

## On the expected discounted penalty function for a risk model with two classes of claims and random incomes

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### Abstract

In this paper, we consider a risk model with two independent classes of insurance risks and random incomes. We assume that the two independent claim counting processes are, respectively, the Poisson and the Erlang(2) process. When the individual premium sizes are exponentially distributed, the explicit expressions for the Laplace transforms of the expected discounted penalty functions are derived. We prove that the expected discounted penalty functions satisfy some defective renewal equations. By employing an associated compound geometric distribution, the analytic expressions for the solutions of the defective renewal equations are obtained. Assuming that the distributions of premium sizes have rational Laplace transforms, we also give the explicit representations for the Laplace transforms of the expected discounted penalty functions.

*2000 AMS Classification:* 60H10, 62P05.

**Keywords:** Compound Poisson risk model, Erlang risk process, Expected discounted penalty function, Random income.

Received 18 /11 /2013 : Accepted 03 /05 /2014 Doi : 10.15672/HJMS.2014157466

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## 1. Introduction

In the actuarial literature, the surplus process of an insurance company is often modeled by the following classic risk process

$$U(t) = u + ct - \sum_{i=1}^{N(t)} Y_i, \quad t \geq 0, \quad (1.1)$$

where  $u = U(0) \geq 0$  is the initial surplus,  $c > 0$  is the constant premium income rate,  $N(t)$  counting the number of claims that occurred before time  $t$  is a Poisson process, and  $\{Y_i\}_{i \geq 1}$  is a sequence of strictly positive random variables (r.v.) representing the claim amounts.

Classic risk models rely on assumption that there is only one class of claims. Although this hypothesis simplifies the study of many risk quantities, it has been proven to be too restrictive in different contexts. In recent years, many authors have studied various aspects of the so-called correlated aggregate claims risk model. Yuen et al. [18] considered the non-ruin probability for a correlated risk process involving two dependent classes of insurance risks, with exponential claims, which can be transformed into a surplus process with two independent classes of insurance risks, for which one claim number process is Poisson and the other is a renewal process with Erlang(2) claim inter-arrival times. Li and Garrido [9] considered a risk process with two classes of independent risks, namely, the compound Poisson process and the renewal process with generalized Erlang(2) inter-arrival times. A system of integro-differential equations for the non-ruin probabilities was derived and explicit results for claim amounts having distributions belonging to the rational family were obtained. A further extension was given by Li and Lu [10]. They derived a system of integro-differential equations for the Gerber-Shiu discounted penalty functions, when the ruin is caused by a claim belonging either to the first or to the second class and obtained explicit results when the claim sizes are exponentially distributed. Recently, Zhang et al. [19] extended the model of Li and Lu [10], by considering the claim number process of the second class to be a renewal process with generalized Erlang( $n$ ) inter-arrival times. The authors derived an integro-differential equation system for the Gerber-Shiu functions, and obtained their Laplace transforms when the corresponding Lundberg equation has distinct roots. Chadjicostantinidis and Papaioannou [4] studied a risk model with two independent classes of insurance risks in the presence of a constant dividend barrier. A system of integro-differential equations with certain boundary conditions for the Gerber-Shiu function was derived and solved. Using systems of integro-differential equations for the moment-generating function as well as for the arbitrary moments of the discounted sum of the dividend payments until ruin, a matrix version of the dividends-penalty was derived. Under the risk models involving two classes of insurance risks described above, the premiums are assumed to be received at a constant rate over time.

Sometimes, the insurance company may have lump sums of income. In order to describe the stochastic income, Boucherie et al. [3] added a compound Poisson process with positive jumps to the Cramér-Lundberg model. The (non-)ruin probabilities for the risk models with stochastic premiums were studied in Boikov [2] and Temnov [12]. Assuming that the premium process is a Poisson process, Bao [1] studied the Gerber-Shiu function in the compound Poisson risk model. Yang and Zhang [17] extended the compound Poisson risk model in Bao [1] to a Sparre Andersen risk model with generalized Erlang( $n$ ) interclaim time distribution. Labbe and Sendova [7] considered a risk model with stochastic premiums income, where both the premium size distribution and the claim size distribution are non-lattice. Zhang and Yang [20] extended the model in Labbe and Sendova [7] by assuming that there exists a specific dependence structure

among the claim sizes, interclaim times and premium sizes. Xie and Zou [16] construct a risk model with a dependence setting where there exists a specific structure among the time between two claim occurrences, premium sizes and claim sizes. When the claims are subexponentially distributed, the asymptotic formulae for ruin probabilities are obtained.

All risk models with random incomes described in the paragraph above focus on risk model with only one class of insurance risk. Motivated by these papers, we explore analogue problems, but in a risk model with random incomes involving two independent classes of insurance risks. Moreover, we assume that the two independent claim counting processes are, respectively, the Poisson and the Erlang(2) process.

Studying the risk model with two classes of claims and random incomes is of interest in ruin theory for two reasons. The first is to make predictions and give risk measures for smaller business whose premium income is more fluctuant than what is received in well established and large insurance company. The second reason for study this risk model is to provide insight about how the randomness in premiums' process influences the risk process with two classes of claims. Replacing the constant premium income in risk model with two classes of claims by a stochastic income can also be interpreted as a stepping stone for risk models with two classes of claims that are closer to real phenomena.

The paper is structured as follows: the risk model with two independent classes of insurance risks and random incomes is introduced in Section 2. Assuming that the premiums are exponentially distributed, the explicit expressions for the Laplace transforms of the expected discounted penalty are derived in Section 3 and the defective renewal equations for the expected discounted penalty are obtained in Section 4. By employing an associated compound geometric distribution, the analytic expressions for the solutions of the defective renewal equations are also given in Section 4. In Section 5, given that distributions of the premium sizes have rational Laplace transforms, we derive the explicit representations for the Laplace transforms of the expected discounted penalty functions. Finally, in Section 6, two numerical examples are given.

## 2. The model

Let us consider the surplus process  $U(t)$  of an insurance company,

$$U(t) = u + \sum_{j=1}^{M(t)} X_j - S(t), \quad t \geq 0, \quad (2.1)$$

where  $u = U(0)$  is the initial capital and  $X_j$  is the  $j$ th premium income with distribution function  $G$ , probability density function (p.d.f.)  $f_G$ , mean  $\mu_G$  and Laplace transform (LT)  $\tilde{f}_G(s) = \int_0^\infty e^{-sx} f_G(x) dx$ . We assume that  $M(t)$  is a Poisson process with intensity  $\lambda > 0$ , then the corresponding premium income inter-arrival times, denoted by  $\{W_i\}_{i \geq 1}$ , are independent and identically distributed (i.i.d.) exponentially distributed r.v. with parameter  $\lambda$ .

In this paper, we assume that  $S(t)$  is generated by two classes of insurance risks, namely

$$S(t) = S_1(t) + S_2(t) = \sum_{i=1}^{N_1(t)} Y_i + \sum_{i=1}^{N_2(t)} Z_i, \quad t \geq 0, \quad (2.2)$$

where  $S_i(t)$ ,  $i = 1, 2$ , represents the aggregate claims up to time  $t$  from the  $i$ -th class. Although such models are usually studied in the context of correlated aggregate claims, here we assume that  $S_1(t)$  and  $S_2(t)$  are stochastically independent.

The r.v.  $\{Y_i\}_{i \geq 1}$  are the nonnegative claim severities from the first class, which are i.i.d. random variables with common distribution function  $F_1$ , p.d.f.  $f_1$ , mean  $\mu_{F_1}$  and LT  $\tilde{f}_1(s) = \int_0^\infty e^{-sx} f_1(x) dx$ . Similarly,  $\{Z_i\}_{i \geq 1}$  are the positive claim severities from the

second class, also assumed i.i.d. r.v., with common distribution function  $F_2$ , p.d.f.  $f_2$ , mean  $\mu_{F_2}$  and LT  $\tilde{f}_2(s) = \int_0^\infty e^{-sx} f_2(x) dx$ . The claim number process  $N_1(t)$  is assumed to be Poisson with parameter  $\lambda_1 > 0$ . More specifically, the corresponding claim inter-arrival times, denoted by  $\{V_i\}_{i \geq 1}$ , are i.i.d. exponentially distributed r.v. with parameter  $\lambda_1$ . In addition,  $N_2(t)$  is a renewal process with i.i.d. claim inter-arrival times  $\{L_i\}_{i \geq 1}$ , which are independent of  $\{V_i\}_{i \geq 1}$  and Erlang(2) distributed r.v., i.e.  $L_i = L_{i1} + L_{i2}$ , where  $\{L_{ij}\}_{i \geq 1, j \geq 1}$  are i.i.d. exponentially distributed r.v. with parameter  $\lambda_2$ .

We finally assume that  $\{X_i\}_{i \geq 1}$ ,  $\{Y_i\}_{i \geq 1}$  and  $\{Z_i\}_{i \geq 1}$  are mutually independent, also independent of  $M(t)$ ,  $N_1(t)$  and  $N_2(t)$ , and  $\lambda\mu_G > \lambda_1\mu_{F_1} + \lambda_2/2\mu_{F_2}$ , providing a net profit condition.

Denote the ruin time by  $T = \inf\{t \geq 0 : U(t) < 0\}$  and  $\infty$  if  $U(t) \geq 0$  for all  $t \geq 0$ . The ruin probability is defined as  $\phi(u) = P(T < \infty | U(0) = u)$ ,  $u \geq 0$ . Further define  $J$  to be the cause-of-ruin r.v., i.e.,  $J = j$ , if the ruin is caused by a claim of class  $j$ ,  $j = 1, 2$ , then ruin probability  $\phi(u)$  can be decomposed as  $\phi(u) = \phi_1(u) + \phi_2(u)$ , where  $\phi_j(u) = P(T < \infty, J = j | U(0) = u)$ ,  $u \geq 0$ ,  $j = 1, 2$ , is the ruin probability due to a claim of class  $j$ . For  $\delta \geq 0$ , and  $j = 1, 2$ , the expected discounted penalty (Gerber-Shiu) function at ruin, if the ruin is caused by a claim of class  $j$  is defined as

$$\Phi_j(u) = E[e^{-\delta T} w_j(U(T-), |U(T)|) I(T < \infty, J = j) | U(0) = u], \quad u \geq 0, \quad (2.3)$$

where  $\delta \geq 0$  is interpreted as the force of interest,  $U(T-)$  is the surplus immediately before ruin,  $|U(T)|$  is the deficit at ruin,  $I(\cdot)$  is the indicator function, and  $w_j(x_1, x_2)$ ,  $0 \leq x_1, x_2 < \infty$ ,  $j = 1, 2$ , be the non-negative measurable function defined on  $[0, \infty) \times (0, \infty)$ . The financial explanations on  $w(x_1, x_2)$  can be found in Gerber and Shiu [6]. It is easy to see that choosing different forms of the function  $w_j(x_1, x_2)$  in Eq.(2.3) yields different information relating to the deficit at ruin and the surplus immediately before ruin.

