The Weibull-Lomax distribution: properties and applications

M. H. Tahir[*] , Gauss M. Cordeiro[†] , M. Mansoor[‡] and M. Zubair[§]

Abstract

We introduce a new model called the Weibull-Lomax distribution which extends the Lomax distribution and has increasing and decreasing shapes for the hazard rate function. Various structural properties of the new distribution are derived including explicit expressions for the moments and incomplete moments, Bonferroni and Lorenz curves, mean deviations, mean residual life, mean waiting time, probability weighted moments, generating and quantile function. The Rényi and q entropies are also obtained. We provide the density function of the order statistics and their moments. The model parameters are estimated by the method of maximum likelihood and the observed information matrix is determined. The potentiality of the new model is illustrated by means of two real life data sets. For these data, the new model outperforms the McDonald-Lomax, Kumaraswamy-Lomax, gamma-Lomax, beta-Lomax, exponentiated Lomax and Lomax models.

[*]Department of Statistics, The Islamia University of Bahawalpur, Bahawalpur 63100, Pakistan.
Email: `mtahir.stat@gmail.com; mht@iub.edu.pk` Corresponding Author.
[†]Department of Statistics, Federal University of Pernambuco, 50740-540, Recife, PE, Brazil.
Email:`gauss@de.ufpe.br; gausscordeiro@gmail.com`
[‡]Punjab College, 26-C Shabbir Shaheed Road, Model Town-A, Bahawalpur, Pakistan.
Email:`mansoor.abbasi143@gmail.com`
[§]Government Degree College Kahrorpacca, Lodhran, Pakistan.
Email:`zubair.stat@yahoo.com`

1. Introduction

The Lomax or Pareto II (the shifted Pareto) distribution was pioneered to model business failure data by Lomax [45]. This distribution has found wide application in a variety of fields such as income and wealth inequality, size of cities, actuarial science, medical and biological sciences, engineering, lifetime and reliability modeling. It has been applied to model data obtained from income and wealth [37, 16], firm size [23], size distribution of computer files on servers [40], reliability and life testing [38], receiver operating characteristic (ROC) curve analysis [21] and Hirsch-related statistics [34].

The characterization of the Lomax distribution is described in a number of ways. It is known as a special form of Pearson type VI distribution and has also considered as a mixture of exponential and gamma distributions. In the lifetime context, the Lomax model belongs to the family of decreasing failure rate [24] and arises as a limiting distribution of residual lifetimes at great age [18]. This distribution has been suggested as heavy tailed alternative to the exponential, Weibull and gamma distributions [19]. Further, it is related to the Burr family of distributions [55] and as a special case can be obtained from compound gamma distributions [30]. Some details about the Lomax distribution and Pareto family are given in Arnold [12] and Johnson *et al.* [41].

The distributional properties, estimation and inference of the Lomax distribution are described in the literature as follows. In record value theory, some properties and moments for the Lomax distribution have been discussed in [7, 17, 43, 11]. The comparison of Bayesian and non-Bayesian estimation from the Lomax distribution based on record values have been made in [4, 49]. The moments and inference for the order statistics and generalized order statistics (gos) are given in [52, 25] and [47], respectively. The estimation of parameters in case of progressive and hybrid censoring have been investigated in [13, 28, 10, 39] and [14]. The problem of Bayesian prediction bounds for future observation based on uncensored and type-I censored sample from the Lomax model are dealt in [3] and [9]. Further, the Bayesian and non-Bayesian estimators of the sample size in case of type-I censored samples for the Lomax distribution are obtained in [1], and the estimation under step-stress accelerated life testing for the Lomax distribution is considered in [38]. The parameter estimation through generalized probability weighted moments (PWMs) is addressed in [2]. More recently, the second-order bias and bias-correction for the maximum likelihood estimators (MLEs) of the parameters of the Lomax distribution are determined in [33].

The main aim of this paper is to provide another extension of the Lomax distribution using the *Weibull-G* generator defined by Bourguignon *et al.* [20]. So, we propose the new *Weibull-Lomax* ("WL" for short) distribution by adding two extra shape parameters to the Lomax model. The objectives of the research are to study some structural properties of the proposed distribution.

A random variable Z has the Lomax distribution with two parameters α and β, if it has cumulative distribution function (cdf) (for $x > 0$) given by

$$(1.1) \qquad H_{\alpha,\beta}(x) = 1 - \left[1 + \left(\frac{x}{\beta}\right)\right]^{-\alpha},$$

where $\alpha > 0$ and $\beta > 0$ are the shape and scale parameters, respectively. The probability density function (pdf) corresponding to (1.1) reduces to

$$(1.2) \qquad h_{\alpha,\beta}(x) = \frac{\alpha}{\beta} \left[1 + \left(\frac{x}{\beta}\right)\right]^{-(\alpha+1)}.$$

The survival function $S(t)$ and the hazard rate function (hrf) $h(t)$ at time t for the Lomax distribution are given by

$$S(t) = \left[1 + \left(\frac{x}{\beta} \right) \right]^{-\alpha} \quad \text{and} \quad h(t) = \frac{\alpha}{\beta} \left[1 + \left(\frac{x}{\beta} \right) \right]^{-1},$$

respectively.

The rth moment of Z, (for $r < \alpha$) comes from (1.2) as $\mu'_{rZ} = \alpha \beta^r B(r+1, \alpha-r)$, where $B(p,q) = \int_0^1 w^{p-1} (1-w)^{q-1} dw$ is the complete beta function. The mean of Z can be expressed as $E(Z) = \beta/(\alpha - 1)$, for $\alpha > 1$, and the variance is $Var(Z) = \beta^2/[(\alpha-1)^2 (\alpha-2)]$, for $\alpha > 2$. As α tends to infinity, the mean tends to β, the variance tends to β^2, the skewness tends to 36 and the excess kurtosis approaches 21.

The trend of parameter(s) induction to the baseline distribution has received increased attention in recent years to explore properties and for efficient estimation of the parameters. In the literature, some extensions of the Lomax distribution are available such as the exponentiated Lomax (EL) [6], Marshall-Olkin extended-Lomax (MOEL) [32, 35], beta-Lomax (BL), Kumaraswamy-Lomax (KwL), McDonald-Lomax (McL) [44] and gamma-Lomax (GL) [27].

The first parameter induction to the Lomax distribution was suggested by [6] using Lehmann alternative type I proposed by Gupta *et al.* [36]. The three-parameter EL cdf (for $x > 0$) is defined by

$$(1.3) \qquad G_{a,\alpha,\beta}(x) = \left\{ 1 - \left[1 + \left(\frac{x}{\beta} \right) \right]^{-\alpha} \right\}^a,$$

where $a > 0$ is a shape power parameter. The pdf corresponding to (1.3) (for $x > 0$) is given by

$$(1.4) \qquad g_{a,\alpha,\beta}(x) = \frac{a\alpha}{\beta} \left[1 + \left(\frac{x}{\beta} \right) \right]^{-(\alpha+1)} \left\{ 1 - \left[1 + \left(\frac{x}{\beta} \right) \right]^{-\alpha} \right\}^{a-1},$$

with two shape parameters and one scale parameter.

Let Y be a random variable having the EL distribution (1.4) with parameters a, α and β. Using the transformation $t = 1 - [1 + (x/\beta)]^{-\alpha}$ and the binomial expansion, the rth moment of Y (for $r < \alpha$) is obtained from (1.4) as

$$(1.5) \qquad \mu'_{r,Y}(x) = a\beta^r \sum_{m=0}^{r} (-1)^m \binom{r}{m} B\left(a, \frac{m-r}{\alpha} + 1\right).$$

The rth incomplete moment of Y is given by

$$(1.6) \qquad \mu'_{(r,Y)}(z) = \int_0^z y^r g_{a,\alpha,\beta}(y) \, dy = a\beta^r \sum_{m=0}^{r} (-1)^m \binom{r}{m} B_y\left(a, \frac{m-r}{\alpha} + 1\right),$$

where $B_y(p,q) = \int_0^y w^{p-1} (1-w)^{q-1} dw$ is the incomplete beta function. Some other mathematical quantities of Y are obtained in [5, 6, 42].

