(j/\beta)]}$$

$$\times\frac{(i/\beta)_q}{[\alpha\{a+c+(m+1)b+2\}+p+q-\{(i+j)/\beta\}]}, \quad m \neq -1 \tag{3.7}$$

$$= \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{(-1)^{p+q}\alpha^{b+2}\sigma^{i+j}b!\ (j/\beta)_{(p)}(i/\beta)_{(q)}}{[\alpha(c+1)+p-(j/\beta)]^{b+1}[\alpha(a+c+2)+p+q-\{(i+j)/\beta\}]}, \quad m = -1 \tag{3.8}$$

where $J_{i,j}(a,b,c)$ is as given in (3.2).

Proof: When $m \neq -1$, we have

$$[h_m(F(y))-h_m(F(x))]^b = \frac{1}{(m+1)^b}[(\bar{F}(x))^{m+1}-(\bar{F}(y))^{m+1}]^b$$

$$= \frac{1}{(m+1)^b}\sum_{v=0}^{b}(-1)^v\binom{b}{v}[\bar{F}(y)]^{v(m+1)}[\bar{F}(x)]^{(b-v)(m+1)}.$$

Now substituting for $[h_m(F(y))-h_m(F(x))]^b$ in equation (3.2), we get

$$J_{i,j}(a,b,c) = \frac{1}{(m+1)^b}\sum_{v=0}^{b}(-1)^v\binom{b}{v}J_{i,j}(a+(b-v)(m+1),0,c+v(m+1)).$$

Making use of the Lemma 3.1, we derive the relation given in (3.7).
When $m = -1$, we have

$J_{i,j}(a,b,c) = \frac{0}{0}$ as $\sum_{v=0}^{b}(-1)^v\binom{b}{v} = 0.$

On applying L' Hospital rule, (3.8) can be proved on the lines of (2.6).

3.3. Theorem. *For type II exponentiated log-logistic distribution as given in (1.2) and $1 \leq r < s \leq n$, $k = 1, 2, \ldots$ and $m \neq -1$*

$$E[X^i(r,n,m,k)X^j(s,n,m,k)] = \frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{r-1}} \sum_{u=0}^{r-1}(-1)^u$$

$$\times \binom{r-1}{u} J_{i,j}(m+u(m+1),s-r-1,\gamma_s-1) \tag{3.9}$$

$$= \frac{\alpha^2\sigma^{i+j}C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1}(-1)^{p+q+u+v} \binom{r-1}{u}$$

$$\times \binom{s-r-1}{v} \frac{(j/\beta)_{(p)}\,(i/\beta)_{(q)}}{[\alpha\gamma_{s-v}+p-(j/\beta)][\alpha\gamma_{r-u}+p+q-\{(i+j)/\beta\}]},$$
$$\beta > max(i,j) \quad and \quad i,j = 0,1,2,\dots. \tag{3.10}$$

Proof: From (1.6), we have

$$E[X^i(r,n,m,k)X^j(s,n,m,k)] = \frac{C_{s-1}}{(r-1)!(s-r-1)!} \int_0^{\infty}\int_x^{\infty} x^i y^j [\bar{F}(x)]^m f(x)$$

$$\times g_m^{r-1}(F(x))[h_m(F(y))-h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s-1}f(y)dydx. \tag{3.11}$$

On expanding $g_m^{r-1}(F(x))$ binomially in (3.11), we get

$$E[X^i(r,n,m,k)X^j(s,n,m,k)] = \frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{r-1}}$$

$$\times \sum_{u=0}^{r-1}(-1)^u \binom{r-1}{u} J_{i,j}(m+u(m+1),s-r-1,\gamma_s-1).$$

Making use of the Lemma 3.2, we derive the relation in (3.10).

Identity 3.1: For $\gamma_r,\gamma_s \geq 1$, $k \geq 1$, $1 \leq r < s \leq n$ and $m \neq -1$

$$\sum_{v=0}^{s-r-1}(-1)^v \binom{s-r-1}{v} \frac{1}{\gamma_{s-v}} = \frac{(s-r-1)!(m+1)^{s-r-1}}{\prod_{t=r+1}^{s}\gamma_t}. \tag{3.12}$$

Proof. At $i=j=0$ in (3.10), we have

$$1 = \frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}} \sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1}(-1)^{u+v}$$

$$\times \binom{r-1}{u} \binom{s-r-1}{v} \frac{1}{\gamma_{s-v}\gamma_{r-u}}.$$

Now on using (2.13), we get the result given in (3.12).

At $r=0$, (3.12) reduces to (2.13).