In the classical risk model, due to the strong Markov property of the surplus process, the expected discounted penalty function is time homogenous, i.e., it is independent of the time at which the surplus process is observed. However, for our risk model, the expected discounted penalty function functions are no longer time homogeneous, due to the assumption that the claim inter-arrival times from the second class are Erlang(2) distributed. Therefore, for the expected discounted penalty functions, defined in (2.3), we assume that a claim from the second class occurs exactly at time 0. More generally, we can define the expected discounted penalty functions, denoted by  $\Phi_j(u, \tau)$ , as bivariate functions of current reserve  $u$  and the length of time  $\tau$ , elapsed since the time of the last claim from the second class (the surplus process renews itself at these points). The quantities we are interested in are  $\Phi_j(u, 0) = \Phi_j(u)$ ,  $j = 1, 2$ , and  $u \geq 0$ ,

$$\Psi_j(u) = E[e^{-\delta(T-t)} w_j(U(T-), |U(T)|) I(T < \infty, J = j) | L_{11} = t, U(t) = u], \quad (2.4)$$

the expected discounted penalty functions at the time of the realization of  $\{L_{i1}\}_{i \geq 1}$ . Then by the law of total probability, for  $j = 1, 2$ , we have

$$\Phi_j(u, \tau) = \Phi_j(u) \Pr(L_{11} > \tau) + \Psi_j(u) \Pr(L_{11} < \tau) = e^{-\lambda_2 \tau} \Phi_j(u) + (1 - e^{-\lambda_2 \tau}) \Psi_j(u).$$

### 3. Laplace transforms

Throughout this paper, we will use a hat  $\hat{\cdot}$  to designate the Laplace transform of a function. Given that the premium size is exponentially distributed, the explicit expressions for the Laplace transforms of the expected discounted penalty functions can be derived. For this purpose, we first consider the integral equation satisfied by the expected discounted penalty function.

Let  $J = \min\{V_1, L_{11}, W_1\}$ , then for  $u \geq 0$ ,

$$\begin{aligned} \Phi_1(u) &= \int_0^\infty \int_0^\infty \Pr(J = t, J = W_1) e^{-\delta t} \Phi_1(u+x) dG(x) dt \\ &+ \int_0^\infty \Pr(J = t, J = V_1) e^{-\delta t} \left[ \int_0^u \Phi_1(u-y) dF_1(y) + \int_u^\infty w_1(u, y-u) dF_1(y) \right] dt \\ &+ \int_0^\infty \Pr(J = t, J = L_{11}) e^{-\delta t} \Psi_1(u) dt. \end{aligned} \quad (3.1)$$

Note that  $\Pr(J = W_1) = \lambda/(\lambda + \lambda_1 + \lambda_2)$ ,  $\Pr(J = V_1) = \lambda_1/(\lambda + \lambda_1 + \lambda_2)$ ,  $\Pr(J = L_{11}) = \lambda_2/(\lambda + \lambda_1 + \lambda_2)$ ,  $\Pr(J > t | J = W_1) = \Pr(J > t | J = V_1) = \Pr(J > t | J = L_{11}) = \exp(-(\lambda + \lambda_1 + \lambda_2)t)$ .

Plugging the expressions above into (3.1) and making some simplifications, we can get

$$\begin{aligned} \Phi_1(u) &= \frac{\lambda}{\lambda^* + \delta} \int_0^\infty \Phi_1(u+x) dG(x) + \frac{\lambda_2}{\lambda^* + \delta} \Psi_1(u) \\ &+ \frac{\lambda_1}{\lambda^* + \delta} \left[ \int_0^u \Phi_1(u-y) dF_1(y) + w_1(u) \right], \end{aligned} \quad (3.2)$$

where  $\lambda^* = \lambda + \lambda_1 + \lambda_2$ ,  $w_1(u) = \int_u^\infty w_1(u, y-u) dF_1(y)$ . Similarly, we derive

$$\begin{aligned} \Psi_1(u) &= \frac{\lambda}{\lambda^* + \delta} \int_0^\infty \Psi_1(u+x) dG(x) + \frac{\lambda_2}{\lambda^* + \delta} \int_0^u \Phi_1(u-y) dF_2(y) \\ &+ \frac{\lambda_1}{\lambda^* + \delta} \left[ \int_0^u \Psi_1(u-y) dF_1(y) + w_1(u) \right], \end{aligned} \quad (3.3)$$

Assume  $A_1(u) = \int_0^\infty \Phi_1(u+x) dG(x)$  and  $\bar{A}_1(u) = \int_0^\infty \Psi_1(u+x) dG(x)$ . Taking Laplace transforms in (3.2) and (3.3) and making some simplifications, we have

$$\tilde{\Phi}_1(s) = \frac{\lambda}{\lambda^* + \delta} \tilde{A}_1(s) + \frac{\lambda_2}{\lambda^* + \delta} \tilde{\Psi}_1(s) + \frac{\lambda_1}{\lambda^* + \delta} \left[ \tilde{\Phi}_1(s) \tilde{f}_1(s) + \tilde{w}_1(s) \right], \quad (3.4)$$

$$\tilde{\Psi}_1(s) = \frac{\lambda}{\lambda^* + \delta} \tilde{\bar{A}}_1(s) + \frac{\lambda_2}{\lambda^* + \delta} \tilde{\Phi}_1(s) \tilde{f}_2(s) + \frac{\lambda_1}{\lambda^* + \delta} \left[ \tilde{\Psi}_1(s) \tilde{f}_1(s) + \tilde{w}_1(s) \right], \quad (3.5)$$

Similar analysis gives

$$\Phi_2(u) = \frac{\lambda}{\lambda^* + \delta} \int_0^\infty \Phi_2(u+x) dG(x) + \frac{\lambda_2}{\lambda^* + \delta} \Psi_2(u) + \frac{\lambda_1}{\lambda^* + \delta} \int_0^u \Phi_2(u-y) dF_1(y), \quad (3.6)$$

$$\begin{aligned} \Psi_2(u) &= \frac{\lambda}{\lambda^* + \delta} \int_0^\infty \Psi_2(u+x) dG(x) + \frac{\lambda_2}{\lambda^* + \delta} \left[ \int_0^u \Phi_2(u-y) dF_2(y) + w_2(u) \right] \\ &+ \frac{\lambda_1}{\lambda^* + \delta} \int_0^u \Psi_2(u-y) dF_1(y). \end{aligned} \quad (3.7)$$

Assume  $A_2(u) = \int_0^\infty \Phi_2(u+x) dG(x)$  and  $\bar{A}_2(u) = \int_0^\infty \Psi_2(u+x) dG(x)$ . Taking Laplace transforms in (3.6) and (3.7) and making some simplifications, we obtain

$$\tilde{\Phi}_2(s) = \frac{\lambda}{\lambda^* + \delta} \tilde{A}_2(s) + \frac{\lambda_2}{\lambda^* + \delta} \tilde{\Psi}_2(s) + \frac{\lambda_1}{\lambda^* + \delta} \tilde{\Phi}_2(s) \tilde{f}_1(s), \quad (3.8)$$

$$\tilde{\Psi}_2(s) = \frac{\lambda}{\lambda^* + \delta} \tilde{\bar{A}}_2(s) + \frac{\lambda_2}{\lambda^* + \delta} \left[ \tilde{\Phi}_2(s) \tilde{f}_2(s) + \tilde{w}_2(s) \right] + \frac{\lambda_1}{\lambda^* + \delta} \tilde{\Psi}_2(s) \tilde{f}_1(s). \quad (3.9)$$

Now, we introduce the Dickson-Hipp operator  $T_r$  provided by Dickson and Hipp [5]. Define the Dickson-Hipp operator  $T_r$  as be

$$T_r f(x) = \int_x^\infty e^{-r(y-x)} f(y) dy, \quad x \geq 0,$$

where  $f(x)$  is a real-valued function,  $r$  is a complex number. It is easy to see that  $T_s f(0) = \tilde{f}(s)$ , and that for distinct  $r_1$  and  $r_2$ ,

$$T_{r_1} T_{r_2} f(x) = T_{r_2} T_{r_1} f(x) = \frac{T_{r_1} f(x) - T_{r_2} f(x)}{r_2 - r_1}, \quad x \geq 0.$$

If  $r_1 = r_2 = r$ ,

$$T_{r_1} T_{r_2} f(x) = \int_x^\infty (y-x)e^{-r(y-x)} f(y) dy, \quad x \geq 0.$$

The properties for the Dickson-Hipp operator can also be found in Dickson and Hipp [5], Li and Garrido [8], Xie and Zou [15].