The second parameter extension to the Lomax model, named the MOEL distribution, was proposed by [32] using a flexible generator pioneered by Marshall and Olkin [46]. The three-parameter MOEL cdf is given by

$$(1.7) \qquad F_{\alpha,\beta,\delta}(x) = \delta \left\{ \left[1 + \left(\frac{x}{\beta} \right) \right]^{\alpha} - \bar{\delta} \right\}^{-1}.$$

The pdf corresponding to (1.7) becomes

$$(1.8) \qquad f_{\alpha,\beta,\delta}(x) = \alpha\beta\delta \left[1 + \left(\frac{x}{\beta} \right) \right]^{\alpha-1} \left\{ \left[1 + \left(\frac{x}{\beta} \right) \right]^{\alpha} - \bar{\delta} \right\}^{-2},$$

where $\bar{\delta} = 1 - \delta$ and $\delta > 0$ is a shape (or tilt) parameter.

The properties and the estimation of the reliability for the MOEL distribution are studied in [32] and [35]. The acceptance sampling plans (double and grouped) based on non-truncated and truncated samples for the MOEL distribution have been considered by [15, 53, 54, 50].

Lemonte and Cordeiro [44] discussed three parameter inductions to the Lomax distributions, namely the BL, KwL and McL by including two, two and three extra shape parameters using the beta-G, Kumaraswamy-G and McDonald-G generators defined by Eugene *et al.* [31], Cordeiro and de Castro [26] and Alexander *et al.* [8], respectively. The cdfs of the BL, KwL and McL distributions are given by

$$(1.9) \qquad F_{BL}(x; a, b, \alpha, \beta) = I_{\left\{1 - \left[1 + \left(\frac{x}{\beta}\right)\right]^{-\alpha}\right\}(x)}(a, b),$$

$$(1.10) \qquad F_{KwL}(x; a, b, \alpha, \beta) = 1 - \left[1 - \left\{1 - \left[1 + \left(\frac{x}{\beta}\right)\right]^{-\alpha}\right\}^a\right]^b$$

and

$$(1.11) \qquad F_{McL}(x; a, b, c, \alpha, \beta) = I_{\left\{1 - \left[1 + \left(\frac{x}{\beta}\right)\right]^{-\alpha}\right\}^c(x)}(a, b),$$

respectively, where $I_w(p, q) = B_x(p, q)/B(p, q)$ is the incomplete beta function ratio, and $a > 0$, $b > 0$ and $c > 0$ are extra shape parameters whose role is to govern the skewness and tail weights.

The density functions corresponding to (1.9), (1.10) and (1.11) are given by

$$(1.12) \qquad f_{BL}(x; a, b, \alpha, \beta) = \frac{\alpha}{\beta \, B(a, b)} \left[1 + \left(\frac{x}{\beta}\right)\right]^{-(\alpha b + 1)} \left\{1 - \left[1 + \left(\frac{x}{\beta}\right)\right]^{-\alpha}\right\}^{a-1},$$

$$(1.13) \qquad f_{KwL}(x; a, b, \alpha, \beta) = \frac{a \, b \, \alpha}{\beta} \left[1 + \left(\frac{x}{\beta}\right)\right]^{-(\alpha+1)} \left\{1 - \left[1 + \left(\frac{x}{\beta}\right)\right]^{-\alpha}\right\}^{a-1}$$
$$\times \left[1 - \left\{1 - \left[1 + \left(\frac{x}{\beta}\right)\right]^{-\alpha}\right\}^a\right]^{b-1}$$

and

$$(1.14) \qquad f_{McL}(x; a, b, c, \alpha, \beta) = \frac{c \, \alpha}{\beta \, B(a \, c^{-1}, b)} \left[1 + \left(\frac{x}{\beta}\right)\right]^{-(\alpha+1)}$$
$$\times \left\{1 - \left[1 + \left(\frac{x}{\beta}\right)\right]^{-\alpha}\right\}^{a-1} \left[1 - \left\{1 - \left[1 + \left(\frac{x}{\beta}\right)\right]^{-\alpha}\right\}^c\right]^{b-1},$$

respectively.

Recently, Cordeiro *et al.* [27] introduced a three-parameter *gamma-Lomax* (GL) distribution based on a versatile and flexible gamma generator proposed by Zagrafos and Balakrishnan [56] using Stacy's generalized gamma distribution and record value theory. The GL cdf is given by

$$(1.15) \qquad F(a, \alpha, \beta)(x) = \frac{\Gamma\left(a, \alpha \, \log\left[1 + \left(\frac{x}{\beta}\right)\right]\right)}{\Gamma(a)}, \quad x > 0,$$

where $\alpha > 0$ and $a > 0$ are shape parameters and $\beta > 0$ is a scale parameter. The pdf corresponding to (1.15) is given by

$$(1.16) \qquad f(a, \alpha, \beta)(x) = \frac{\alpha^a}{\beta \, \Gamma(a)} \left[1 + \left(\frac{x}{\beta}\right)\right]^{-(\alpha+1)} \left\{\log\left[1 + \left(\frac{x}{\beta}\right)\right]\right\}^{a-1}, \quad x > 0.$$

More recently, Bourguignon *et al.* [20] proposed the Weibull-G class influenced by the Zografos-Balakrishnan-G class. Let $G(x; \Theta)$ and $g(x; \Theta)$ denote the cumulative and density functions of the baseline model with parameter vector Θ and consider the Weibull cdf $F_W(x) = 1 - e^{-a \, x^b}$ (for $x > 0$ and $a, b > 0$). Bourguignon *et al.* [20] replaced the

argument x by $G(x;\Theta)/\overline{G}(x;\Theta)$, where $\overline{G}(x;\Theta) = 1 - G(x;\Theta)$, and defined their class of distributions, say Weibull-G(a, b, Θ), by the cdf

$$(1.17) \quad F(x; a, b, \Theta) = a\,b \int_0^{\left[\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}\right]} x^{b-1} \exp\left(-a\,x^b\right) dx = 1 - \exp\left\{-a \left[\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}\right]^b\right\}.$$

The *Weibull-G* density function is given by

$$(1.18) \quad f(x; a, b, \Theta) = a\,b\,g(x;\Theta) \left[\frac{G(x;\Theta)^{b-1}}{\overline{G}(x;\Theta)^{b+1}}\right] \exp\left\{-a \left[\frac{G(x;\Theta)}{\overline{G}(x;\Theta)}\right]^b\right\}, \quad x \in \Re.$$

In this context, we propose and study the WL distribution based on equations (1.17) and (1.18). The paper is outlined as follows. In Section 2, we define the WL distribution. We provide a mixture representation for its density function in Section 3. Structural properties such as the ordinary and incomplete moments, Bonferroni and Lorenz curves, mean deviations, mean residual life, mean waiting time, probability weighted moments, generating function and quantile function are derived in Section 4. In Section 5, we obtain the Rényi and q entropies. The density of the order statistics is determined in Section 6. The maximum likelihood estimation of the model parameters is discussed in Section 7. We explore its usefulness by means of two real data sets in Section 8. Finally, Section 9 offers some concluding remarks.