Special cases:

i) Putting $m=0$, $k=1$ in (3.10), the explicit formula for the product moments of order statistics of the type II exponentiated log-logistic distribution can be obtained as

$$E(X_{r:n}^i X_{s:n}^j) = \alpha^2\sigma^{i+j}C_{r,s:n} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1}(-1)^{p+q+u+v} \binom{n-s}{u}$$

$$\times \binom{s-r-1}{v} \frac{(j/\beta)_{(p)}}{[\alpha(n-s+1+v)+p-(j/\beta)]}$$

$$\times \frac{(i/\beta)_{(q)}}{[\alpha(n-r+1+u)+p+q-\{(i+j)/\beta\}]}.$$

where,

$$C_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}.$$

ii)Putting $m = -1$ in (3.10), we deduce the explicit expression for the product moments of upper k record values for the type II exponentiated log-logistic distribution in view of (3.9) and (3.8) in the form

$$E[(X_{U(r)}^{(k)})^i (X_{U(s)}^{(k)})^j] = (\alpha k)^s \sigma^{i+j} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{(-1)^{p+q} (j/\beta)_{(p)}}{[\alpha k + p - (j/\beta)]^{s-r}}$$

$$\times \frac{(i/\beta)_{(q)}}{[\alpha k + p + q - \{(i+j)/\beta\}]^r}$$

and hence for upper records

$$E(X_{U(r)}^i X_{U(s)}^j) = \alpha^s \sigma^{i+j} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{(-1)^{p+q} (j/\beta)_{(p)} (i/\beta)_{(q)}}{[\alpha + p - (j/\beta)]^{s-r} [\alpha + p + q - \{(i+j)/\beta\}]^r}.$$

Remark 3.1   At $j = 0$ in (3.10), we have

$$E[X^i(r,n,m,k) = \frac{\alpha \sigma^i C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}} \sum_{q=0}^{\infty} \sum_{u=0}^{r-1} \sum_{v=0}^{s-r-1} (-1)^{q+u+v}$$

$$\times \binom{r-1}{u} \binom{s-r-1}{v} \frac{(i/\beta)_q}{\gamma_{s-v} [\alpha \gamma_{r-u} + q - (i/\beta)]}. \qquad (3.12)$$

Making use of (3.12) in (3.13) and simplifying the resulting expression, we get

$$E[X^i(r,n,m,k) = \frac{\alpha \sigma^i C_{r-1}}{(r-1)!(m+1)^{s-1}} \sum_{q=0}^{\infty} \sum_{u=0}^{r-1} (-1)^{q+u}$$

$$\times \binom{r-1}{u} \frac{(i/\beta)_q}{[\alpha \gamma_{r-u} + q - (i/\beta)]},$$

as obtained in (2.12).

Making use of (1.6), we can derive recurrence relations for product moments of *gos* from (1.2).

3.4. Theorem. *For the given type II exponentiated log-logistic distribution and $n \in N$, $m \in \Re$, $1 \le r < s \le n-1$*

$$\left(1 - \frac{\sigma j}{\alpha \beta \gamma_s}\right) E[X^i(r,n,m,k) X^j(s,n,m,k)] = E[X^i(r,n,m,k) X^j(s-1,n,m,k)]$$

$$+ \frac{j \sigma^{\beta+1}}{\alpha \beta \gamma_s} E[X^i(r,n,m,k) X^{j-\beta}(s,n,m,k)]. \qquad (3.14)$$

Proof: From (1.6), we have

$$E[X^i(r,n,m,k) X^j(s,n,m,k)] = \frac{C_{s-1}}{(r-1)!(s-r-1)!}$$

$$\times \int_0^{\infty} x^i [\bar{F}(x)]^m f(x) g_m^{r-1}(F(x)) I(x) dx, \qquad (3.15)$$

where

$$I(x) = \int_x^{\infty} y^j [\bar{F}(y)]^{\gamma_s - 1} [h_m(F(y)) - h_m(F(x))]^{s-r-1} f(y) dy.$$

Solving the integral in $I(x)$ by parts and substituting the resulting expression in (3.15), we get

$$E[X^i(r,n,m,k)X^j(s,n,m,k)] - E[X^i(r,n,m,k)X^j(s-1,n,m,k)]$$
$$= \frac{jC_{s-1}}{\gamma_s(r-1)!(s-r-1)!} \int_0^\infty \int_x^\infty x^i y^j [\bar{F}(x)]^m f(x) g_m^{r-1}(F(x))$$
$$\times [h_m(F(y)) - h_m(F(x))]^{s-r-1} [\bar{F}(y)]^{\gamma_s} dy dx$$

the constant of integration vanishes since the integral in $I(x)$ is a definite integral. On using the relation (1.3), we obtain