Suppose the premium sizes are exponentially distributed, i.e.,  $G(x) = 1 - e^{-\frac{x}{\mu_G}}$ , for  $\mu_G > 0$ . Taking Laplace transform of  $A_i(u)$ ,  $i = 1, 2$ , and using the Dickson-Hipp operator, we can get

$$\begin{aligned} \tilde{A}_i(s) &= \int_0^\infty e^{-su} \int_0^\infty \Phi_i(u+x) \frac{e^{-\frac{x}{\mu_G}}}{\mu_G} dx du = \int_0^\infty \int_0^\infty e^{-su} \Phi_i(u+x) du \frac{e^{-\frac{x}{\mu_G}}}{\mu_G} dx \\ &= \int_0^\infty T_s \Phi_i(x) \frac{e^{-\frac{x}{\mu_G}}}{\mu_G} dx = \frac{1}{\mu_G} T_{\frac{1}{\mu_G}} T_s \Phi_i(0) = \frac{\tilde{\Phi}_i(s) - \tilde{\Phi}_i(\frac{1}{\mu_G})}{1 - s\mu_G}, \quad i = 1, 2. \end{aligned} \quad (3.10)$$

Similarly,

$$\tilde{\tilde{A}}_i(s) = \frac{\tilde{\Psi}_i(s) - \tilde{\Psi}_i(\frac{1}{\mu_G})}{1 - s\mu_G}, \quad i = 1, 2. \quad (3.11)$$

Combining the above results with (3.4), (3.5), (3.8) and (3.9), respectively, we derive

$$\tilde{\Phi}_1(s) = \frac{-\left(1 - \frac{\lambda}{(\lambda^* + \delta)(1 - s\mu_G)} - \frac{\lambda_1 \tilde{f}_1(s)}{\lambda^* + \delta}\right) \left(\frac{\lambda \tilde{\Phi}_1(\frac{1}{\mu_G})}{(\lambda^* + \delta)(1 - s\mu_G)} - \frac{\lambda_1 \tilde{\omega}_1(s)}{\lambda^* + \delta}\right) - \frac{\lambda_2 \varpi_1(s)}{\lambda^* + \delta}}{\left(1 - \frac{\lambda}{(\lambda^* + \delta)(1 - s\mu_G)} - \frac{\lambda_1 \tilde{f}_1(s)}{\lambda^* + \delta}\right)^2 - \frac{\lambda_2^2 \tilde{f}_2(s)}{(\lambda^* + \delta)^2}}, \quad (3.12)$$

$$\tilde{\Phi}_2(s) = \frac{-\frac{\lambda \tilde{\Phi}_2(\frac{1}{\mu_G})(1 - \frac{\lambda}{(\lambda^* + \delta)(1 - s\mu_G)} - \frac{\lambda_1 \tilde{f}_1(s)}{\lambda^* + \delta})}{(\lambda^* + \delta)(1 - s\mu_G)} - \frac{\lambda_2 \varpi_2(s)}{\lambda^* + \delta}}{\left(1 - \frac{\lambda}{(\lambda^* + \delta)(1 - s\mu_G)} - \frac{\lambda_1 \tilde{f}_1(s)}{\lambda^* + \delta}\right)^2 - \frac{\lambda_2^2 \tilde{f}_2(s)}{(\lambda^* + \delta)^2}}. \quad (3.13)$$

where  $\varpi_i(s) = \frac{\lambda \tilde{\Psi}_i(\frac{1}{\mu_G})}{(\lambda^* + \delta)(1 - s\mu_G)} - \frac{\lambda_i \tilde{\omega}_i(s)}{\lambda^* + \delta}$ ,  $i = 1, 2$ .

To obtain  $\tilde{\Phi}_1(s)$  and  $\tilde{\Phi}_2(s)$ , we still have to determine  $\tilde{\Phi}_1(\frac{1}{\mu_G})$ ,  $\tilde{\Psi}_1(\frac{1}{\mu_G})$ ,  $\tilde{\Phi}_2(\frac{1}{\mu_G})$ , and  $\tilde{\Psi}_2(\frac{1}{\mu_G})$ . For this purpose, we discuss analytically the zeros of the common denominators of (3.12) and (3.14), i.e., the roots of following equation

$$\left(1 - \frac{\lambda}{(\lambda^* + \delta)(1 - s\mu_G)} - \frac{\lambda_1 \tilde{f}_1(s)}{\lambda^* + \delta}\right)^2 - \frac{\lambda_2^2 \tilde{f}_2(s)}{(\lambda^* + \delta)^2} = 0. \quad (3.14)$$

**3.1. Lemma.** 1. For  $\delta > 0$ , Eq.(3.14) has exactly two roots, say,  $\rho_1(\delta)$ , and  $\rho_2(\delta)$ , in the right half complex plane, i.e.,  $\mathbf{Re} \rho_i(\delta) > 0$  for  $i = 1, 2$ .

*Proof.* Eq.(3.14) can be simplified as

$$\left(1 - s\mu_G - \frac{\lambda}{\lambda^* + \delta} - \frac{\lambda_1(1 - s\mu_G)\tilde{f}_1(s)}{\lambda^* + \delta}\right)^2 - \frac{(\lambda_2(1 - s\mu_G))^2 \tilde{f}_2(s)}{(\lambda^* + \delta)^2} = 0.$$

Let  $r > 0$  be a sufficiently large number, and define  $\mathbf{C}_r$  as the contour containing the imaginary axis running from  $-ir$  to  $ir$  and a half circle with radius  $r$  running clockwise from  $ir$  to  $-ir$ . Firstly, we apply Rouché's theorem to prove that equation

$$1 - s\mu_G - \frac{\lambda}{\lambda^* + \delta} - \frac{\lambda_1(1 - s\mu_G)\tilde{f}_1(s)}{\lambda^* + \delta} = 0,$$

has exactly one root inside  $\mathbf{C}_r$ . When  $s$  on the imaginary axis, we have,

$$\frac{\left| \frac{\lambda_1(1-s\mu_G)\tilde{f}_1(s)}{\lambda^*+\delta} \right|}{\left| 1-s\mu_G - \frac{\lambda}{\lambda^*+\delta} \right|} = \frac{\left| \frac{\lambda_1(1-s\mu_G)}{\lambda^*+\delta} \right| |\tilde{f}_1(s)|}{\left| \frac{\lambda_1+\lambda_2+\delta}{\lambda^*+\delta} - s\mu_G \right|} \leq \frac{\left| \frac{\lambda_1(1-s\mu_G)}{\lambda^*+\delta} \right|}{\left| \frac{\lambda_1+\lambda_2+\delta}{\lambda^*+\delta} - s\mu_G \right|} < 1.$$

For  $s$  on the half circle, we have for  $\forall \varepsilon > 0$ ,

$$\frac{\left| \frac{1}{\mu_G} - s \right|}{\left| \frac{\lambda_1+\lambda_2+\delta}{(\lambda^*+\delta)\mu_G} - s \right|} < 1 + \varepsilon,$$

when  $r$  is sufficiently large. In particular, for  $\varepsilon = \frac{\lambda+\lambda_2+\delta}{\lambda_1}$ , there exists  $r_0 > 0$  such that when  $r > r_0$ , we get

$$\frac{\left| \frac{\lambda_1(1-s\mu_G)\tilde{f}_1(s)}{\lambda^*+\delta} \right|}{\left| 1-s\mu_G - \frac{\lambda}{\lambda^*+\delta} \right|} \leq \frac{\lambda_1}{\lambda^*+\delta} \frac{\left| \frac{1}{\mu_G} - s \right|}{\left| \frac{\lambda_1+\lambda_2+\delta}{(\lambda^*+\delta)\mu_G} - s \right|} < \frac{\lambda_1}{\lambda^*+\delta} (1 + \varepsilon) \leq 1.$$

That is to say, we show that for  $s \in \mathbf{C}_r$ , the module  $|1-s\mu_G - \frac{\lambda}{\lambda^*+\delta}| > \left| \frac{\lambda_1(1-s\mu_G)\tilde{f}_1(s)}{\lambda^*+\delta} \right|$ . Using Rouché's theorem, we conclude that the number of roots of the equation  $1-s\mu_G - \frac{\lambda}{\lambda^*+\delta} - \frac{\lambda_1(1-s\mu_G)\tilde{f}_1(s)}{\lambda^*+\delta} = 0$  equals the number of roots of the equation  $1-s\mu_G - \frac{\lambda}{\lambda^*+\delta} = 0$  inside  $\mathbf{C}_r$ . Moreover, the latter has exactly one root inside  $\mathbf{C}_r$ . It follows that  $1-s\mu_G - \frac{\lambda}{\lambda^*+\delta} - \frac{\lambda_1(1-s\mu_G)\tilde{f}_1(s)}{\lambda^*+\delta} = 0$  has exactly one positive real root inside  $\mathbf{C}_r$ .

Secondly, we apply Rouché's theorem and the result above to prove this Lemma. When  $s$  on the imaginary axis

$$\begin{aligned} & \left| \left( 1-s\mu_G - \frac{\lambda}{\lambda^*+\delta} - \frac{\lambda_1(1-s\mu_G)\tilde{f}_1(s)}{\lambda^*+\delta} \right)^2 \right| \\ & \geq \left( \left| \frac{\lambda_1+\lambda_2+\delta}{\lambda^*+\delta} - s\mu_G \right| - \left| \frac{\lambda_1(1-s\mu_G)\tilde{f}_1(s)}{\lambda^*+\delta} \right| \right)^2 \\ & > \left( \frac{\lambda_2|1-s\mu_G|}{\lambda^*+\delta} \right)^2 \geq \left| \frac{(\lambda_2(1-s\mu_G))^2 \tilde{f}_2(s)}{(\lambda^*+\delta)^2} \right|. \end{aligned}$$

For  $s$  on the half circle, we get for  $\varepsilon = \frac{\lambda}{\lambda_1+\lambda_2+\delta}$ , there exists  $r_1 > 0$  such that when  $r > r_1$ ,

$$\frac{\left| \frac{(\lambda_1+\lambda_2+\delta)(1-s\mu_G)}{\lambda^*+\delta} \right|}{\left| 1-s\mu_G - \frac{\lambda}{\lambda^*+\delta} \right|} \leq \frac{\lambda_1+\lambda_2+\delta}{\lambda^*+\delta} \frac{\left| \frac{1}{\mu_G} - s \right|}{\left| \frac{\lambda_1+\lambda_2+\delta}{(\lambda^*+\delta)\mu_G} - s \right|} < \frac{\lambda_1+\lambda_2+\delta}{\lambda^*+\delta} (1 + \varepsilon) \leq 1,$$

then

$$\begin{aligned} & \left| \left( 1-s\mu_G - \frac{\lambda}{\lambda^*+\delta} - \frac{\lambda_1(1-s\mu_G)\tilde{f}_1(s)}{\lambda^*+\delta} \right)^2 \right| \\ & > \left( \frac{(\lambda_1+\lambda_2+\delta)|1-s\mu_G|}{\lambda^*+\delta} - \frac{\lambda_1|1-s\mu_G|}{\lambda^*+\delta} \right)^2 > \left| \frac{(\lambda_2(1-s\mu_G))^2 \tilde{f}_2(s)}{(\lambda^*+\delta)^2} \right|. \end{aligned}$$

That is to say, for  $s \in \mathbf{C}_r$ ,

$$\left| \left( 1-s\mu_G - \frac{\lambda}{\lambda^*+\delta} - \frac{\lambda_1(1-s\mu_G)\tilde{f}_1(s)}{\lambda^*+\delta} \right)^2 \right| > \left| \frac{(\lambda_2(1-s\mu_G))^2 \tilde{f}_2(s)}{(\lambda^*+\delta)^2} \right|.$$