2. The WL distribution

Inserting (1.1) in equation (1.17) yields the four-parameter WL cdf

$$(2.1) \quad F(x; a, b, \alpha, \beta) = 1 - \exp\left\{-a\left\{\left[1 + \left(\frac{x}{\beta}\right)\right]^\alpha - 1\right\}^b\right\}.$$

The pdf corresponding to (2.1) is given by

$$f(x; a, b, \alpha, \beta) = \frac{ab\,\alpha}{\beta}\left[1 + \left(\frac{x}{\beta}\right)\right]^{b\,\alpha-1}\left\{1 - \left[1 + \left(\frac{x}{\beta}\right)\right]^{-\alpha}\right\}^{b-1}$$

$$(2.2) \qquad\qquad \times\ \exp\left\{-a\left\{\left[1 + \left(\frac{x}{\beta}\right)\right]^\alpha - 1\right\}^b\right\},$$

where $a > 0$ and $b > 0$ are two additional shape parameters.

Plots of the WL pdf for some parameter values are displayed in Figure 1. Henceforth, we denote by $X \sim$WL(a, b, α, β) a random variable having the pdf (2.2). The survival function (sf) $(S(x))$, hrf $(h(x))$, reversed-hazard rate function (rhrf) $(r(x))$ and cumulative hazard rate function (chrf) $(H(x))$ of X are given by

$$(2.3) \quad S(x; a, b, \alpha, \beta) = \exp\left\{-a\left\{\left[1 + \left(\frac{x}{\beta}\right)\right]^\alpha - 1\right\}^b\right\},$$

$$h(x) = \frac{a\,b\,\alpha}{\beta}\left[1 + \left(\frac{x}{\beta}\right)\right]^{b\,\alpha-1}\left\{1 - \left[1 + \left(\frac{x}{\beta}\right)\right]^{-\alpha}\right\}^{b-1},$$

$$r(x) = \frac{\frac{ab\,\alpha}{\beta}\left[1 + \left(\frac{x}{\beta}\right)\right]^{b\,\alpha-1}\left\{1 - \left[1 + \left(\frac{x}{\beta}\right)\right]^{-\alpha}\right\}^{b-1}\exp\left\{-a\left\{\left[1 + \left(\frac{x}{\beta}\right)\right]^\alpha - 1\right\}^b\right\}}{1 - \exp\left\{-a\left\{\left[1 + \left(\frac{x}{\beta}\right)\right]^\alpha - 1\right\}^b\right\}}$$

and

$$H(x) = -a\left\{\left[1 + \left(\frac{x}{\beta}\right)\right]^\alpha - 1\right\}^b,$$

respectively. Plots of the WL hrf for some parameter values are displayed in Figure 2.
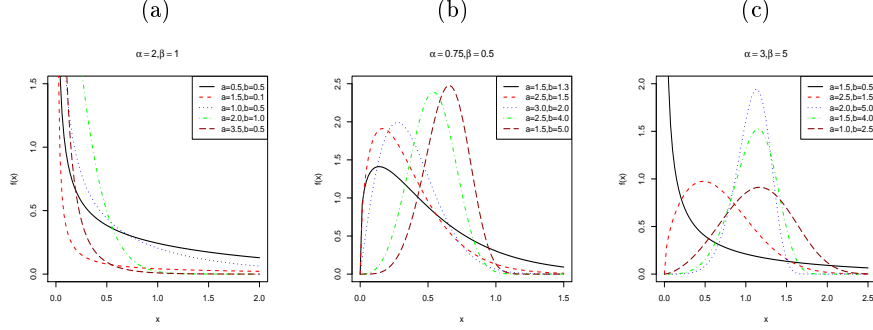
(a) (b) (c)

FIGURE 1. Plots of the WL pdf for some parameter values
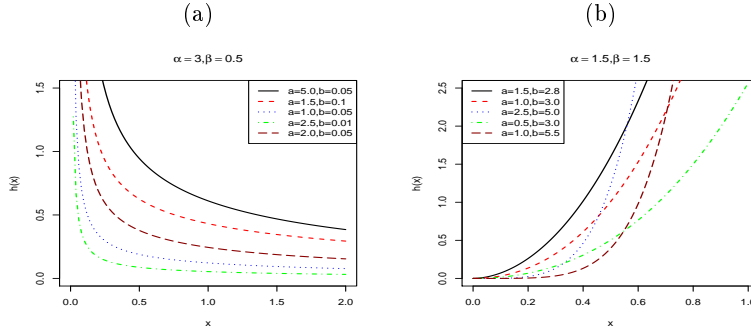
(a) (b)

FIGURE 2. Plots of the WL hrf for some parameter values

3. Mixture representation

The WL density function can be expressed as

$$(3.1) \qquad f(x; a, b, \alpha, \beta) = a\, b\, g(x)\, \frac{G(x)^{b-1}}{\overline{G}(x)^{b+1}} \exp\left\{ -a \left[\frac{G(x)}{\overline{G}(x)} \right]^b \right\}.$$

Inserting (1.1) and (1.2) in equation (3.1), we obtain

$$f(x; a, b, \alpha, \beta) = \frac{ab\alpha}{\beta} \left[1 + \left(\frac{x}{\beta} \right) \right]^{-(\alpha+1)} \frac{\left\{ 1 - \left[1 + \left(\frac{x}{\beta} \right) \right]^{-\alpha} \right\}^{b-1}}{\left[1 - \left\{ 1 - \left[1 + \left(\frac{x}{\beta} \right) \right]^{-\alpha} \right\} \right]^{b+1}}$$

$$(3.2) \qquad \times \quad \exp\left\{ -a \left\{ \frac{1 - \left[1 + \left(\frac{x}{\beta} \right) \right]^{-\alpha}}{1 - \left\{ 1 - \left[1 + \left(\frac{x}{\beta} \right) \right]^{-\alpha} \right\}} \right\}^b \right\}.$$

In order to derive a simple form for the WL pdf, we can expand (3.1) in power series.

Let $A = \exp\left\{ -a \left\{ \frac{1 - \left[1 + \left(\frac{x}{\beta} \right) \right]^{-\alpha}}{1 - \left\{ 1 - \left[1 + \left(\frac{x}{\beta} \right) \right]^{-\alpha} \right\}} \right\}^b \right\}.$

By expanding the exponential function in A, we have

$$A = \sum_{k=0}^{\infty} \frac{(-1)^k a^k}{k!} \frac{\left\{1 - \left[1 + \left(\frac{x}{\beta}\right)\right]^{-\alpha}\right\}^{kb}}{\left[1 - \left\{1 - \left[1 + \left(\frac{x}{\beta}\right)\right]^{-\alpha}\right\}\right]^{kb}}.$$

Inserting this expansion in (3.2) and, after some algebra, we obtain

$$
\begin{aligned}
f(x; a, b, \alpha\beta) &= \sum_{k=0}^{\infty} \frac{(-1)^k a^k}{k!} \frac{a\,b\alpha}{\beta} \left[1 + \left(\frac{x}{\beta}\right)\right]^{-(\alpha+1)} \\
&\times \left\{1 - \left[1 + \left(\frac{x}{\beta}\right)\right]^{-\alpha}\right\}^{b(k+1)-1} \\
&\times \underbrace{\left[1 - \left\{1 - \left[1 + \left(\frac{x}{\beta}\right)\right]^{-\alpha}\right\}\right]^{-[b(k+1)-1]}}_{B_k}.
\end{aligned}
$$

After a power series expansion, the quantity B_k in the last equation becomes

$$B_k = \sum_{j=0}^{\infty} (-1)^j \binom{-[(k+1)b + 1]}{j} \left\{1 - \left[1 + \left(\frac{x}{\beta}\right)\right]^{-\alpha}\right\}^j.$$