$$E[X^i(r,n,m,k)X^j(s,n,m,k)] - E[X^i(r,n,m,k)X^j(s-1,n,m,k)]$$
$$= \frac{j\sigma C_{s-1}}{\alpha\beta\gamma_s(r-1)!(s-r-1)!} \int_0^\infty \int_x^\infty x^i y^j [\bar{F}(x)]^m f(x) g_m^{r-1}(F(x))$$
$$\times [h_m(F(y)) - h_m(F(x))]^{s-r-1} [\bar{F}(y)]^{\gamma_s-1} f(y) dy dx$$
$$+ \frac{j\sigma^{\beta+1} C_{s-1}}{\alpha\beta\gamma_s(r-1)!(s-r-1)!} \int_0^\infty \int_x^\infty x^i y^{j-\beta} [\bar{F}(x)]^m f(x) g_m^{r-1}(F(x))$$
$$\times [h_m(F(y)) - h_m(F(x))]^{s-r-1} [\bar{F}(y)]^{\gamma_s-1} f(y) dy dx$$

and hence the result given in (3.14).

Remark 3.2 Setting $m = 0$, $k = 1$ in (3.14), we obtain recurrence relations for product moments of order statistics of the type II exponentiated log-logistic distribution in the form

$$\left(1 - \frac{\sigma j}{\alpha\beta(n-s+1)}\right) E[X_{r,s:n}^{i,j}] = E[X_{r,s-1:n}^{i,j}] + \frac{j\sigma^{\beta+1}}{\alpha\beta(n-s+1)} E[X_{r,s:n}^{i,j-\beta}].$$

Remark 3.3 Putting $m = -1$, $k \geq 1$ in (3.5), we get the recurrence relations for product moments of upper k records of the type II exponentiated log-logistic distribution in the form

$$\left(1 - \frac{\sigma j}{\alpha\beta k}\right) E[(X_{U(r)}^{(k)})^i (X_{U(s)}^{(k)})^j] = E[(X_{U(r)}^{(k)})^i (X_{U(s-1)}^{(k)})^j]$$
$$+ \frac{j\sigma^{\beta+1}}{\alpha\beta k} E[(X_{U(r)}^{(k)})^i (X_{U(s-1)}^{(k)})^{j-\beta}].$$

Ratio moments of *gos* from type II exponentiated log-logistic distribution can be obtain by the following Theorem.

3.5. Theorem. *For type II exponentiated log-logistic distribution as given in (1.2)*

$$E[X^i(r,n,m,k)X^{j-\beta}(s,n,m,k)] = \sum_{p=0}^\infty \sum_{q=0}^\infty \frac{(-1)^{p+q} \sigma^{i+j-\beta} \Gamma\left(\frac{j}{\beta}\right) \Gamma\left(\frac{i}{\beta}+1\right)}{p!q!\Gamma\left(\frac{j}{\beta}-p\right)\Gamma\left(\frac{i}{\beta}+1-p\right)}$$

$$\times \frac{1}{\prod_{a=1}^r \left(1 + \frac{p+q-((i+j)/\beta)}{\alpha\gamma_a}\right) \prod_{b=r+1}^s \left(1 + \frac{p+1-(j/\beta)}{\alpha\gamma_b}\right)}, \quad \beta > j. \qquad (3.16)$$

Proof From (1.6), we have

$$E[X^i(r,n,m,k)X^{j-\beta}(s,n,m,k)] = \frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}$$

$$\times \sum_{u=0}^{r-1} \sum_{v=0}^{s-r-1} (-1)^{u+v} \binom{r-1}{u} \binom{s-r-1}{v}$$

$$\times \int_0^\infty x^i [\bar{F}(x)]^{(s-r+u-v)(m+1)-1} f(x) J(x) dx, \qquad (3.17)$$

where

$$J(x) = \int_x^\infty y^{j-\beta}[\bar{F}(y)]^{\gamma_{s-v}-1}f(y)dy. \tag{3.18}$$

By setting $z = [\bar{F}(y)]^{1/\alpha}$ in (3.18), we find that

$$J(x) = \sigma^{j-\beta}\sum_{p=0}^\infty \frac{(-1)^p\Gamma\left(\frac{j}{\beta}\right)[\bar{F(x)}]^{\gamma_{s-v}+\frac{p+1-(j/\beta)}{\alpha}}}{p!\Gamma\left(\frac{j}{\beta}-p\right)\left[\gamma_{s-v}+\frac{p+1-(j/\beta)}{\alpha}\right]}.$$

On substituting the above expression of $J(x)$ in (3.17), we get

$$E[X^i(r,n,m,k)X^{j-\beta}(s,n,m,k)] = \frac{\sigma^{j-\beta}C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\sum_{p=0}^\infty\sum_{u=0}^{r-1}$$

$$\times \sum_{v=0}^{s-r-1}(-1)^{u+v+p}\left(\begin{array}{c}r-1\\u\end{array}\right)\left(\begin{array}{c}s-r-1\\v\end{array}\right)\frac{\Gamma\left(\frac{j}{\beta}\right)}{p!\Gamma\left(\frac{j}{\beta}-p\right)}$$

$$\times \frac{1}{\left[\gamma_{s-v}+\frac{p+1-(j/\beta)}{\alpha}\right]}\int_0^\infty x^i[\bar{F}(x)]^{\gamma_{r-u}+\frac{p+1-(j/\beta)-1}{\alpha}-1}dx. \tag{3.19}$$