Using Rouché's theorem, we conclude that the number of roots of the equation

$$\left( 1-s\mu_G - \frac{\lambda}{\lambda^*+\delta} - \frac{\lambda_1(1-s\mu_G)\tilde{f}_1(s)}{\lambda^*+\delta} \right)^2 - \frac{(\lambda_2(1-s\mu_G))^2 \tilde{f}_2(s)}{(\lambda^*+\delta)^2} = 0$$

equals the number of roots of the equation  $\left(1 - s\mu_G - \frac{\lambda}{\lambda^* + \delta} - \frac{\lambda_1(1-s\mu_G)\tilde{f}_1(s)}{\lambda^* + \delta}\right)^2 = 0$  inside  $\mathbf{C}_r$ . Moreover, by the discussion in the paragraph above, the latter has exactly two roots with positive real parts inside  $\mathbf{C}_r$ . It follows from all above that Eq.(3.14) has exactly two distinct positive real roots, say,  $\rho_1(\delta)$ , and  $\rho_2(\delta)$ , inside  $\mathbf{C}_r$ . Finally, letting  $r \rightarrow \infty$  completes the proof.  $\square$

Denote the root with the smaller module by  $\rho_1(\delta)$ . It is easily seen that  $\rho_1(\delta) \rightarrow 0^+$  as  $\delta \rightarrow 0^+$ . In the rest of the paper, we denote these two roots  $\rho_i(\delta)$  by  $\rho_i$ ,  $i = 1, 2$ , for simplicity.

Since  $\tilde{\Phi}(s)$  is finite for all  $s$  with  $\mathbf{Re} s \geq 0$ , we know  $\rho_1$  and  $\rho_2$  must be zeros of the numerators of (3.12) and (3.13). From Eq.(3.12), we can give the following equations for  $\tilde{\Phi}_1(\frac{1}{\mu_G})$  and  $\tilde{\Psi}_1(\frac{1}{\mu_G})$ ,

$$\begin{aligned} & -\left(1 - \frac{\lambda}{(\lambda^* + \delta)(1 - \rho_i\mu_G)} - \frac{\lambda_1\tilde{f}_1(\rho_i)}{\lambda^* + \delta}\right)\left(\frac{\lambda\tilde{\Phi}_1(\frac{1}{\mu_G})}{(\lambda^* + \delta)(1 - \rho_i\mu_G)} - \frac{\lambda_1\tilde{w}_1(\rho_i)}{\lambda^* + \delta}\right) \\ & = \frac{\lambda_2\left(\frac{\lambda\tilde{\Psi}_1(\frac{1}{\mu_G})}{(\lambda^* + \delta)(1 - \rho_i\mu_G)} - \frac{\lambda_1\tilde{w}_1(\rho_i)}{\lambda^* + \delta}\right)}{\lambda^* + \delta}, \end{aligned} \quad (3.15)$$

where  $i = 1, 2$ . From Eq.(3.13), we can give the following equations for  $\tilde{\Phi}_2(\frac{1}{\mu_G})$  and  $\tilde{\Psi}_2(\frac{1}{\mu_G})$ ,

$$-\frac{\lambda\tilde{\Phi}_2(\frac{1}{\mu_G})\left(1 - \frac{\lambda}{(\lambda^* + \delta)(1 - \rho_i\mu_G)} - \frac{\lambda_1\tilde{f}_1(\rho_i)}{\lambda^* + \delta}\right)}{(\lambda^* + \delta)(1 - \rho_i\mu_G)} = \frac{\lambda_2\left(\frac{\lambda\tilde{\Psi}_2(\frac{1}{\mu_G})}{(\lambda^* + \delta)(1 - \rho_i\mu_G)} - \frac{\lambda_2\tilde{w}_2(\rho_i)}{\lambda^* + \delta}\right)}{\lambda^* + \delta}. \quad (3.16)$$

By solving linear equations (3.15) and (3.16), we can get  $\tilde{\Phi}_i(\frac{1}{\mu_G})$  and  $\tilde{\Psi}_i(\frac{1}{\mu_G})$ ,  $i = 1, 2$ . Then  $\tilde{\Phi}_1(s)$  and  $\tilde{\Phi}_2(s)$  can also be obtained.

#### 4. Defective renewal equations

In this section, we study the defective renewal equations satisfied by the two expected discounted penalty functions in the risk model with two classes of claims and random income.

Based on the results of (3.12) and (3.13), the Laplace transforms of  $\Phi_1(u)$  and  $\Phi_2(u)$  can be simplified as

$$\tilde{\Phi}_1(s) = \frac{\tilde{f}_{1,1}(s) + \tilde{f}_{1,2}(s)}{\tilde{h}_1(s) - \tilde{h}_2(s)}, \quad (4.1)$$

$$\tilde{\Phi}_2(s) = \frac{\tilde{f}_{2,1}(s) + \tilde{f}_{2,2}(s)}{\tilde{h}_1(s) - \tilde{h}_2(s)}, \quad (4.2)$$

where  $\tilde{h}_1(s) = (1 - s\mu_G - \frac{\lambda}{\lambda^* + \delta})^2$ ,  $\tilde{h}_2(s) = \frac{s^2\mu_G^2}{(\lambda^* + \delta)^2}(2\lambda_1(\lambda^* + \delta)\tilde{f}_1(s) + \lambda_2^2\tilde{f}_2(s) - \lambda_1^2\tilde{f}_1^2(s)) + \frac{2s\mu_G}{(\lambda^* + \delta)^2}((2\lambda_1(\lambda^* + \delta) - \lambda\lambda_1)\tilde{f}_1(s) + \lambda_2^2\tilde{f}_2(s) - \lambda_1^2\tilde{f}_1^2(s)) + \frac{1}{(\lambda^* + \delta)^2}((2\lambda_1(\lambda^* + \delta) - 2\lambda\lambda_1)\tilde{f}_1(s) + \lambda_2^2\tilde{f}_2(s) - \lambda_1^2\tilde{f}_1^2(s))$ ,  $\tilde{f}_{1,1}(s) = -(1 - s\mu_G - \frac{\lambda}{\lambda^* + \delta})\frac{\lambda\tilde{\Phi}_1(\frac{1}{\mu_G})}{\lambda^* + \delta} - \frac{\lambda\lambda_2(1-s\mu_G)\tilde{\Psi}_1(\frac{1}{\mu_G})}{(\lambda^* + \delta)^2}$ ,  $\tilde{f}_{1,2}(s) = \frac{\lambda_1}{(\lambda^* + \delta)^2}((\lambda^* + \lambda_2 - \lambda_1 + \delta)\tilde{w}_1(s) - \lambda_1\tilde{f}_1(s)\tilde{w}_1(s) + \lambda\tilde{\Phi}_1(\frac{1}{\mu_G})\tilde{f}_1(s)) - \frac{s\mu_G\lambda_1}{(\lambda^* + \delta)^2}((2(\lambda^* + \lambda_2 + \delta) - \lambda)\tilde{w}_1(s) - 2\lambda_1\tilde{f}_1(s)\tilde{w}_1(s) + \lambda\tilde{\Phi}_1(\frac{1}{\mu_G})\tilde{f}_1(s)) + \frac{s^2\mu_G^2\lambda_1}{(\lambda^* + \delta)^2}((\lambda^* + \lambda_2 + \delta)\tilde{w}_1(s) - \lambda_1\tilde{f}_1(s)\tilde{w}_1(s))$ ,  $\tilde{f}_{2,1}(s) = -(1 - s\mu_G - \frac{\lambda}{\lambda^* + \delta})\frac{\lambda\tilde{\Phi}_2(\frac{1}{\mu_G})}{\lambda^* + \delta} - \frac{\lambda\lambda_2(1-s\mu_G)\tilde{\Psi}_2(\frac{1}{\mu_G})}{(\lambda^* + \delta)^2}$ ,  $\tilde{f}_{2,2}(s) = \frac{1}{(\lambda^* + \delta)^2}(\lambda_2^2\tilde{w}_2(s) + \lambda\lambda_1\tilde{f}_1(s)\tilde{\Phi}_2(\frac{1}{\mu_G})) - \frac{s\mu_G}{(\lambda^* + \delta)^2}(2\lambda_2^2\tilde{w}_2(s) + \lambda\lambda_1\tilde{f}_1(s)\tilde{\Phi}_2(\frac{1}{\mu_G})) + \frac{s^2\mu_G^2\lambda_2^2\tilde{w}_2(s)}{(\lambda^* + \delta)^2}$ . Define  $f_{1,1}(u)$ ,  $f_{1,2}(u)$ ,  $f_{2,1}(u)$ ,  $f_{2,2}(u)$ ,  $h_1(u)$  and  $h_2(u)$  as the inverse image functions of  $\tilde{f}_{1,1}(s)$ ,  $\tilde{f}_{1,2}(s)$ ,



$\tilde{f}_{2,1}(s)$ ,  $\tilde{f}_{2,2}(s)$ ,  $\tilde{h}_1(s)$ , and  $\tilde{h}_2(s)$ , i.e.,  $T_s f_{i,j}(0) = \tilde{f}_{i,j}(s)$  and  $T_s h_i(0) = \tilde{h}_i(s)$ ,  $i = 1, 2$ ,  $j = 1, 2$ .