Combining the last two results, we can write

$$
\begin{aligned}
f(x; a, b, \alpha, \beta) &= \sum_{k,j=0}^{\infty} \underbrace{\frac{(-1)^k a^{k+1}}{k!\,j!} \frac{b}{[(k+1)b + j]} \frac{\Gamma([k+1]b + j + 1)}{\Gamma([k+1]b + 1)}}_{v_{k,j}} \\
&\times \underbrace{[(k+1)b + j] \frac{\alpha}{\beta} \left[1 + \left(\frac{x}{\beta}\right)\right]^{-(\alpha+1)} \left\{1 - \left[1 + \left(\frac{x}{\beta}\right)\right]^{-\alpha}\right\}^{[(k+1)b+j]-1}}_{g_{a,\alpha,(k+1)b+j}}.
\end{aligned}
$$

The last equation can be rewritten as

$$(3.3) \qquad f(x; a, b, \alpha, \beta) = \sum_{k,j=0}^{\infty} v_{k,j}\, g_{a,\alpha,(k+1)b+j}(x).$$

Equation (3.3) reveals that the WL density function has a double mixture representation of EL densities. So, several of its structural properties can be derived form those of the EL distribution. The coefficients $v_{k,j}$ depend only on the generator parameters. This equation is the main result of this section.

4. Some Structural Properties

Established algebraic expansions to determine some structural properties of the WL distribution can be more efficient than computing those directly by numerical integration of its density function, which can be prone to rounding off errors among others.

4.1. Quantile Function. Quantile functions are in widespread use in general statistics and often find representations in terms of lookup tables for key percentiles. The quantile function (qf) of X is obtained by inverting (2.1) as

$$(4.1) \qquad Q(u) = \beta \left\{ \left[\left\{-a^{-1} \log(1 - u)\right\}^{1/b} + 1 \right]^{1/\alpha} - 1 \right\}.$$

Simulating the WL random variable is straightforward. If U is a uniform variate on the unit interval $(0, 1)$, then $X = Q(U)$ follows (2.2), i.e. $X \sim \text{WL}(a, b, \alpha, \beta)$.

4.2. Moments. Some of the most important features and characteristics of a distribution can be studied through moments (e.g. tendency, dispersion, skewness and kurtosis). The rth moment of X can be obtained from (3.3) as

$$\mu'_r = E(X^r) = \sum_{k,j=0}^{\infty} v_{k,j} \int_0^{\infty} x^r \, g_{a,\alpha,(k+1)b+j}(x) \, dx.$$

Using (3.3), we obtain (for $r \leq \alpha$)

$$(4.2) \qquad \mu'_r = \beta^r \sum_{m=0}^{r} \sum_{k,j=0}^{\infty} (-1)^m \, [(k+1)b+j] \binom{r}{m} v_{k,j} \, B\left([k+1]b+j, \tfrac{m-r}{\alpha}+1\right).$$

Setting $r = 1$ in (4.2), we have the mean of X. Further, the central moments (μ_n) and cumulants (κ_n) of X are obtained from (4.2) as

$$\mu_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \, \mu'^k_1 \, \mu'_{n-k} \qquad \text{and} \qquad \kappa_n = \mu'_n - \sum_{k=1}^{n-1} \binom{n-1}{k-1} \kappa_k \, \mu'_{n-k},$$

respectively, where $\kappa_1 = \mu'_1$. Thus, $\kappa_2 = \mu'_2 - \mu'^2_1$, $\kappa_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2\mu'^3_1$, etc. The skewness and kurtosis measures can be calculated from the ordinary moments using well-known relationships.

The nth descending factorial moment of X (for $n = 1, 2, \ldots$) is

$$\mu'_{(n)} = E(X^{(n)}) = E[X(X-1) \times \cdots \times (X-n+1)] = \sum_{j=0}^{n} s(n,j) \, \mu'_j,$$

where $s(n,j) = (j!)^{-1} \, [d^j j^{(n)}/dx^j]_{x=0}$ is the Stirling number of the first kind.

4.3. Incomplete moments. The answers to many important questions in economics require more than just knowing the mean of the distribution, but its shape as well. This is obvious not only in the study of econometrics but in other areas as well. The rth incomplete moment of X $(r \leq \alpha)$ follows from (3.3) as

$$(4.3) \qquad m_r(z) = \beta^r \sum_{m=0}^{r} \sum_{k,j=0}^{\infty} (-1)^m \, [(k+1)b+j] \, v_{k,j} \binom{r}{m} B_z\left([k+1]b+j, \tfrac{m-r}{\alpha}+1\right).$$

The main application of the first incomplete moment refers to the Bonferroni and Lorenz curves. These curves are very useful in economics, reliability, demography, insurance and medicine. For a given probability π, they are defined by $B(\pi) = m_1(q)/(\pi \, \mu'_1)$ and $L(\pi) = m_1(q)/\mu'_1$, respectively, where $m_1(q)$ can be determined from (4.3) with $r = 1$ and $q = Q(\pi)$ is calculated from (4.1).

The amount of scatter in a population is measured to some extent by the totality of deviations from the mean and median defined by $\delta_1 = \int_0^{\infty} |x - \mu| f(x) dx$ and $\delta_2(x) = \int_0^{\infty} |x - M| f(x) dx$, respectively, where $\mu'_1 = E(X)$ is the mean and $M = Q(0.5)$ is the median. These measures can be determined from $\delta_1 = 2\mu'_1 F(\mu'_1) - 2m_1(\mu'_1)$ and $\delta_2 = \mu'_1 - 2m_1(M)$, where $F(\mu'_1)$ comes from (2.1).

A further application of the first incomplete moment is related to the mean residual life and the mean waiting time given by $m(t; a, b, \alpha, \beta) = [1 - m_1(t)]/S(t) - t$ and $\mu(t; a, b, \alpha, \beta) = t - [m_1(t)/F(t; a, b, \alpha, \beta)]$, respectively, where $F(\cdot; \cdot)$ and $S(\cdot; \cdot) = 1 - F(\cdot; \cdot)$ are obtained from (2.1).

4.4. Probability weighted moments. The probability weighted moments (PWMs) are used to derive estimators of the parameters and quantiles of generalized distributions. These moments have low variances and no severe biases, and they compare favorably with estimators obtained by the maximum likelihood method. The (s, r)th PWM of X (for $r \geq 1, s \geq 0$) is formally defined by $\rho_{r,s} = E[X^r F(X)^s] = \int_0^\infty x^r F(x)^s f(x) dx$. We can write from (2.1)

$$F(x; a, b, \alpha, \beta)^s = \sum_{i=0}^\infty (-1)^i \binom{s}{i} \exp\left\{-i\, a\left\{\left(1 + \frac{x}{\beta}\right)^\alpha - 1\right\}^b\right\}.$$

Then, we can express $\rho_{s,r}$ after some algebra from (2.1) and (2.2) as

$$\rho_{r,s} = \sum_{i=0}^\infty \frac{(-1)^i \binom{s}{i}}{i+1} \int_0^\infty x^r f(x; (i+1)a, b, \alpha, \beta) dx.$$

By using (4.2), we obtain (for $r < \alpha$)

$$\rho_{r,s} = \beta^r \sum_{i,j,k=0}^\infty \frac{(-1)^i \binom{s}{i}}{(i+1)} s_{i,k,j} \sum_{m=0}^r B\left([k+1]b + j, \frac{(m-r)}{\alpha} + 1\right),$$

where

$$s_{i,k,j} = \frac{(-1)^k\, a^k\, (i+1)^k\, \Gamma((k+1)b + j + 1)}{[(b+1)k + 1]\, \Gamma((k+1)b + 1)\, j!\, k!}.$$