Again by setting $t = [\bar{F}(x)]^{1/\alpha}$ in (3.19), we get

$$E[X^i(r,n,m,k)X^{j-\beta}(s,n,m,k)] = \frac{\sigma^{i+j-\beta}C_{s-1}}{(m+1)^s}\sum_{p=0}^\infty\sum_{q=0}^\infty(-1)^{p+q}$$

$$\times \frac{\Gamma\left(\frac{j}{\beta}\right)\Gamma\left(\frac{i}{\beta}+1\right)\Gamma\left[\frac{\alpha\{k+(n-r)(m+1)\}+p+q-\{(i+j)/\beta\}}{\alpha(m+1)}\right]}{p!q!\,\Gamma\left(\frac{j}{\beta}-p\right)\Gamma\left(\frac{i}{\beta}+1-q\right)\Gamma\left[\frac{\alpha\{k+n(m+1)\}+p+q-\{(i+j)/\beta\}}{\alpha(m+1)}\right]}$$

$$\times \frac{\Gamma\left[\frac{\alpha\{k+(n-s)(m+1)\}+p+1-(j/\beta)}{\alpha(m+1)}\right]}{\Gamma\left[\frac{\alpha\{k+(n-r)(m+1)\}+p+1-(j/\beta)}{\alpha(m+1)}\right]} \tag{3.20}$$

and hence the result given in (3.16).

Special cases

iii) Putting $m = 0$, $k = 1$ in (3.20), the explicit formula for the ratio moments of order statistics of the type II exponentiated log-logistic distribution can be obtained as

$$E[X_{r:n}^i X_{s:n}^{j-\beta}] = \frac{n!\sigma^{i+j-\beta}}{(n-s)!}\sum_{p=0}^\infty\sum_{q=0}^\infty(-1)^{p+q}\frac{\Gamma\left(\frac{j}{\beta}\right)\Gamma\left(\frac{i}{\beta}+1\right)}{p!q!\,\Gamma\left(\frac{j}{\beta}-p\right)\Gamma\left(\frac{i}{\beta}+1-q\right)}$$

$$\times \frac{\Gamma[\alpha(n-r+1)+p+q-\{(i+j)/\beta\}]\Gamma[\alpha(n-s+1)+p+1-(j/\beta)]}{\Gamma[\alpha(n+1)+p+q-\{(i+j)/\beta\}]\Gamma[\alpha(n-r+1)+p+1-(j/\beta)]}.$$

iv) Putting $m = -1$ in (3.16), the explicit expression for the ratio moments of upper k record values for the type II exponentiated log-logistic distribution can be obtained as

$$E[(X_{U(r)}^{(k)})^i(X_{U(s)}^{(k)})^{j-\beta}] = \sigma^{i+j-\beta}\sum_{p=0}^\infty\sum_{q=0}^\infty\frac{(-1)^{p+q}}{p!q!}\frac{\Gamma\left(\frac{j}{\beta}\right)\Gamma\left(\frac{i}{\beta}+1\right)}{\Gamma\left(\frac{j}{\beta}-p\right)\Gamma\left(\frac{i}{\beta}+1-q\right)}$$

$$\times \frac{1}{\left(1+\frac{p+q-\{(i+j)/\beta\}}{\alpha k}\right)^r\left(1+\frac{p+1-(j/\beta)}{\alpha k}\right)^{s-r}}.$$

Making use of (1.6), we can derive recurrence relations for ratio moments of *gos* from (1.2).

3.6. Theorem. *For type II exponentiated log-logistic distribution*

$$\left(1 - \frac{\sigma(j - \beta)}{\alpha\beta\gamma_s}\right)E[X^i(r, n, m, k)X^{j-\beta}(s, n, m, k)]$$

$$= E[X^i(r, n, m, k)X^{j-\beta}(s - 1, n, m, k)]$$

$$+ \frac{(j - \beta)\sigma^{\beta+1}}{\alpha\beta\gamma_s}E[X^i(r, n, m, k)X^{j-2\beta}(s, n, m, k)], \quad \beta > j. \tag{3.21}$$

Proof The proof is easy.