**4.1. Proposition.** 1. *The Laplace transform  $\tilde{\Phi}_i(s)$  of the expected discounted penalty function satisfies*

$$\tilde{\Phi}_i(s) = \frac{T_s T_{\rho_2} T_{\rho_1} h_2(0)}{\mu_G^2} \tilde{\Phi}_i(s) + \frac{T_s T_{\rho_2} T_{\rho_1} f_{i,2}(0)}{\mu_G^2}, \quad i = 1, 2. \quad (4.3)$$

*Proof.* Since  $\tilde{\Phi}_i(s)$  ( $i = 1, 2$ ) is analytic for all  $s$  with  $\mathbf{Re} s \geq 0$ , we know  $\rho_1$  and  $\rho_2$  are zeros of the numerators of which means that  $\tilde{f}_{i,1}(\rho_j) = -\tilde{f}_{i,2}(\rho_j)$  for  $i = 1, 2$ ,  $j = 1, 2$ . Because  $\tilde{f}_{i,1}(s)$  is a polynomial of degree 1, applying the Lagrange interpolating theorem, we have

$$\tilde{f}_{i,1}(s) = -\frac{\tilde{f}_{i,2}(\rho_1)(s - \rho_2) - \tilde{f}_{i,2}(\rho_2)(s - \rho_1)}{\rho_1 - \rho_2},$$

which yields

$$\begin{aligned} \tilde{f}_{i,1}(s) + \tilde{f}_{i,2}(s) &= \frac{(s - \rho_2) \left( \tilde{f}_{i,2}(s) - \tilde{f}_{i,2}(\rho_1) \right) - (s - \rho_1) \left( \tilde{f}_{i,2}(s) - \tilde{f}_{i,2}(\rho_2) \right)}{\rho_1 - \rho_2} \\ &= (s - \rho_1)(s - \rho_2) T_s T_{\rho_2} T_{\rho_1} f_{i,2}(0). \end{aligned} \quad (4.4)$$

Obviously, an simple expression for the denominator of  $\tilde{\Phi}_i(s)$ ,  $i = 1, 2$ , can be dealt with in a similar way. Due to Lemma 1, we get that  $\tilde{h}_1(\rho_i) = \tilde{h}_2(\rho_i)$  for  $i = 1, 2$ . Similarly, because  $\tilde{h}_1(s)$  is a polynomial of degree 2, using the Lagrange interpolating theorem, we have

$$\begin{aligned} \tilde{h}_1(s) &= \tilde{h}_1(0) \frac{(s - \rho_1)(s - \rho_2)}{\rho_1 \rho_2} + s \left( \frac{\tilde{h}_1(\rho_1)}{\rho_1} \frac{s - \rho_2}{\rho_1 - \rho_2} + \frac{\tilde{h}_1(\rho_2)}{\rho_2} \frac{s - \rho_1}{\rho_2 - \rho_1} \right) \\ &= \tilde{h}_1(0) \frac{(s - \rho_1)(s - \rho_2)}{\rho_1 \rho_2} + (s - \rho_1)(s - \rho_2) \left( \frac{\tilde{h}_2(\rho_1)}{\rho_1} \frac{1}{\rho_1 - \rho_2} + \frac{\tilde{h}_2(\rho_2)}{\rho_2} \frac{1}{\rho_2 - \rho_1} \right) \\ &\quad + \tilde{h}_2(\rho_1) \frac{s - \rho_2}{\rho_1 - \rho_2} + \tilde{h}_2(\rho_2) \frac{s - \rho_1}{\rho_2 - \rho_1}. \end{aligned}$$

Using the result above and recalling the Property 6 of the Dickson-Hipp operator derived in Li and Garrido [8],  $\tilde{h}_1(s) - \tilde{h}_2(s)$  can be rewritten as

$$\begin{aligned} \tilde{h}_1(s) - \tilde{h}_2(s) &= \tilde{h}_1(0) \frac{(s - \rho_1)(s - \rho_2)}{\rho_1 \rho_2} + (s - \rho_1)(s - \rho_2) \left( \frac{\tilde{h}_2(\rho_1)}{\rho_1(\rho_1 - \rho_2)} + \frac{\tilde{h}_2(\rho_2)}{\rho_2(\rho_2 - \rho_1)} \right) \\ &\quad - \left( \tilde{h}_2(s) - \tilde{h}_2(\rho_1) \frac{s - \rho_2}{\rho_1 - \rho_2} - \tilde{h}_2(\rho_2) \frac{s - \rho_1}{\rho_2 - \rho_1} \right) \\ &= (s - \rho_1)(s - \rho_2) (T_0 T_{\rho_2} T_{\rho_1} h_1(0) - T_s T_{\rho_2} T_{\rho_1} h_2(0)). \end{aligned} \quad (4.5)$$

It is easy to check that  $T_0 T_{\rho_2} T_{\rho_1} h_1(0) = \mu_G^2$  which makes (4.5) become

$$\tilde{h}_1(s) - \tilde{h}_2(s) = (s - \rho_1)(s - \rho_2) (\mu_G^2 - T_s T_{\rho_2} T_{\rho_1} h_2(0)). \quad (4.6)$$

Invoking (4.4) and (4.6) into (4.1) and (4.2), we can derive  $\tilde{\Phi}_i(s) = \frac{T_s T_{\rho_2} T_{\rho_1} f_{i,2}(0)}{\mu_G^2 - T_s T_{\rho_2} T_{\rho_1} h_2(0)}$  which gives (4.3). The result of Proposition 1 is proved.  $\square$

Now, we are ready to obtain the defective renewal equations for  $\Phi_i(u)$ ,  $i = 1, 2$ .

**4.2. Proposition. 2.**  $\Phi_i(u)$  satisfies the following defective renewal equation

$$\Phi_i(u) = \kappa_\delta \int_0^u \Phi_i(u-y)\varsigma(y)dy + \xi_i(u), \quad i = 1, 2, \quad (4.7)$$

where

$$\begin{aligned} \kappa_\delta &= \frac{(2\lambda_1(\lambda^* + \delta) - 2\lambda\lambda_1)T_0T_{\rho_2}T_{\rho_1}f_1(0) + \lambda_2^2T_0T_{\rho_2}T_{\rho_1}f_2(0) - \lambda_1^2T_0T_{\rho_2}T_{\rho_1}f_1 * f_1(0)}{(\lambda^* + \delta)^2\mu_G^2} \\ &\quad + \frac{2}{(\lambda^* + \delta)^2\mu_G} ((2\lambda_1(\lambda^* + \delta) - \lambda\lambda_1)(\rho_1T_0T_{\rho_2}T_{\rho_1}f_1(0) - T_0T_{\rho_2}f_1(0)) \\ &\quad + \lambda_2^2(\rho_1T_0T_{\rho_2}T_{\rho_1}f_2(0) - T_0T_{\rho_2}f_2(0)) - \lambda_1^2(\rho_1T_0T_{\rho_2}T_{\rho_1}f_1 * f_1(0) - T_0T_{\rho_2}f_1 * f_1(0))) \\ &\quad + \frac{1}{(\lambda^* + \delta)^2} (2\lambda_1(\lambda^* + \delta)(1 - (\rho_2 + \rho_1)T_0T_{\rho_2}f_1(0) \\ &\quad + \rho_1^2T_0T_{\rho_2}T_{\rho_1}f_1(0)) + \lambda_2^2(1 - (\rho_2 + \rho_1)T_0T_{\rho_2}f_2(0) + \rho_1^2T_0T_{\rho_2}T_{\rho_1}f_2(0)) \\ &\quad - \lambda_1^2(1 - (\rho_2 + \rho_1)T_0T_{\rho_2}f_1 * f_1(0) + \rho_1^2T_0T_{\rho_2}T_{\rho_1}f_1 * f_1(0))), \\ \varsigma(y) &= \frac{1}{\kappa_\delta} \left\{ \frac{(2\lambda_1(\lambda^* + \delta) - 2\lambda\lambda_1)T_{\rho_2}T_{\rho_1}f_1(y) + \lambda_2^2T_{\rho_2}T_{\rho_1}f_2(y) - \lambda_1^2T_{\rho_2}T_{\rho_1}f_1 * f_1(y)}{(\lambda^* + \delta)^2\mu_G^2} \right. \\ &\quad + \frac{2}{(\lambda^* + \delta)^2\mu_G} ((2\lambda_1(\lambda^* + \delta) - \lambda\lambda_1)(\rho_1T_{\rho_2}T_{\rho_1}f_1(y) - T_{\rho_2}f_1(y)) \\ &\quad + \lambda_2^2(\rho_1T_{\rho_2}T_{\rho_1}f_2(y) - T_{\rho_2}f_2(y)) - \lambda_1^2(\rho_1T_{\rho_2}T_{\rho_1}f_1 * f_1(y) - T_{\rho_2}f_1 * f_1(y))) \\ &\quad + \frac{1}{(\lambda^* + \delta)^2} (2\lambda_1(\lambda^* + \delta)(f_1(y) - (\rho_2 + \rho_1)T_{\rho_2}f_1(y) \\ &\quad + \rho_1^2T_{\rho_2}T_{\rho_1}f_1(y)) + \lambda_2^2(f_2(y) - (\rho_2 + \rho_1)T_{\rho_2}f_2(y) + \rho_1^2T_{\rho_2}T_{\rho_1}f_2(y)) \\ &\quad \left. - \lambda_1^2(f_1 * f_1(y) - (\rho_2 + \rho_1)T_{\rho_2}f_1 * f_1(y) + \rho_1^2T_{\rho_2}T_{\rho_1}f_1 * f_1(y))) \right\}. \end{aligned}$$

and

$$\begin{aligned} \xi_1(u) &= \frac{\lambda_1((\lambda^* + \lambda_2 - \lambda_1 + \delta)T_{\rho_2}T_{\rho_1}w_1(u) - \lambda_1T_{\rho_2}T_{\rho_1}f_1 * w_1(u) + \lambda\tilde{\Phi}_1(\frac{1}{\mu_G})T_{\rho_2}T_{\rho_1}f_1(u))}{(\lambda^* + \delta)^2\mu_G^2} \\ &\quad - \frac{\lambda_1}{(\lambda^* + \delta)^2\mu_G} (((2(\lambda^* + \lambda_2 + \delta) - \lambda)(\rho_1T_{\rho_2}T_{\rho_1}w_1(u) - T_{\rho_2}w_1(u)) \\ &\quad - 2\lambda_1(\rho_1T_{\rho_2}T_{\rho_1}f_1 * w_1(u) - T_{\rho_2}f_1 * w_1(u)) + \lambda\tilde{\Phi}_1(\frac{1}{\mu_G})(\rho_1T_{\rho_2}T_{\rho_1}f_1(u) - T_{\rho_2}f_1(u))) \\ &\quad + \frac{\lambda_1}{(\lambda^* + \delta)^2} ((\lambda^* + \lambda_2 + \delta)(w_1(u) - (\rho_2 + \rho_1)T_{\rho_2}w_1(u) + \rho_1^2T_{\rho_2}T_{\rho_1}w_1(u)) \\ &\quad - \lambda_1(f_1 * w_1(u) - (\rho_2 + \rho_1)T_{\rho_2}f_1 * w_1(u) + \rho_1^2T_{\rho_2}T_{\rho_1}f_1 * w_1(u))), \\ \xi_2(u) &= \frac{1}{(\lambda^* + \delta)^2\mu_G^2} (\lambda_2^2T_{\rho_2}T_{\rho_1}w_2(u) + \lambda\lambda_1\tilde{\Phi}_2(\frac{1}{\mu_G})T_{\rho_2}T_{\rho_1}f_1(u)) \\ &\quad - \frac{2\lambda_2^2(\rho_1T_{\rho_2}T_{\rho_1}w_2(u) - T_{\rho_2}w_2(u)) + \lambda\lambda_1\tilde{\Phi}_2(\frac{1}{\mu_G})(\rho_1T_{\rho_2}T_{\rho_1}f_1(u) - T_{\rho_2}f_1(u))}{(\lambda^* + \delta)^2\mu_G} \\ &\quad + \frac{\lambda_2^2}{(\lambda^* + \delta)^2} (w_2(u) - (\rho_2 + \rho_1)T_{\rho_2}w_2(u) + \rho_1^2T_{\rho_2}T_{\rho_1}w_2(u)). \end{aligned}$$