4.5. Generating function. The moment generating function (mgf) $M_X(t)$ of a random variable X provides the basis of an alternative route to analytical results compared with working directly with the pdf and cdf of X. We obtain the mgf of the WL distribution from equation (3.3) as

$$\begin{aligned}
M_X(t) &= \sum_{k,j=0}^\infty v_{k,j} \int_0^\infty [(k+1)b + j]\, \frac{\alpha}{\beta}\left[1 + \left(\frac{x}{\beta}\right)\right]^{-(\alpha+1)} \\
&\times \left\{1 - \left[1 + \left(\frac{x}{\beta}\right)\right]^{-\alpha}\right\}^{[(k+1)b+j]-1} e^{tx}\, dx.
\end{aligned}$$

By expanding the binomial terms, we can write

$$\begin{aligned}
M_X(t) &= \frac{\alpha}{\beta} \sum_{k,j=0}^\infty v_{k,j} \sum_{m=0}^\infty (-1)^m \binom{[(k+1)b+j]-1}{m} \int_0^\infty [(k+1)b + j] \\
&\times \left(1 + \frac{x}{\beta}\right)^{-(m+1)\alpha - 1} e^{tx}\, dx.
\end{aligned}$$

By expanding the binomial terms again, we obtain (for $t < 0$)

$$\begin{aligned}
M_X(t) &= \alpha \sum_{k,j,m,n=0}^\infty \frac{(-1)^m\, [(k+1)b+j]\, v_{k,j}\, n!}{\beta^{n+1}} \binom{[(k+1)b+j]-1}{m} \\
&\times \binom{-(1+m)\alpha - 1}{n} (-t)^{-(n+1)},
\end{aligned}$$

which is the main result of this section.

5. Rényi and q-Entropies

The entropy of a random variable X is a measure of the uncertain variation. The Rényi entropy is defined by

$$I_R(\delta) = \frac{1}{1-\delta} \log\left[I(\delta)\right],$$

where $I(\delta) = \int_{\Re} f^{\delta}(x)\,dx, \delta > 0$ and $\delta \neq 1$. We have

$$
\begin{aligned}
I(\delta) &= \left(\frac{a\,b\,\alpha}{\beta}\right)^{\delta} \int_0^{\infty} \left(1+\frac{x}{\beta}\right)^{\delta(b\alpha-1)} \left\{1 - \left(1+\frac{x}{\beta}\right)^{-\alpha}\right\}^{\delta(b-1)} \\
&\times \quad \exp\left\{-a\delta\left\{\left(1+\frac{x}{\beta}\right)^{\alpha} - 1\right\}^{b}\right\} dx.
\end{aligned}
$$

By expanding the exponential term of the above integrand, we can write

$$
\begin{aligned}
I(\delta) &= \left(\frac{a\,b\,\alpha}{\beta}\right)^{\delta} \int_0^{\infty} \left(1+\frac{x}{\beta}\right)^{\delta(b\alpha-1)} \left\{1 - \left(1+\frac{x}{\beta}\right)^{-\alpha}\right\}^{\delta(b-1)} \\
&\times \sum_{k=0}^{\infty} \frac{(-1)^k\,(\delta\,a)^k}{k!} \left\{\left(1+\frac{x}{\beta}\right)^{\alpha} - 1\right\}^{bk} dx.
\end{aligned}
$$

Using the binomial expansion twice in the last equation and integrating, we obtain

$$(5.1) \qquad I(\delta) = \left(\frac{a\,b\,\alpha}{\beta}\right)^{\delta} \sum_{m=0}^{\infty} t_m.$$

Hence, the Rényi entropy reduces to

$$(5.2) \qquad I_R(\delta) = \frac{1}{1-\delta} \log\left[\left(\frac{a\,b\,\alpha}{\beta}\right)^{\delta} \left(\sum_{m=0}^{\infty} t_m\right)\right],$$

where

$$t_m = \sum_{k,j=0}^{\infty} \frac{(-1)^k\,\beta^{m+1}\,\Gamma(\delta(b+1)+bk+j)\,\Gamma(m-\delta(b-1)+bk+j)}{k!\,j!\,m!\,[m\alpha+\delta(\alpha+1)-1]\,\Gamma(\delta(b+1)+bk)\,\Gamma(bk-\delta(b+1)+j)}.$$

The q-entropy, say $H_q(f)$, is defined by

$$H_q(f) = \frac{1}{q-1} \log\left[1 - I_q(f)\right],$$

where $I_q(f) = \int_{\Re} f^q(x)\,dx, q > 0$ and $q \neq 1$. From equation (5.2), we can easily obtain

$$H_q(f) = \frac{1}{q-1} \log\left[1 - \left(\frac{a\,b\,\alpha}{\beta}\right)^q \left(\sum_{m=0}^{\infty} t_m\right)\right].$$

6. Order Statistics

Here, we provide the density of the ith order statistic $X_{i:n}$, $f_{i:n}(x)$ say, in a random sample of size n from the WL distribution. By suppressing the parameters, we have (for $i = 1, \ldots, n$)

$$(6.1) \qquad f_{i:n}(x) = \frac{f(x)}{B(i, n-i+1)} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} F(x)^{i+j-1}.$$

Thus, we can write

$$F(x)^{i+j-1} = \sum_{k=0}^{\infty} (-1)^k \binom{i+j-1}{k} \exp\left\{-ak\left\{\left(1+\frac{x}{\beta}\right)^{\alpha} - 1\right\}^{b}\right\}$$

and then by inserting (2.2) in equation (6.1), we obtain

$$(6.2) \qquad f_{i:n}(x) = \sum_{m=0}^{\infty} t_{m+1} \, f(x; (m+1)a, b, \alpha, \beta),$$

where

$$t_{m+1} = \frac{1}{(m+1) \, B(i, n-i+1)} \sum_{j=0}^{n-i} (-1)^{j+m} \binom{n-i}{j} \binom{i+j-1}{m}$$

and $f(x; (m+1)a, b, \alpha, \beta)$ denotes the WL density function with parameters $(m+1)a, b$, α and β. So, the density function of the WL order statistics is a mixture of WL densities. Based on equation (6.2), we can obtain some structural properties of $X_{i:n}$ from those WL properties.

The rth moment of $X_{i:n}$ (for $r < \alpha$) follows from (4.2) and (6.2) as

$$(6.3) \quad E(X_{i:n}^r) = \beta^r \sum_{m=0}^{r} (-1)^m \binom{r}{m} t_{m+1} \sum_{k,j=0}^{\infty} [(k+1)b+j] \, v_{k,j} \, B\Big([k+1]b+j, \tfrac{m-r}{\alpha}+1\Big),$$

where $v_{k,j}$ is given in Section 3.

The L-moments are analogous to the ordinary moments but can be estimated by linear combinations of order statistics. They exist whenever the mean of the distribution exists, even though some higher moments may not exist, and are relatively robust to the effects of outliers. Based upon the moments (6.3), we can derive explicit expressions for the L-moments of X as infinite weighted linear combinations of the means of suitable WL distributions. They are linear functions of expected order statistics defined by (for $s \geq 1$) $\lambda_s = s^{-1} \sum_{p=0}^{s-1} (-1)^p \binom{s-1}{p} E(X_{s-p:p})$.

The first four L-moments are: $\lambda_1 = E(X_{1:1})$, $\lambda_2 = \frac{1}{2} E(X_{2:2} - X_{1:2})$, $\lambda_3 = \frac{1}{3} E(X_{3:3} - 2X_{2:3} + X_{1:3})$ and $\lambda_4 = \frac{1}{4} E(X_{4:4} - 3X_{3:4} + 3X_{2:4} - X_{1:4})$. We can easily obtain the λ's for X from (6.3) with $r = 1$.