Remark 3.4 Setting $m = 0$, $k = 1$ in (3.21), we obtain a recurrence relation for Ratio moments of order statistics for type II exponentiated log-logistic distribution in the form

$$\left(1 - \frac{\sigma(j - \beta)}{\alpha\beta(n - s + 1)}\right)E[X^i_{r:n}X^{j-\beta}_{s:n}] = E[X^i_{r:n}X^{j-\beta}_{s-1:n}] + \frac{(j - \beta)\sigma^{\beta+1}}{\alpha\beta(n - s + 1)}E[X^i_{r:n}X^{j-2\beta}_{s:n}].$$

Remark 3.5 Putting $m = -1$, in Theorem 3.6, we get a recurrence relation for ratio moments of upper k record values from type II exponentiated log-logistic distribution in the form

$$\left(1 - \frac{\sigma(j - \beta)}{\alpha\beta k}\right)E[(X^{(k)}_{U(r)})^i(X^{(k)}_{U(s)})^{j-\beta}] = E[(X^{(k)}_{U(r)})^i(X^{(k)}_{U(s-1)})^{j-\beta}]$$

$$+ \frac{(j - \beta)\sigma^{\beta+1}}{\alpha\beta k}E[(X^{(k)}_{U(r)})^i(X^{(k)}_{U(s)})^{j-2\beta}].$$

Remark 3.6 At $\gamma_r = n - r + 1 + \sum_{i=r}^{j} m_i$, $1 \le r \le j \le n$, $m_i \in N$, $k = m_n + 1$ in (3.16) the product moment of progressive type II censored order statistics of type II exponentiated log-logistic distribution can be obtained.

Remark 3.7 The result is more general in the sense that by simply adjusting $j - \beta$ in (3.16), we can get interesting results. For example if $j - \beta = -1$ then $E\left[\frac{X(r,n,m,k)}{X(s,n,m,k)}\right]^i$ gives the moments of quotient. For $j - \beta > 0$, $E[X^i(r, n, m, k) X^{j-\beta}(s, n, m, k)]$ represent product moments, whereas for $j < \beta$, it is moment of the ratio of two generalized order statistics of different powers.

4. Characterization

This Section contains characterization of type II exponentiated log-logistic distribution by using the conditional expectation of *gos* .
Let $X(r, n, m, k)$, $r = 1, 2, \ldots, n$ be *gos*, then from a continuous population with *cdf* $F(x)$ and *pdf* $f(x)$, then the conditional *pdf* of $X(s, n, m, k)$ given $X(r, n, m, k) = x$, $1 \le r < s \le n$, in view of (1.5) and (1.6), is

$$f_{X(s,n,m,k)|X(r,n,m,k)}(y|x) = \frac{C_{s-1}}{(s - r - 1)!C_{r-1}}$$

$$\times \frac{[h_m(F(y)) - h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s-1}}{[\bar{F}(x)]^{\gamma_r+1}}f(y). \quad x < y \tag{4.1}$$

4.1. Theorem. *Let X be a non-negative random variable having an absolutely continuous distribution function $F(x)$ with $F(0) = 0$ and $0 < F(x) < 1$ for all $x > 0$, then*

$$E[X(s,n,m,k)|X(r,n,m,k)=x] = \sigma \sum_{p=0}^{\infty}(1/\beta)_{(p)}[1+(x/\sigma)^{\beta}]^p$$

$$\times \prod_{j=1}^{s-r}\Big(\frac{\gamma_{r+j}}{\gamma_{r+j}-p/\alpha}\Big) \tag{4.2}$$

if and only if

$$\bar{F}(x) = \Big(1+\Big\{\frac{x}{\sigma}\Big\}^{\beta}\Big)^{-\alpha}, \quad x \geq 0, \ \alpha, \ \sigma > 0, \ \beta > 1.$$

Proof From (4.1), we have

$$E[X(s,n,m,k)|X(r,n,m,k)=x] = \frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}}$$

$$\times \int_{x}^{\infty} y\Big[1-\Big(\frac{\bar{F}(y)}{\bar{F}(x)}\Big)^{m+1}\Big]^{s-r-1}\Big(\frac{\bar{F}(y)}{\bar{F}(x)}\Big)^{\gamma_s-1}\frac{f(y)}{\bar{F}(x)}dy. \tag{4.3}$$

By setting $u = \frac{\bar{F}(y)}{\bar{F}(x)} = \Big(\frac{1+(x/\sigma)^{\beta}}{1+(y/\sigma)^{\beta}}\Big)^{\alpha}$ from (1.2) in (4.3), we obtain

$$E[X(s,n,m,k)|X(r,n,m,k)=x] = \frac{\sigma C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}}$$

$$\times \int_{0}^{1}[\{1+(x/\sigma)^{\beta}\}u^{-1/\alpha}-1]^{1/\beta}u^{\gamma_s-1}(1-u^{m+1})^{s-r-1}du$$

$$= \frac{\sigma C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}}\sum_{p=0}^{\infty}(1/\beta)_{(p)}[1+(x/\sigma)^{\beta}]^p$$