*Proof.* By employing the property of the Dickson-Hipp operator, we deduce

$$\frac{\frac{\tilde{f}(s)-\tilde{f}(\rho_2)}{s-\rho_2} - \frac{\tilde{f}(s)-\tilde{f}(\rho_1)}{s-\rho_1}}{\rho_2 - \rho_1} = \frac{T_s T_{\rho_1} f(0) - T_s T_{\rho_2} f(0)}{\rho_2 - \rho_1} = T_s T_{\rho_2} T_{\rho_1} f(0), \quad (4.8)$$

$$\frac{\frac{s\tilde{f}(s)-\rho_2\tilde{f}(\rho_2)}{s-\rho_2} - \frac{s\tilde{f}(s)-\rho_1\tilde{f}(\rho_1)}{s-\rho_1}}{\rho_2 - \rho_1} = \rho_1 T_s T_{\rho_2} T_{\rho_1} f(0) - T_s T_{\rho_2} f(0), \quad (4.9)$$

$$\frac{\frac{s^2\tilde{f}(s)-\rho_2^2\tilde{f}(\rho_2)}{s-\rho_2} - \frac{s^2\tilde{f}(s)-\rho_1^2\tilde{f}(\rho_1)}{s-\rho_1}}{\rho_2 - \rho_1} = \tilde{f}(s) - (\rho_2 + \rho_1)T_s T_{\rho_2} f(0) + \rho_1^2 T_s T_{\rho_2} T_{\rho_1} f(0). \quad (4.10)$$

Recalling the definition of the Dickson-Hipp operator  $T_r$  and together with (4.8)-(4.10), one finds

$$\begin{aligned} & T_s T_{\rho_2} T_{\rho_1} h_2(0) \\ &= \frac{(2\lambda_1(\lambda^* + \delta) - 2\lambda\lambda_1)T_s T_{\rho_2} T_{\rho_1} f_1(0) + \lambda_2^2 T_s T_{\rho_2} T_{\rho_1} f_2(0) - \lambda_1^2 T_s T_{\rho_2} T_{\rho_1} f_1 * f_1(0)}{(\lambda^* + \delta)^2} \\ &+ \frac{2\mu_G}{(\lambda^* + \delta)^2} ((2\lambda_1(\lambda^* + \delta) - \lambda\lambda_1)(\rho_1 T_s T_{\rho_2} T_{\rho_1} f_1(0) - T_s T_{\rho_2} f_1(0)) \\ &+ \lambda_2^2(\rho_1 T_s T_{\rho_2} T_{\rho_1} f_2(0) - T_s T_{\rho_2} f_2(0)) - \lambda_1^2(\rho_1 T_s T_{\rho_2} T_{\rho_1} f_1 * f_1(0) - T_s T_{\rho_2} f_1 * f_1(0))) \\ &+ \frac{\mu_G^2}{(\lambda^* + \delta)^2} (2\lambda_1(\lambda^* + \delta)(\tilde{f}_1(s) - (\rho_2 + \rho_1)T_s T_{\rho_2} f_1(0)) \\ &+ \rho_1^2 T_s T_{\rho_2} T_{\rho_1} f_1(0)) + \lambda_2^2(\tilde{f}_2(s) - (\rho_2 + \rho_1)T_s T_{\rho_2} f_2(0) + \rho_1^2 T_s T_{\rho_2} T_{\rho_1} f_2(0)) \\ &- \lambda_1^2(\tilde{f}_1^2(s) - (\rho_2 + \rho_1)T_s T_{\rho_2} f_1 * f_1(0) + \rho_1^2 T_s T_{\rho_2} T_{\rho_1} f_1 * f_1(0))). \end{aligned} \quad (4.11)$$

Similarly, we find

$$\begin{aligned} T_s T_{\rho_2} T_{\rho_1} f_{1,2}(0) &= \frac{\lambda_1}{(\lambda^* + \delta)^2} ((\lambda^* + \lambda_2 - \lambda_1 + \delta)T_s T_{\rho_2} T_{\rho_1} w_1(0) \\ &- \lambda_1 T_s T_{\rho_2} T_{\rho_1} f_1 * w_1(0) + \lambda\tilde{\Phi}_1(\frac{1}{\mu_G})T_s T_{\rho_2} T_{\rho_1} f_1(0)) \\ &- \frac{\mu_G \lambda_1}{(\lambda^* + \delta)^2} (((2(\lambda^* + \lambda_2 + \delta) - \lambda)(\rho_1 T_s T_{\rho_2} T_{\rho_1} w_1(0) - T_s T_{\rho_2} w_1(0)) \\ &- 2\lambda_1(\rho_1 T_s T_{\rho_2} T_{\rho_1} f_1 * w_1(0) - T_s T_{\rho_2} f_1 * w_1(0)) \\ &+ \lambda\tilde{\Phi}_1(\frac{1}{\mu_G})(\rho_1 T_s T_{\rho_2} T_{\rho_1} f_1(0) - T_s T_{\rho_2} f_1(0))) + \frac{\mu_G^2 \lambda_1}{(\lambda^* + \delta)^2} ((\lambda^* + \lambda_2 + \delta)(\tilde{w}_1(s) \\ &- (\rho_2 + \rho_1)T_s T_{\rho_2} w_1(0) + \rho_1^2 T_s T_{\rho_2} T_{\rho_1} w_1(0)) - \lambda_1(\tilde{f}_1(s)\tilde{w}_1(s) \\ &- (\rho_2 + \rho_1)T_s T_{\rho_2} f_1 * w_1(0) + \rho_1^2 T_s T_{\rho_2} T_{\rho_1} f_1 * w_1(0))) = \mu_G^2 T_s \xi_1(0), \end{aligned} \quad (4.12)$$

and

$$\begin{aligned} T_s T_{\rho_2} T_{\rho_1} f_{2,2}(0) &= \frac{1}{(\lambda^* + \delta)^2} (\lambda_2^2 T_s T_{\rho_2} T_{\rho_1} w_2(0) + \lambda\lambda_1\tilde{\Phi}_2(\frac{1}{\mu_G})T_s T_{\rho_2} T_{\rho_1} f_1(0)) \\ &- \frac{\mu_G}{(\lambda^* + \delta)^2} (2\lambda_2^2(\rho_1 T_s T_{\rho_2} T_{\rho_1} w_2(0) - T_s T_{\rho_2} w_2(0)) \\ &+ \lambda\lambda_1\tilde{\Phi}_2(\frac{1}{\mu_G})(\rho_1 T_s T_{\rho_2} T_{\rho_1} f_1(0) - T_s T_{\rho_2} f_1(0))) \\ &+ \frac{\mu_G^2 \lambda_2^2}{(\lambda^* + \delta)^2} (\tilde{w}_2(s) - (\rho_2 + \rho_1)T_s T_{\rho_2} w_2(0) + \rho_1^2 T_s T_{\rho_2} T_{\rho_1} w_2(0)) = \mu_G^2 T_s \xi_2(0), \end{aligned} \quad (4.13)$$

where the operator  $*$  is denoted as convolution.

Therefore, plugging (4.11), (4.12) and (4.13) into (4.3), we can get

$$\tilde{\Phi}_i(s) = \frac{T_s T_{\rho_2} T_{\rho_1} h_2(0)}{\mu_G^2} \tilde{\Phi}_i(s) + T_s \xi_i(0), \quad i = 1, 2. \quad (4.14)$$

Inverting the Laplace transform in (4.14) gives

$$\Phi_i(u) = \frac{T_0 T_{\rho_2} T_{\rho_1} h_2(0)}{\mu_G^2} \int_0^u \Phi_i(u-y) \frac{T_{\rho_2} T_{\rho_1} h_2(y)}{T_0 T_{\rho_2} T_{\rho_1} h_2(0)} dy + \xi_i(u),$$

which corresponds to (4.7).