7. Estimation

Here, we consider the estimation of the unknown parameters of the WL distribution by the maximum likelihood method. Let $x_1, \ldots, x_n$ be a sample of size n from the WL distribution given by (2.2). The log-likelihood function for the vector of parameters $\Theta = (a, b, \alpha, \beta)^{\intercal}$ can be expressed as

$$(7.1) \qquad \ell = \ell(\Theta) = n \log (ab\,\alpha) - n \log \beta - (\alpha - 1) \sum_{i=1}^{n} \log \Big[1 + \frac{x_i}{\beta}\Big]$$

$$+ (b-1) \sum_{i=1}^{n} \log \Big\{ \Big[1 + \frac{x_i}{\beta}\Big]^{\alpha} - 1 \Big\} - a \sum_{i=1}^{n} \Big\{ \Big[1 + \frac{x_i}{\beta}\Big]^{\alpha} - 1 \Big\}^{b}.$$

Let $z_i = \big(1 + \frac{x_i}{\beta}\big)^{\alpha} - 1$. Then, we can write ℓ as

$$(7.2) \qquad \ell = n \log (ab\alpha) - n \log \beta - \Big(1 - \tfrac{1}{\alpha}\Big) \sum_{i=1}^{n} \log\big(z_i + 1\big) + (b-1) \sum_{i=1}^{n} \log(z_i) - a \sum_{i=1}^{n} z_i^{b}.$$

The log-likelihood function can be maximized either directly by using the R-package (AdequecyModel), SAS (PROC NLMIXED) or the Ox program (sub-routine MaxBFGS) (see Doornik, [29]) or by solving the nonlinear likelihood equations obtained by differentiating (7.1) or (7.2). In the *AdequecyModel* package, there exist many maximization algorithms like NR (Newton-Raphson), BFGS (Broyden-Fletcher-Goldfarb-Shanno), BHHH (Berndt-Hall-Hall-Hausman), NM (Nelder-Mead), SANN (Simulated-Annealing)

and Limited-Memory quasi-Newton code for Bound-Constrained Optimization (L-BFGS-B). However, the MLEs here are computed using L-BFGS-B method.

The components of the score vector $U(\Theta)$ are given by

$$U_a \;=\; \frac{n}{a} - \sum_{i=1}^{n} z_i^b,$$

$$U_b \;=\; \frac{n}{b} - \sum_{i=1}^{n} \log z_i - a \sum_{i=1}^{n} z_i^b \log z_i,$$

$$U_\alpha \;=\; \frac{n}{\alpha} + \frac{1}{\alpha^2} \sum_{i=1}^{n} \log(z_i+1) + \left(1 - \frac{1}{\alpha}\right) \sum_{i=1}^{n} \log(z_i+1)^{\frac{1}{\alpha}}$$

$$+(b-1)\sum_{i=1}^{n} \left[\left(1 + z_i^{-1}\right) \log(z_i+1)^{\frac{1}{\alpha}} \right]$$

$$-ab \sum_{i=1}^{n} \left(z_i^b + z_i^{b-1} \right) \log(z_i+1)^{\frac{1}{\alpha}},$$

$$U_\beta \;=\; -\frac{n}{\beta} - \frac{\alpha}{\beta^2} \left(1 - \frac{1}{\alpha}\right) \sum_{i=1}^{n} (z_i+1)^{-\frac{1}{\alpha}}$$

$$-\frac{\alpha(b-1)}{\beta^2} \sum_{i=1}^{n} z_i^{-1} (z_i+1)^{1-\frac{1}{\alpha}} + \frac{ab\alpha}{\beta^2} \sum_{i=1}^{n} z_i^{b-1} (z_i+1)^{1-\frac{1}{\alpha}}.$$

Setting these above equations to zero and solving them simultaneously also yield the MLEs of the four parameters.

For interval estimation of the model parameters, we require the 4×4 observed information matrix $J(\Theta) = \{J_{rs}\}$ (for $r, s = a, b, \alpha, \beta$) given in Appendix A. Under standard regularity conditions, the multivariate normal $N_4(0, J(\widehat{\Theta})^{-1})$ distribution can be used to construct approximate confidence intervals for the model parameters. Here, $J(\widehat{\Theta})$ is the total observed information matrix evaluated at $\widehat{\Theta}$. Then, the $100(1-\gamma)\%$ confidence intervals for a, b, α and β are given by $\hat{a} \pm z_{\gamma/2} \times \sqrt{var(\hat{a})}$, $\hat{b} \pm z_{\gamma/2} \times \sqrt{var(\hat{b})}$, $\hat{\alpha} \pm z_{\gamma/2} \times \sqrt{var(\hat{\alpha})}$ and $\hat{\beta} \pm z_{\gamma/2} \times \sqrt{var(\hat{\beta})}$, respectively, where the $var(\cdot)$'s denote the diagonal elements of $J(\widehat{\Theta})^{-1}$ corresponding to the model parameters, and $z_{\gamma/2}$ is the quantile $(1 - \gamma/2)$ of the standard normal distribution. Two problems that can be addressed in a future research are: (i) how large are the correlations between the parameter estimates? and (ii) how about the sample size required in order for the asymptotic standard errors to be reasonable approximations? The answer to problem (i) could be investigated through simulation studies. The answer to (ii) is related to the adequacy of the normal approximation to the MLE $\widehat{\Theta}$. Clearly, some asymptotic techniques could be adopted to improve the normal approximation for $\widehat{\Theta}$.

The likelihood ratio (LR) statistic can be used to check if the fitted new distribution is strictly "superior" to the fitted Lomax distribution for a given data set. Then, the test of $H_0 : a = b = 1$ versus $H_1 : H_0$ *is not true* is equivalent to compare the WL and Lomax distributions and the LR statistic becomes $w = 2\{\ell(\widehat{a}, \widehat{b}, \widehat{\alpha}, \widehat{\beta}) - \ell(1, 1, \tilde{\alpha}, \tilde{\beta})\}$, where $\widehat{a}$, $\widehat{b}$, $\widehat{\alpha}$ and $\widehat{\beta}$ are the MLEs under H_1 and $\tilde{\alpha}$ and $\tilde{\beta}$ are the estimates under H_0.

8. Applications

In this section, we illustrate the usefulness of the WL model. We fit the WL distribution to two data sets and compare the results with those of the fitted McL, KwL, GL, BL, EL and Lomax models.

8.1. Aircraft Windshield data sets. The windshield on a large aircraft is a complex piece of equipment, comprised basically of several layers of material, including a very strong outer skin with a heated layer just beneath it, all laminated under high temperature and pressure. Failures of these items are not structural failures. Instead, they typically involve damage or delamination of the nonstructural outer ply or failure of the heating system. These failures do not result in damage to the aircraft but do result in replacement of the windshield.

We consider the data on failure and service times for a particular model windshield given in Table 16.11 of Murthy *et al.* [48]. These data were recently studied by Ramos *et al.* [51]. The data consist of 153 observations, of which 88 are classified as failed windshields, and the remaining 65 are service times of windshields that had not failed at the time of observation. The unit for measurement is 1000 h.

First data set: Failure times of 84 Aircraft Windshield
0.040, 1.866, 2.385, 3.443, 0.301, 1.876, 2.481, 3.467, 0.309, 1.899, 2.610, 3.478, 0.557,

1.911, 2.625, 3.578, 0.943, 1.912, 2.632, 3.595, 1.070, 1.914, 2.646, 3.699, 1.124, 1.981, 2.661, 3.779,1.248, 2.010, 2.688, 3.924, 1.281, 2.038, 2.82,3, 4.035, 1.281, 2.085, 2.890, 4.121, 1.303, 2.089, 2.902, 4.167, 1.432, 2.097, 2.934, 4.240, 1.480, 2.135, 2.962, 4.255, 1.505, 2.154, 2.964, 4.278, 1.506, 2.190, 3.000, 4.305, 1.568, 2.194, 3.103, 4.376, 1.615, 2.223, 3.114, 4.449, 1.619, 2.224, 3.117, 4.485, 1.652, 2.229, 3.166, 4.570, 1.652, 2.300, 3.344, 4.602, 1.757, 2.324, 3.376, 4.663.