$$\times \int_{0}^{1}u^{\gamma_s-(p/\alpha)-1}(1-u^{m+1})^{s-r-1}du \tag{4.4}$$

Again by setting $t = u^{m+1}$ in (4.4), we get

$$E[X(s,n,m,k)|X(r,n,m,k)=x]$$

$$= \frac{\sigma C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r}}\sum_{p=0}^{\infty}(1/\beta)_{(p)}[1+(x/\sigma)^{\beta}]^p$$

$$\times \int_{0}^{1}t^{\frac{k-(p/\alpha)}{m+1}+n-s-1}(1-t)^{s-r-1}dt$$

$$= \frac{\sigma C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r}}\sum_{p=0}^{\infty}(1/\beta)_{(p)}[1+(x/\sigma)^{\beta}]^p$$

$$\times \frac{\Gamma\Big(\frac{k-(p/\alpha)}{m+1}+n-s\Big)\Gamma(s-r)}{\Gamma\Big(\frac{k-(p/\alpha)}{m+1}+n-r\Big)}$$

$$= \frac{\sigma C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r}}\sum_{p=0}^{\infty}(1/\beta)_{(p)}[1+(x/\sigma)^{\beta}]^p$$

$$\times \frac{(m+1)^{s-r}\Gamma(s-r)}{\prod_{j=1}^{s-r}(\gamma_{r+j}-(p/\alpha))}$$

and hence the relation in (4.2).

To prove sufficient part, we have from (4.1) and (4.2)

$$\frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}}\int_{x}^{\infty}y[(\bar{F}(x))^{m+1}-(\bar{F}(y))^{m+1}]^{s-r-1}$$

$$\times [\bar{F}(y)]^{\gamma_s-1} f(y) dy = [\bar{F}(x)]^{\gamma_{r+1}} H_r(x), \tag{4.8}$$

where

$$H_r(x) = \sigma \sum_{p=0}^{\infty} (1/\beta)_{(p)} [1 + (x/\sigma)^\beta]^p \prod_{j=1}^{s-r} \left(\frac{\gamma_{r+j}}{\gamma_{r+j} - p/\alpha} \right).$$

Differentiating (4.5) both sides with respect to x and rearranging the terms, we get

$$-\frac{C_{s-1}[\bar{F}(x)]^m f(x)}{(s-r-2)! C_{r-1}(m+1)^{s-r-2}} \int_x^\infty y[(\bar{F}(x))^{m+1} - (\bar{F}(y))^{m+1}]^{s-r-2}$$
$$\times [\bar{F}(y)]^{\gamma_s-1} f(y) dy = H_r'(x)[\bar{F}(x)]^{\gamma_{r+1}} - \gamma_{r+1} H_r(x)[\bar{F}(x)]^{\gamma_{r+1}-1} f(x)$$

or

$$-\gamma_{r+1} H_{r+1}(x)[\bar{F}(x)]^{\gamma_{r+2}+m} f(x)$$
$$= H_r'(x)[\bar{F}(x)]^{\gamma_{r+1}} + \gamma_{r+1} H_r(x)[\bar{F}(x)]^{\gamma_{r+1}-1} f(x).$$

Therefore,

$$\frac{f(x)}{\bar{F}(x)} = -\frac{H_r'(x)}{\gamma_{r+1}[H_{r+1}(x) - H_r(x)]} = \frac{\alpha\beta(x/\sigma)^{\beta-1}}{\sigma[1 + (x/\sigma)^\beta]}$$

which proves that

$$\bar{F}(x) = \left(1 + \left\{ \frac{x}{\sigma} \right\}^\beta \right)^{-\alpha}, \quad x \geq 0, \ \alpha, \ \sigma > 0, \ \beta > 1.$$

Remark For $m = 0$, $k = 1$ and $m = -1$, $k = 1$, we obtain the characterization results of the type II exponentiated log-logistic distribution based on order statistics and record values respectively.

5. Applications

In this Section, we suggest some applications based on moments discussed in Section 2. Order statistics, record values and their moments are widely used in statistical inference [see for example Balakrishnan and Sandhu [11], Sultan and Moshref [38] and Mahmoud *et al.* [31], among several others].

i) Estimation: The moments of order statistics and record values given in Section 2 can be used to obtain the best linear unbiased estimate of the parameters of the type II exponentiated log-logistic distribution. Some works of this nature based on *gos* have been done by Ahsanullah and habibullah [6], Malinowska et al. [32] and Burkchat *et al.* [16].

ii) Characterization: The type II exponentiated log-logistic distribution given in (1.2) can be characterized by using recurrence of single moment of *gos* as follows:

Let $L(a, b)$ stand for the space of all integrable functions on (a, b). A sequence $(f_n) \subset L(a, b)$ is called complete on $L(a, b)$ if for all functions $g \in L(a, b)$ the condition

$$\int_a^b g(x) f_n(x) dx = 0, \quad n \in N,$$

implies $g(x) = 0$ a.e. on (a, b). We start with the following result of Lin [30].