To show that (4.7) to be a defective renewal equation, we need to verified  $\kappa_\delta < 1$ . We first consider the case  $\delta > 0$ . Comparing (4.11) at  $s = 0$  to the expression of  $\kappa_\delta$  gives  $\kappa_\delta = \frac{T_0 T_{\rho_1} T_{\rho_2} h_2(0)}{\mu_G^2}$ . Because of  $\rho_1(\delta) > 0$  and  $\rho_2(\delta) > 0$ , putting (4.6) at  $s = 0$  in (4.6), one deduces

$$\kappa_\delta = \frac{T_0 T_{\rho_1} T_{\rho_2} h_2(0)}{\mu_G^2} = 1 - \frac{\tilde{h}_1(0) - \tilde{h}_2(0)}{\mu_G^2 \rho_1 \rho_2} = 1 - \frac{\delta^2 + 2\lambda_2 \delta}{\mu_G^2 (\lambda^* + \delta)^2 \rho_1 \rho_2} < 1.$$

For  $\delta = 0$ , putting  $s = \rho_1(\delta)$  in (3.14) yields

$$\begin{aligned} & \left( \lambda^* + \delta - (\lambda^* + \delta) \mu_G \rho_1(\delta) - \lambda - \lambda_1 (1 - \mu_G \rho_1(\delta)) \tilde{f}_1(\rho_1(\delta)) \right)^2 \\ & = \lambda_2^2 (1 - \mu_G \rho_1(\delta))^2 \tilde{f}_2(\rho_1(\delta)). \end{aligned}$$

Note the fact that  $\rho_1(0) = 0$ . Differentiating the equation above with respect to  $\delta$  and then putting  $\delta = 0$ , one finds

$$\rho_1'(0) = \frac{1}{\lambda \mu_G - \lambda_1 \mu_{F_1} - \frac{\lambda_2 \mu_{F_2}}{2}} > 0,$$

where the inequality above follows from the net profit condition. Then taking the limit  $\delta \rightarrow 0^+$  in  $\kappa_\delta$  and using L'Hôpital's rule, we can get

$$\begin{aligned} \kappa_0 & = \frac{T_0 T_0 T_{\rho_2(0)} h_2(0)}{\mu_G^2} = 1 - \frac{1}{\mu_G^2 (\lambda^*)^2 \rho_2} \times \lim_{\delta \rightarrow 0^+} \frac{\delta^2 + 2\lambda_2 \delta}{\rho_1(\delta)} \\ & = 1 - \frac{2\lambda_2}{\mu_G^2 (\lambda^*)^2 \rho_2 \rho_1'(0)} < 1. \end{aligned}$$

Thus, Eq.(4.7) is defective renewal equation. This completes the proof.  $\square$

In order to derive the analytic expression for  $\Phi_i(u)$ , an associated compound geometric distribution function are defined as

$$\bar{H}(u) = \frac{\zeta}{1 + \zeta} \sum_{n=1}^{\infty} \left( \frac{1}{1 + \zeta} \right)^n \bar{K}^{*n}(u), \quad u \geq 0,$$

where  $\zeta = (1 - \kappa_\delta) / \kappa_\delta$ ,  $\bar{K}^{*n}(u)$  is the tail of the  $n$ -fold convolution of  $K(u) = 1 - \bar{K}(u) = \int_0^u \zeta(y) dy$ . By employing the Theorem 2.1 of Lin and Willmot [11], we can derive the following Proposition.

**4.3. Proposition. 3.** *The expected discounted penalty function  $\Phi_i(u)$  satisfying the defective renewal equation (4.7) can be rewritten as*

$$\Phi_i(u) = \frac{1}{\zeta} \int_0^u [1 - \bar{H}(u-y)] dB_i(y) + \frac{B_i(0)}{\zeta} [1 - \bar{H}(u)], \quad i = 1, 2, \quad (4.15)$$

or

$$\Phi_i(u) = \frac{1}{\zeta} \int_0^u B_i(u-y) dH(y) + \frac{1}{1 + \zeta} B_i(u), \quad i = 1, 2, \quad (4.16)$$

where  $B_i(u) = \xi_i(u) / \kappa_\delta$ .

*Proof.* Using the Eq.(4.7) and the result of Theorem 2.1 obtained in Lin and Willmot [11], the proof is straightforward.  $\square$

## 5. Premium sizes with rational Laplace transforms

In this section, we consider the situation in which the premium size have the following rational Laplace transforms, i.e.,

$$\tilde{f}_G(s) = \frac{\varrho(s)}{\prod_{i=1}^N (s + \varrho_i)^{n_i}}, \quad (5.1)$$

where  $N, n_i \in \mathbf{N}^+$  with  $n_1 + n_2 + \cdots + n_N = n$ ,  $\varrho_i > 0$ ,  $i = 1, 2, \dots, N$ , and  $\varrho_i \neq \varrho_j$  for  $i \neq j$ .  $\varrho(s)$  is a polynomial function of degree  $n-1$  or less and satisfying  $\varrho(0) = \prod_{i=1}^N \varrho_i^{n_i}$ . Using partial fraction, Eq.(5.1) can be rewritten as

$$\tilde{f}_G(s) = \sum_{i=1}^N \sum_{j=1}^{n_i} \frac{\alpha_{ij} \varrho_i^j}{(s + \varrho_i)^j}, \quad (5.2)$$

where

$$\alpha_{ij} = \frac{1}{\varrho_i^j (n_i - j)!} \left. \frac{d^{n_i-j}}{ds^{n_i-j}} \left\{ \prod_{k=1, k \neq i}^N \frac{\varrho(s)}{(s + \varrho_k)^{n_k}} \right\} \right|_{s=-\varrho_i}$$

Taking the inverse Laplace transform of Eq.(5.2), one deduces

$$f_G(x) = \sum_{i=1}^N \sum_{j=1}^{n_i} \alpha_{ij} \frac{x^{j-1} \varrho_i^j e^{-\varrho_i x}}{(j-1)!}, \quad (5.3)$$

which is a density function of a combination of Erlangs. Define

$$\beta_{ij}(x) = \frac{x^{j-1} \varrho_i^j e^{-\varrho_i x}}{(j-1)!}, \quad x > 0, \quad j \in \mathbf{N}^+,$$

as the density function of Erlang( $j$ ) with parameter  $\varrho_i$ ,  $\chi_{ij}$  is a random variable with density function  $\beta_{ij}(x)$ . Thus,  $\chi_{ij}$  can be defined as  $\chi_{ij} = \vartheta_{i1} + \vartheta_{i2} + \cdots + \vartheta_{ij}$ , where  $\vartheta_{i1}, \vartheta_{i2}, \dots, \vartheta_{ij}$  are i.i.d. exponentials with mean  $1/\varrho_i$ . For  $\mathbf{Re} s > \max(\varrho_i)$ , we get, for  $k = 1, 2$ ,

$$\begin{aligned} \tilde{A}_k(s) &= \int_0^\infty e^{-su} \int_0^\infty \Phi_k(u+x) f_G(x) dx du \\ &= \sum_{i=1}^N \sum_{j=1}^{n_i} \alpha_{ij} \int_0^\infty \beta_{ij}(x) T_s \Phi_k(x) dx = \sum_{i=1}^N \sum_{j=1}^{n_i} \alpha_{ij} \mathbf{E}[T_s \Phi_k(\vartheta_{i1} + \vartheta_{i2} + \cdots + \vartheta_{ij})] \\ &= \sum_{i=1}^N \sum_{j=1}^{n_i} \alpha_{ij} \varrho_i^j \mathbf{E} \left[ T_s T_{\varrho_i}^j \Phi_k(0) \right], \end{aligned}$$

where  $T_{\varrho_i}^j = \underbrace{T_{\varrho_i} \cdots T_{\varrho_i}}_j$ . Furthermore, by the Property 5 of the Dickson-Hipp operator provided in Li and Garrido [8], we get,  $k = 1, 2$ ,

$$\tilde{A}_k(s) = \sum_{i=1}^N \sum_{j=1}^{n_i} \alpha_{ij} \varrho_i^j \left( \frac{\tilde{\Phi}_k(s)}{(\varrho_i - s)^j} - \sum_{l=1}^j \frac{T_{\varrho_i}^l \Phi_k(0)}{(\varrho_i - s)^{j+1-l}} \right) = \tilde{f}_G(-s) \tilde{\Phi}_k(s) - Q_k(s), \quad (5.4)$$

where  $Q_k(s) = \sum_{i=1}^N \sum_{j=1}^{n_i} \alpha_{ij} \varrho_i^j \sum_{l=1}^j \frac{T_{\varrho_i}^l \Phi_k(0)}{(\varrho_i - s)^{j+1-l}}$ . Similarly, after some careful calculations, we can find  $k = 1, 2$ ,

$$\tilde{\tilde{A}}_k(s) = \sum_{i=1}^N \sum_{j=1}^{n_i} \alpha_{ij} \varrho_i^j \left( \frac{\tilde{\Psi}_k(s)}{(\varrho_i - s)^j} - \sum_{l=1}^j \frac{T_{\varrho_i}^l \Psi_k(0)}{(\varrho_i - s)^{j+1-l}} \right) = \tilde{f}_G(-s) \tilde{\Psi}_k(s) - \bar{Q}_k(s), \quad (5.5)$$

where  $\bar{Q}_k(s) = \sum_{i=1}^N \sum_{j=1}^{n_i} \alpha_{ij} \varrho_i^j \sum_{l=1}^j \frac{T_{\varrho_i}^l \Psi_k(0)}{(\varrho_i - s)^{j+1-l}}$ .

Substituting (5.4) and (5.5) into (3.4), (3.5), (3.8) and (3.9), one finds

$$\tilde{\Phi}_1(s) = \frac{\lambda_1 L(s) \tilde{w}_1(s) - \lambda L(s) Q_1(s) - \lambda \lambda_2 \bar{Q}_1(s) + \lambda_1 \lambda_2 \tilde{w}_1(s)}{L^2(s) - \lambda_2^2 \tilde{f}_2(s)}, \quad (5.6)$$

$$\tilde{\Phi}_2(s) = \frac{-\lambda L(s) Q_2(s) - \lambda \lambda_2 \bar{Q}_2(s) + \lambda_2^2 \tilde{w}_2(s)}{L^2(s) - \lambda_2^2 \tilde{f}_2(s)}, \quad (5.7)$$

where  $L(s) = \lambda^* + \delta - \lambda \tilde{f}_G(-s) - \lambda \tilde{f}_1(s)$ .

Note that the common denominator of (5.6) and (5.7) is analytic for  $s$  in the right half complex plane except the points  $\varrho_i$ 's. To make it analytic for all  $s$  with  $\mathbf{Re} s \geq 0$ , we assume  $\Lambda(s) = \prod_{i=1}^N (s - \varrho_i)^{n_i}$  and multiply both the numerators and denominators of (5.6) and (5.7) by  $\Lambda(s)$ . Then, one finds

$$\tilde{\Phi}_1(s) = \frac{\lambda_1 L(s) \Lambda(s) \tilde{w}_1(s) - \lambda L(s) \Lambda(s) Q_1(s) - \lambda \lambda_2 \Lambda(s) \bar{Q}_1(s) + \lambda_1 \lambda_2 \Lambda(s) \tilde{w}_1(s)}{L^2(s) \Lambda(s) - \lambda_2^2 \tilde{f}_2(s) \Lambda(s)}, \quad (5.8)$$

$$\tilde{\Phi}_2(s) = \frac{-\lambda L(s) \Lambda(s) Q_2(s) - \lambda \lambda_2 \Lambda(s) \bar{Q}_2(s) + \lambda_2^2 \Lambda(s) \tilde{w}_2(s)}{L^2(s) \Lambda(s) - \lambda_2^2 \tilde{f}_2(s)}. \quad (5.9)$$

From (5.8) and (5.9), in order to determine  $\tilde{\Phi}_1(s)$  and  $\tilde{\Phi}_2(s)$ , we need to find  $\Lambda(s) Q_k(s)$  and  $\Lambda(s) \bar{Q}_k(s)$ ,  $k = 1, 2$ . Note that  $\Lambda(s) Q_k(s)$  and  $\Lambda(s) \bar{Q}_k(s)$ ,  $k = 1, 2$  are polynomials of degree  $n - 1$ , i.e.,

$$\Lambda(s) Q_k(s) = \sum_{i=1}^n L_{k,i} s^{i-1}, \quad \Lambda(s) \bar{Q}_k(s) = \sum_{i=1}^n \bar{L}_{k,i} s^{i-1}.$$

Then, we need to find  $n$  unknown coefficients  $L_{k,i}$ 's and  $n$  unknown coefficients  $\bar{L}_{k,i}$ 's. For this purpose, we give without proof the following Lemma. The result of the following Lemma can be proved by the same technique provided in Lemma 1.