Second data set: Service times of 63 Aircraft Windshield
0.046, 1.436, 2.592, 0.140, 1.492, 2.600, 0.150, 1.580, 2.670, 0.248, 1.719, 2.717, 0.280,

1.794, 2.819, 0.313, 1.915, 2.820, 0.389, 1.920, 2.878, 0.487, 1.963, 2.950, 0.622, 1.978, 3.003, 0.900, 2.053, 3.102, 0.952, 2.065, 3.304, 0.996, 2.117, 3.483, 1.003, 2.137, 3.500, 1.010, 2.141, 3.622, 1.085, 2.163, 3.665, 1.092, 2.183, 3.695, 1.152, 2.240, 4.015, 1.183, 2.341, 4.628, 1.244, 2.435, 4.806, 1.249, 2.464, 4.881, 1.262, 2.543, 5.140.

We estimate the unknown parameters of each model by maximum likelihood using L-BFGS-B method and the goodness-of-fit statistics Akaike information criterion (AIC), Bayesian information criterion (BIC), consistent Akaike information criterion (CAIC), Hannan-Quinn information criterion (HQIC), Anderson-Darling (A^*) and Cramér–von Mises (W^*) are used to compare the five models. The statistics $A^*$and W^* are described in details in [22]. In general, the smaller the values of these statistics, the better the fit to the data. The required computations are carried out using the R-script AdequacyModel developed by Pedro Rafael Diniz Marinho, Cícero Rafael Barros Dias and Marcelo Bourguignon. It is freely available from
http://cran.r-project.org/web/packages/AdequacyModel/AdequacyModel.pdf.

Tables 1 and 3 give the MLEs and their corresponding standard errors (in parentheses) of the model parameters. The model selection is carried out using the AIC , BIC, CAIC and HQIC statistics defined by:

$$AIC = -2\ell(\cdot) + 2p, \quad BIC = -2\ell(\cdot) + p\log(n),$$

$$CAIC = -2\ell(\cdot) + \frac{2pn}{n-p-1}, \quad \text{and} \quad HQIC = 2\log\left[\log(n)\left(k - 2\,\ell(\cdot)\right)\right],$$

where $\ell(\cdot)$ denotes the log-likelihood function evaluated at the MLEs, p is the number of parameters, and n is the sample size. The figures in Tables 1 and 3 indicate that the fitted Lomax models have huge parameter estimates, although they are accurate compared with their standard errors. Sometimes, the log-likelihood can become quite flat by fitting special models of the WL distribution leading to numerical maximization problems. For these cases, we can obtain different MLEs for the model parameters using

alternative algorithms of maximization since they correspond to local maximums of the log-likelihood function. Thus, it is important to investigate the global maximum.

The values of the AIC, CAIC, BIC, HQIC, A^* and W^* are listed in Tables 2 and 4. Tables 2 and 4 compare the WL model with the McL, KwL, GL, BL, EL and

TABLE 1. MLEs and their standard errors (in parentheses) for failure times of 84 Aircraft Windshield data

Distribution	a	b	c	α	β
WL	0.0128	0.5969	-	6.7753	1.5324
	(0.0114)	(0.3590)	-	(3.9049)	(1.3863)
McL	2.1875	119.1751	12.4171	19.9243	75.6606
	(0.5211)	(140.2970)	(20.8446)	(38.9601)	(147.2422)
KwL	2.6150	100.2756	-	5.2771	78.6774
	(0.3822)	(120.4856)	-	(9.8116)	(186.0052)
GL	3.5876	-	-	52001.4994	37029.6583
	(0.5133)	-	-	(7955.0003)	(81.1644)
BL	3.6036	33.6387	-	4.8307	118.8374
	(0.6187)	(63.7145)	-	(9.2382)	(428.9269)
EL	3.6261	-	-	20074.5097	26257.6808
	(0.6236)	-	-	(2041.8263)	(99.7417)
Lomax	-	-	-	51425.3500	131789.7800
	-	-	-	(5933.4892)	(296.1198)

TABLE 2. The statistics $\ell(\cdot)$, AIC, BIC , CAIC , HQIC, A^* and W^* for failure times of 84 Aircraft Windshield data

Distribution	$\ell(\cdot)$	AIC	CAIC	BIC	HQIC	A^*	W^*
WL	**-127.8652**	**263.7303**	**264.2303**	**273.5009**	**267.6603**	**0.6185**	0.0932
McL	-129.8023	269.6045	270.3640	281.8178	274.5170	0.6672	0.0858
KwL	-132.4048	272.8096	273.3096	282.5802	276.7396	0.6645	0.0658
GL	-138.4042	282.8083	283.1046	290.1363	285.7559	1.3666	0.1618
BL	-138.7177	285.4354	285.9354	295.2060	289.3654	1.4084	0.1680
EL	-141.3997	288.7994	289.0957	296.1273	291.7469	1.7435	0.2194
Lomax	-164.9884	333.9767	334.1230	338.8620	335.9417	1.3976	0.1665

Lomax models. We note that the WL model gives the lowest values for the AIC, BIC and CAIC, HQIC and A^* statistics (except W^* for the first data set) among all fitted models. So, the WL model could be chosen as the best model. The histogram of the data and the estimated pdfs and cdfs for the fitted models are displayed in Figure 3. It is clear from Tables 2 and 4 and Figure 3 that the WL distribution provides a better fit to the histogram and therefore could be chosen as the best model for both data sets.

TABLE 3. MLEs and their standard errors (in parentheses) for service times of 63 Aircraft Windshield data

| Distribution	a	b	c	α	β
WL	0.1276	0.9204	-	3.9136	3.0067
	(0.2964)	(0.4277)	-	(3.8489)	(8.2769)
McL	1.3230	53.7712	5.7144	7.4371	42.8972
	(0.2517)	(199.2803)	(5.3853)	(34.7310)	(150.8150)
KwL	1.6691	60.5673	-	2.5649	65.0640
	(0.2570)	(86.0131)	-	(4.7589)	(177.5919)
GL	1.9073	-	-	35842.4330	39197.5715
	(0.3213)	-	-	(6945.0743)	(151.6530)
BL	1.9218	31.2594	-	4.9684	169.5719
	(0.3184)	(316.8413)	-	(50.5279)	(339.2067)
EL	1.9145	-	-	22971.1536	32881.9966
	(0.3482)	-	-	(3209.5329)	(162.2299)
Lomax	-	-	-	99269.78 00	207019.3700
	-	-	-	(11863.5222)	(301.2366)

TABLE 4. The statistics $\ell(\cdot)$, AIC, BIC, CAIC, HQIC, A^* and W^* for service times of 63 Aircraft Windshield data

Distribution	$\ell(.)$	AIC	CAIC	BIC	HQIC	A^*	W^*
WL	**-98.11712**	**204.2342**	**204.9239**	**212.8068**	**207.6059**	**0.2417**	**0.0356**
McL	-98.5883	207.1766	208.2292	217.8923	211.3911	0.3560	0.0573
KwL	-100.8676	209.7353	210.4249	218.3078	213.1069	0.7391	0.1219
GL	-102.8332	211.6663	212.0731	218.0958	214.1951	1.112	0.1836
BL	-102.9611	213.9223	214.6119	222.4948	217.2939	1.1336	0.1872
EL	-103.5498	213.0995	213.5063	219.5289	215.6282	1.2331	0.2037
Lomax	-109.2988	222.5976	222.7976	226.8839	224.2834	1.1265	0.1861