Proposition 5.1 Let n_0 be any fixed non-negative integer, $-\infty \leq a < b \leq \infty$ and $g(x) \geq 0$ an absolutely continuous function with $g'(x) \neq 0$ a.e. on (a, b). Then the sequence of functions $\{(g(x))^n e^{-g(x)}, \ n \geq n_0\}$ is complete in $L(a, b)$ iff $g(x)$ is strictly monotone on (a, b).

Using the above Proposition we get a stronger version of Theorem 2.4.

5.1. Theorem. *A necessary and sufficient conditions for a random variable X to be distributed with pdf given by (1.1) is that*

$$\left(1 - \frac{\sigma j}{\alpha\beta\gamma_r}\right)E[X^j(r,n,m,k)] = E[X^j(r-1,n,m,k)]$$

$$+\frac{j\sigma^{\beta+1}}{\alpha\beta\gamma_r}E[X^{j-\beta}(r,n,m,k)]. \tag{5.1}$$

Proof The necessary part follows immediately from (2.14) on the other hand if the recurrence relation (5.1) is satisfied then on using (1.5), we have

$$\frac{C_{r-1}}{(r-1)!}\int_0^\infty x^j[\bar{F}(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx$$

$$= \frac{C_{r-1}}{\gamma_r(r-2)!}\int_0^\infty x^j[\bar{F}(x)]^{\gamma_r+m}f(x)g_m^{r-2}(F(x))dx$$

$$+\frac{\sigma j C_{r-1}}{\alpha\beta\gamma_r(r-1)!}\int_0^\infty x^j[\bar{F}(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx$$

$$+\frac{j\sigma^{\beta+1}C_{r-1}}{\alpha\beta\gamma_r(r-1)!}\int_0^\infty x^{j-\beta}[\bar{F}(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx. \tag{5.2}$$

Integrating the first integral on the right-hand side of the above equation by parts and simplifying the resulting expression, we get

$$\frac{jC_{r-1}}{\gamma_r(r-1)!}\int_0^\infty x^{j-1}[\bar{F}(x)]^{\gamma_r-1}g_m^{r-1}(F(x))$$

$$\times\left\{\bar{F}(x) - \frac{\sigma x}{\alpha\beta}f(x) - \frac{\sigma^{\beta+1}}{\alpha\beta x^{\beta-1}}f(x)\right\}dx = 0.$$

It now follows from Proposition 5.1, we get

$$\alpha\beta\bar{F}(x) = \sigma[1 + (x/\sigma)^\beta]xf(x),$$

which proves that $f(x)$ has the form (1.1).

6. Concluding Remarks

In the study presented above, we established some new explicit expressions and recurrence relations between the single and product moments of *gos* from the type II exponentiated log-logistic distribution. In addition ratio and inverse moments of type II exponentiated log-logistic distribution are also established. Further, the conditional expectation of *gos* is used to characterize the distribution.

Acknowledgements

References

[1] Aggarwala, R. and Balakrishnan, N. *Recurrence relations for progressive type II right censored order statistics from exponential truncated exponential distributions*, Ann. Instit. Statist. Math., **48**, 757-771, (1996).

[2] Ahmad, A.A. and Fawzy, M. *Recurrence relations for single moments of generalized order statistics from doubly truncated distribution*, J. Statist. Plann. Inference, **177**, 241-249, (2003).

[3] Ahmad, A.A. *Relations for single and product moments of generalized order statistics from doubly truncated Burr type XII distribution*, J. Egypt. Math. Soc., **15**, 117-128, (2007).

[4] Ahsanullah, M. *Record Statistics*, Nova Science Publishers, New York, (1995).

[5] Ahsanullah, M. *Generalized order statistics from exponential distribution*, J. Statist. Plann. Inference, **85**, 85-91, (2000).

[6] Ahsanullah, M. and Habibullah, M. *Estimation of parameters of a Pareto distribution by generalized order statistics*, Comm. Statist. Theory Methods, **29**, 1597-1609, (2000).

[7] Arnold, B.C. and Balakrishnan, N. *Relations, Bounds and Approximations for Order Statistics. Lecture Notes in Statistics*, Springer-Verlag, Berlin, **53**, (1989).

[8] Arnold, B.C., Balakrishnan, N. and Nagaraja, H.N. *A First Course in Order Statistics*, John Wiley and Sons, New York (1992).

[9] Arnold, B.C., Balakrishnan, N. and Nagaraja, H.N. *Records*, . John Wiley, New York, (1998).

[10] Balakrishnan, N. and Aggarwala, R. *Progressive censoring: Theory Methods and Applications*, Birkhauser Berlin, (2000).