**5.1. Lemma.** *2 For  $\delta > 0$ , the common denominator of (5.8) and (5.9) has exactly  $2n$  zeros, say  $\rho_1(\delta), \dots, \rho_{2n}(\delta)$ , in the right half complex plane.*

Assume that  $\rho_1(\delta), \dots, \rho_{2n}(\delta)$  are distinct. Since  $\tilde{\Phi}_1(s)$  and  $\tilde{\Phi}_2(s)$  are analytic for all  $s$  with  $\mathbf{Re} s \geq 0$ , then the roots  $\rho_1(\delta), \dots, \rho_{2n}(\delta)$  are zeros of the numerators of (5.8) and (5.9). Thus we can get the following  $2n$  linear equations satisfied by  $L_{1,i}$  and  $\bar{L}_{1,i}$ ,  $i = 1, 2, \dots, 2n$ ,

$$\begin{aligned} \lambda_1 L(\rho_i(\delta)) \Lambda(\rho_i(\delta)) \tilde{w}_1(\rho_i(\delta)) - \lambda L(\rho_i(\delta)) \Lambda(\rho_i(\delta)) Q_1(\rho_i(\delta)) - \lambda \lambda_2 \Lambda(\rho_i(\delta)) \bar{Q}_1(\rho_i(\delta)) \\ + \lambda_1 \lambda_2 \Lambda(\rho_i(\delta)) \tilde{w}_1(\rho_i(\delta)) = 0. \end{aligned} \quad (5.10)$$

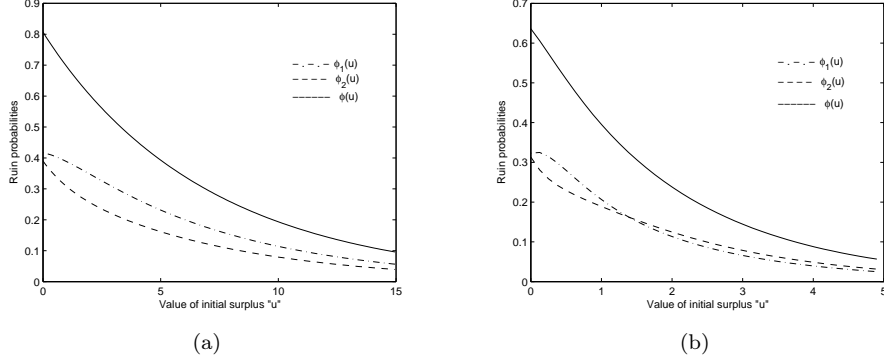
Similarly, we also get  $2n$  linear equations satisfied by  $L_{2,i}$  and  $\bar{L}_{2,i}$ ,  $i = 1, 2, \dots, 2n$ ,

$$-\lambda L(\rho_i(\delta)) \Lambda(\rho_i(\delta)) Q_2(\rho_i(\delta)) - \lambda \lambda_2 \Lambda(\rho_i(\delta)) \bar{Q}_2(\rho_i(\delta)) + \lambda_2^2 \Lambda(\rho_i(\delta)) \tilde{w}_2(\rho_i(\delta)) = 0. \quad (5.11)$$

After solving linear equations (5.10) and (5.11),  $L_{k,i}$  and  $\bar{L}_{k,i}$ ,  $k = 1, 2, i = 1, 2, \dots, n$  can be determined. Then, we can derive the Laplace transforms (5.8) and (5.9).

## 6. Numerical examples

In this section, we give two numerical examples to show how to find the ruin probabilities  $\phi_1(u)$ ,  $\phi_2(u)$  and  $\phi(u)$  and illustrate the behavior of these ruin probabilities.



**Figure 1.** (a) Ruin probabilities in Example 1. (b) Ruin probabilities in Example 2.

**6.1. Example 1.** We illustrate the ruin probabilities when the claim sizes and the premium sizes are exponentially distributed. For illustration purpose, we set  $\mu_G = 1.8$ ,  $\mu_{F_1} = 1.5$ ,  $\mu_{F_2} = 1$ ,  $\lambda = 2.5$ ,  $\lambda_1 = 1$  and  $\lambda_2 = 3$ . The net profit condition is obviously fulfilled. Let  $\delta = 0$ ,  $w_i(x_1, x_2) = 1 (i = 1, 2)$ , then the expected penalty function  $\Phi_i(u) (i = 1, 2)$  simplifies to the ruin probability  $\phi_i(u) (i = 1, 2)$ .

Eq. (3.14) can be simplified as

$$\left(1 - \frac{5}{13(1 - 1.8s)} - \frac{1}{6.5(1 + 1.5s)}\right)^2 = \left(\frac{3}{6.5}\right)^2 \frac{1}{(1 + s)}.$$

After solving the equation above, we obtain five roots, 0, 0.390477, -0.879945, -0.600289, -0.141669. Then, we derive  $\tilde{\Phi}_1(\frac{1}{\mu_G}) = 0.640091$ ,  $\tilde{\Psi}_1(\frac{1}{\mu_G}) = 0.559909$ ,  $\tilde{\Phi}_2(\frac{1}{\mu_G}) = 0.510574$ , and  $\tilde{\Psi}_1(\frac{1}{\mu_G}) = 0.689426$ . Finally, the inversion of the Laplace transforms in (3.12) and (3.13) yields

$$\begin{aligned} \phi_1(u) &= -0.071144e^{-0.879945u} + 0.018003e^{-0.600289u} + 0.469982e^{-0.141669u}, \\ \phi_2(u) &= 0.079266e^{-0.879945u} - 0.016749e^{-0.600289u} + 0.327589e^{-0.141669u}. \end{aligned}$$

Figure 1(a) shows the behavior of ruin probabilities  $\phi_1(u)$ ,  $\phi_2(u)$  and  $\phi(u)$  in Example 1, for different values of  $u \in [0, 15]$ .

**6.2. Example 2.** In this numerical example, we illustrate the ruin probabilities ruin probabilities when claim sizes from one class are distributed as Erlang(2) and claim sizes from the other class are distributed as a mixture of two exponentials, i.e.,  $f_1(x) = 6.76xe^{-2.6x}$  and  $f_2(x) = 0.15e^{-x} + 1.5e^{-2x}$ , for  $x \geq 0$ . For illustration purpose, we also assume that  $\delta = 0$ ,  $w_i(x_1, x_2) = 1 (i = 1, 2)$  and the premium sizes are exponentially distributed with  $\mu_G = 1.8$ . Let  $\lambda = 2.5$ ,  $\lambda_1 = 1$ ,  $\lambda_2 = 3$ . Solving Eq.(3.14) yields eight roots, 0, 0.390985, -1.194764, -0.489621, -3.578668-0.234714i, -3.578668+0.234714i, -1.946361-0.1240087i, -1.946361+0.1240087i. Then, we derive  $\tilde{\Phi}_1(\frac{1}{\mu_G}) = 0.307941$ ,  $\tilde{\Psi}_1(\frac{1}{\mu_G}) = 0.307444$ ,  $\tilde{\Phi}_2(\frac{1}{\mu_G}) = 0.453832$ , and  $\tilde{\Psi}_1(\frac{1}{\mu_G}) = 0.296168$ . Furthermore, the inversion of the Laplace transforms in (3.12) and (3.13) gives

$$\begin{aligned} \phi_1(u) &= 0.081003e^{-1.946361u} \cos(0.124009u) + 0.088605e^{-1.946361u} \sin(0.124009u) \\ &\quad - 0.125463e^{-3.578668u} \cos(0.234714u) + 0.117397e^{-3.578668u} \sin(0.234714u) \end{aligned}$$

$$\begin{aligned}
& +0.093966e^{-1.194764u} + 0.273590e^{-0.489621u}. \\
\phi_2(u) = & -0.072828e^{-1.946361u} \cos(0.124009u) - 0.084349e^{-1.946361u} \sin(0.124009u) \\
& +0.084415e^{-3.578668u} \cos(0.234714u) - 0.108023e^{-3.578668u} \sin(0.234714u) \\
& -0.046824e^{-1.194764u} + 0.348229e^{-0.489621u}.
\end{aligned}$$

Figure 1(b) shows the behavior of the ruin probabilities  $\phi_1(u)$ ,  $\phi_2(u)$  and  $\phi(u)$  in Example 2, for different values of  $u \in [0, 5]$ .

## 7. Concluding remarks

In this paper, we analyze the ruin problems in a risk model with two independent classes of insurance risks and random incomes, one is from the classical risk process, the other is from a Erlang(2) risk process. The expected discounted penalty functions are studied through some analytic methods. Assuming that the premium sizes are exponentially distributed, we show the defective renewal equations for the expected discounted penalty functions can be derived. While for the distributions of premium sizes have rational Laplace transforms, the Laplace transforms for the discounted penalty functions are also be derived.

The model considered in this paper can be extended in the more general framework. For example, the model can be a risk process with two independent classes, one being compound Poisson, the other being generalized Erlang(2), and such extension will only lead to a little computation involvement.

## Acknowledgements

The author thanks the referees for their valuable comments and suggestions which led to the improvement of the paper. This research is supported by the Natural Science Foundation of China under Grant No. 11201217 and the Natural Science Foundation of JiangXi Province under Grant No. 20132BAB211010. This research is also supported by the Natural Science Foundation of China under Grant No. 71301067 and the Natural Science Foundation of JiangXi Province under Grant No. 20142BAB211015.

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