9. Concluding remarks

In this paper, we propose a four-parameter Weibull-Lomax (WL) distribution. We study some structural properties of the WL distribution including an expansion for the density function and explicit expressions for the ordinary and incomplete moments, mean residual life, mean waiting time, probability weighted moments, generating function and quantile function. Further, the explicit expressions for the Rényi entropy, q entropy and order statistics are also derived. The maximum likelihood method is employed for estimating the model parameters. We also obtain the observed information matrix. We fit the WL model to two real life data sets to show the usefulness of the proposed distribution. The new model provides consistently a better fit than the other models, namely: the McDonald-Lomax, Kumaraswamy-Lomax, gamma-Lomax, beta-Lomax, exponentiated-Lomax and Lomax distributions. We hope that the proposed model will attract wider application in areas such as engineering, survival and lifetime data, hydrology, economics (income inequality) and others.

(a) Estimated pdfs for the first data set   (b) Estimated cdfs for the first data set.
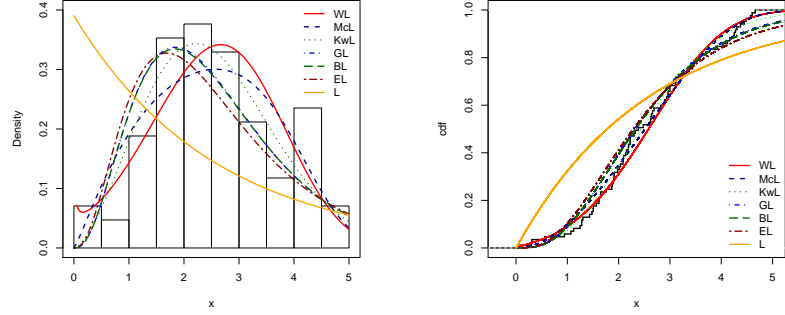
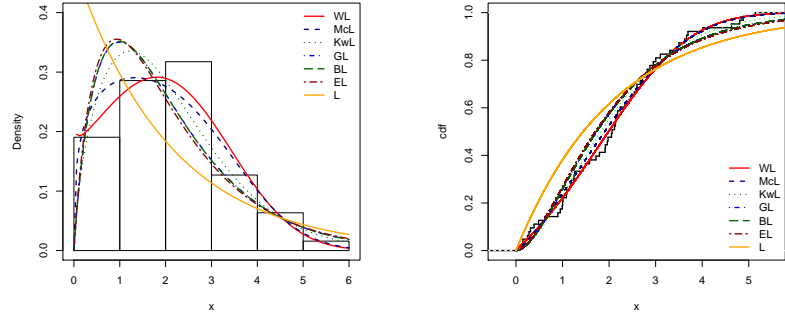(c) Estimated pdfs for the second data set (d) Estimated cdfs for the second data set

FIGURE 3. Plots of the estimated pdfs and cdfs for the WL, McL, KwL, GL, BL, EL and Lomax models to the first and second data sets.

Acknowledgments

Appendix A

The elements of the 4×4 observed information matrix $J(\Theta) = \{J_{rs}\}$ (for $r, s = a, b, \alpha, \beta$) are given by

$$J_{aa} = -\frac{n}{a^2},$$

$$J_{ab} = -\sum_{i=1}^{n} z_i^b \log z_i,$$

$$J_{a\alpha} = -b \sum_{i=1}^{n} z_i^{b-1} (z_i + 1) \log (z_i + 1)^{\frac{1}{\alpha}},$$

$$J_{a\beta} = \frac{\alpha b}{\beta^2} \sum_{i=1}^{n} z_i^{b-1} \left(z_i + 1\right)^{1-\frac{1}{\alpha}},$$

$$J_{bb} = -\frac{n}{b^2} - a \sum_{i=1}^{n} z_i^{b} \left[\log(z_i)\right]^2,$$

$$J_{b\alpha} = \sum_{i=1}^{n} \left(1 + z_i^{-1}\right) \log \left(z_i + 1\right)^{\frac{1}{\alpha}}$$
$$-a(b+1) \sum_{i=1}^{n} \left[z_i^{b-1} \left(z_i + 1\right) \log \left(z_i + 1\right)^{\frac{1}{\alpha}}\right],$$

$$J_{b\beta} = -\frac{\alpha}{\beta^2} \sum_{i=1}^{n} z_i^{-1} \left(z_i + 1\right)^{1-\frac{1}{\alpha}} - \frac{a\alpha}{\beta^2} \sum_{i=1}^{n} z_i^{b-1} \left(z_i + 1\right)^{1-\frac{1}{\alpha}} \left[1 + b \log z_i\right],$$

$$J_{\alpha\alpha} = -\frac{n}{\alpha^2} + \sum_{i=1}^{n} \left(z_i + 1\right) \left[\alpha + (b-1) \left\{\alpha \left(1 + z_i^{-1}\right) - z_i^{-2}\right\}\right.$$
$$\left. -ab \left\{\alpha \left(z_i^{b} + z_i^{b-1}\right) + b z_i^{b} + (b-1) z_i^{b-2}\right\}\right],$$

$$J_{\alpha\beta} = -\frac{1}{\beta^2} \sum_{i=1}^{n} \left(z_i + 1\right)^{-\frac{1}{\alpha}}$$
$$-\sum_{i=1}^{n} \left(z_i + 1\right)^{1-\frac{1}{\alpha}} \left[-\frac{(b-1)}{\beta^2} \left\{z_i^{-1} - z_i^{-2} \log \left(z_i + 1\right)^{\frac{1}{\alpha}}\right\}\right.$$
$$\left. +\frac{ab}{\beta^2} \left\{\left(z_i^{b} + z_i^{b-1}\right) \left(z_i + 1\right)^{-1} + \log \left(z_i + 1\right) \left[b z_i^{b-1} + (b-1) z_i^{b-2}\right]\right\}\right],$$

$$J_{\beta\beta} = -\frac{n}{\beta 2} + \frac{\alpha}{\beta^3} \left(1 - \frac{1}{\alpha}\right) \sum_{i=1}^{n} \left(z_i + 1\right)^{-\frac{1}{\alpha}} \left\{z_i + \frac{1}{\beta} \left(z_i + 1\right)^{1-\frac{1}{\alpha}} \log \left(z_i + 1\right)\right\}$$
$$-\frac{\alpha}{\beta^3} (b-1) \sum_{i=1}^{n} \left(z_i + 1\right)^{1-\frac{1}{\alpha}}$$
$$\left[-2 z_i^{-1} - \frac{\alpha}{\beta} z_i^{-2} \left(z_i + 1\right)^{1-\frac{1}{\alpha}} - \frac{\alpha}{\beta} \left(1 - \frac{1}{\alpha}\right) z_i^{-1} \left(z_i + 1\right)^{-\frac{1}{\alpha}}\right]$$
$$+\frac{ab\alpha}{\beta^3} \sum_{i=1}^{n} \left(z_i + 1\right)^{1-\frac{1}{\alpha}}$$
$$\left[-2 z_i^{b-1} - \frac{(b-1)\alpha}{\beta} z_i^{b-1} \left(z_i + 1\right)^{1-\frac{1}{\alpha}} - \frac{\alpha}{\beta} \left(1 - \frac{1}{\alpha}\right) z_i^{b-1} \left(z_i + 1\right)^{-\frac{1}{\alpha}}\right].$$

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