[11] Balakrishnan, N. and Sandhu, R. A. *Best linear unbiased and maximum likelihood estimation for exponential distribution under general progressive type II censored samples*, Sankhya Ser. B, **58**, 1-9, (1996).

[12] Balakrishnan, N., Chan, P. S. and Ahsanullah, M. *Recurrence relations for moments of record values from generalized extreme value distribution*, Comm. Statis. Theory and Methods, **22**, 1471-1482, (1993).

[13] Balakrishnan, N., Cramer, E. and Kamps, U. *Bounds for means and variances of progressive type II censored order statistics*, Statist. Prob. Letters, **54**, 301-315, (2001).

[14] Balasooriya, U. and Saw, S. *Reliability sampling plans for the two parameter exponential distribution under progressive censoring*, J. Appl. Statist., **25**, 707-714, (1998).

[15] Bieniek, M. and Szynal, D. *Characterizations of distributions via linearity of regression of generalized order statistics*, Metrika, **58**, 259-271, (2003).

[16] Burkschat, M. Cramer, E. and Kamps, U. *Linear estimation of location and scale parameters based on generalized order statistics from generalized Pareto distribution*, In Recent Development in Ordered random Variables, 253-261, (2007).

[17] Cramer, E., Kamps, U. and Keseling, C. *Characterization via linear regression of ordered random variables: a unifying approach*, Comm. Statist. Theory Methods, **33**, 2885-2911, (2004).

[18] Cramer, E. *Contributions to generalized order statistics*, Ph. D. Thesis. University of Oldenburg, (2002).

[19] Cramer, E. and Kamps, U. *Sequential order statistics and k- out-of n systems with sequentially adjusted failure rates*, Ann. Instit. Statist. Math, **48**, 535-549, (1996).

[20] Cramer, E. and Kamps, U. *Relations for expectations of functions of generalized order statistics*, J. Statist. Plann. Inference, **89**, 79-89, (2000).

[21] Glick, N. *Breaking records and breaking boards*, Amer. Math. Monthly , **85**, 2-26, (1978).

[22] Kamps, U. and Cramer, E. *On distributions of generalized order statistics*, Statistics, **35**, 269-280, (2001).

[23] Kamps, U. and Gather, U. *Characteristic property of generalized order statistics for exponential distribution*, Appl. Math. (Warsaw), **24**, 383-391, (1997).

[24] Kamps, U. *A Concept of Generalized Order Statistics*, B.G. Teubner Stuttgart, (1995).

[25] Keseling, C. *Conditional distributions of generalized order statistics and some characterizations*, Metrika, **49**, 27-40, (1999).

[26] Khan, A.H., Khan, R.U. and Yaqub, M. *Characterization of continuous distributions through conditional expectation of functions of generalized order statistics*, J. Appl. Probab. Statist., **1**, 115-131, (2006).

[27] Kumar, D. *Recurrence relations for single and product moments of generalized order statistics from p−th order exponential distribution and its characterization*, Journal of Statistical Research of Iran, **7**, 101-112, (2010).

[28] Kumar, D. *Generalized order statistics from Kumaraswamy distribution and its characterization*, Tamsui Oxford journal of Mathematical Sciences, **27**, 463-476, (2011).

[29] Kumar, D. *On moments of lower generalized order statistics from exponentiated lomax distribution and characterization*, American Journal of Mathematical and Management Sciences, **32**, 238-256, (2013).

[30] Lin, G.D. *On a moment problem*, Tohoku Math. Journal, **38**, 595-598, (1986).

[31] Mahmoud, M. A. W., Sultan, K. S. and Amer, A. M. *Order statistics from inverse Weibull distribution and associated inference*, Computational Statistics and data Analysis, **42**, 149-163, (2003).

[32] Malinowska, I., Pawlas, P. and Szynal, D. *Estimation of location and scale paremeters for the Burr XII distribution using generalized order statistics.* , Linear Algebra and its Applications, **417**, 150-162, (2006).

[33] Nevzorov, V.B. *Records*, Theory probab. Appl, **32**, (1987).

[34] Pawlas, P. and Szynal, D. *Recurrence relations for single and product moments of generalized order statistics from Pareto, generalized Pareto, and Burr distributions*, Comm. Statist. Theory Methods, **30**, 739-746, (2001).

[35] Resnick, S. I. *Record values and related statistics*, Ann. Probab, **2**, 650- 662, (1973).

[36] Ruiz, S. M. *An algebraic identity leading to Wilson's theorem*, Math. Gaz., **80**, 579-582, (1996).

[37] Schenk, N. *Point estimation with sequential order statistics from exponential distributions*, PhD. Thesis, Oldenburg University, Germany, (2001).

[38] Sultan, K. S. and Moshref, M. E. *Record value from generalized Pareto distribution and associated inference*, Metrika, **51**, 105-116, (2000).