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Multi-sample test based on bootstrap methods for second order stochastic dominance

Jianling Zhang* †and Zhongzhan Zhang‡

Abstract

Statistical inferences under second order stochastic dominance for two sample case has a long and rich history. But the $k(\geq 2)$ sample case has not been well studied. In this article we consider $k(\geq 2)$ sample test for the equality of distribution functions against second order stochastic dominance alternative. A test statistic is constructed with isotonic regression estimates of stop-loss transform functions, and the asymptotic distribution of the proposed test is given. A bootstrap procedure is employed to obtain the p-value of the test, and some simulation results are presented to illustrate the proposed test method.

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1. Introduction

Ordering of distribution functions play an important role in many scientific areas including lifetime testing, reliability and economics, (see, for example, Alzaid et al. [1], Boland and Samaniego [6], Li and Lu [14], Shaked and Shanthikumar [18]). Many types of orderings of varying degrees of strength for comparing univariate distributions are discussed in the literature, including likelihood ratio ordering (Dykstra et al. [8]), uniform stochastic ordering or hazard rate ordering (Dykstra et al. [7]), and first- and second-order stochastic ordering (Feng and Wang [11], Klonner [13], Schnid and Trede [17]). Among them, first- and second-order stochastic ordering are the weakest, and used widely

*College of Applied Sciences, Beijing University of Technology, Beijing, China, 100124. Email: zhangjl@emails.bjut.edu.cn

†School of Mathematics and Information Sciences, Weifang University, Weifang, China, 261061. Email: zhangjl@emails.bjut.edu.cn

‡College of Applied Sciences, Beijing University of Technology, Beijing, China, 100124. Email:zzhang@bjut.edu.cn

in practice (see, for example, Fong et al. [12], Klonner [13], Sriboonchita et al. [20], Wong [23]).

Second-order stochastic ordering is often called as second-order stochastic dominance or concave stochastic order, especially in economics. Since the beginning of the 1970's, stochastic dominance rules have been an essential tool in the comparison and analysis of poverty and income inequality. More recently, stochastic dominance has also been employed in the development of the theory of decision under risk and in actuarial sciences. The influential articles by Atkinson [2] and Shorrocks [19] are examples of theoretical works that provided a far-reaching insight into the importance of the stochastic dominance rules. And in economics and finance, second order stochastic dominance plays a major role in developing a general framework to establish a criterion for selecting one option over another. Therefore, it is of major interest to acquire a deep understanding of the meaning and implications of the second order stochastic dominance assumptions. This is why we focus on the statistical test of second order stochastic dominance in this article.

Testing against second order stochastic dominance of two distributions has a rich history and has been studied by many authors, for example, Liu and Wang [15], Bai et al.[3] and Berrendero and Carcamo [5], among others. In practice, we may be faced to compare multiple distributions in the mean of second order stochastic dominance. However, as far as we know, the multi-sample comparisons have not been well studied. In this article, we consider the test of stochastic equality of multiple distributions against the stochastic monotonicity under second order stochastic dominance.

The rest of the article is organized as follows. In section 2, as preparation we define some estimators for the unknown functionals of distribution functions, and discuss their consistency. In section 3 we provide test for the stochastic equality against second order stochastic dominance of k distributions and give the asymptotic distribution of the test statistic. In section 4 we establish a bootstrap procedure to implement the proposed test. In section 5 we present simulation to illustrate the performance of the proposed method. Some conclusion remarks are given in section 6.

2. Preliminaries

In this section, we first recall the definition of second order stochastic dominance for the convenience of statement, and then present estimators of the integrated distribution functions which satisfy the ordering restrictions.

2.1. Second order stochastic dominance.

2.1. Definition. Let X and Y be independent random variables with corresponding cumulative distribution functions F and G respectively. We say that Y dominates X in the sense of second order stochastic dominance, and denote by $X \leq_{SSD} Y$ or $F \leq_{SSD} G$, if for every nondecreasing and concave function $u(\cdot)$, we have

(2.1) $E(u(X)) \leq E(u(Y))$

or if

(2.2) $E(X-t)_- \leq E(Y-t)_-, \quad \forall t \in R.$

The equivalence of (2.1) and (2.2) refers to Stoyan [21]. In addition, a straightforward application of Fubini's theorem leads to yet another equivalent expression. In fact, define the transform W_F associated with a distribution function F by

(2.3) $W_F(t) = \int_{-\infty}^{t} F(y)dy, \quad \forall t \in R,$

then (2.1) is equivalent to

(2.4) $W_F(t) \geq W_G(t), \quad \forall t \in R.$

see Theorem 4.A.2 in Shaked and Shanthikumar [18].

2.2. Isotonic regression estimators of the integrated distribution functions. Assume that there are k independent samples $X_{i1}, X_{i2}, \cdots, X_{in_i}$, where $X_{ij}, j = 1, \cdots, n_i$ have common distribution function $F_i, i = 1, 2, \cdots, k$. We are interested in how to test with the samples that

(2.5) $F_1 \geq_{SSD} F_2 \geq_{SSD} \cdots \geq_{SSD} F_k$

or, equivalently

(2.6) $W_{F_1}(t) \leq W_{F_2}(t) \leq \cdots \leq W_{F_k}(t), \quad \forall t \in R.$

For this purpose, we first estimate the integrated distribution functions $W_{F_i}(t)$.

As is well known, a suitable estimator of F_i is the empirical distribution function

$$\hat{F}_i(x) = \frac{1}{n_i} \sum_{j=1}^{n_i} I_{[X_{ij}, \infty)}(x)$$

where $I_A(\cdot)$ denotes the indicator function associated with the set A. An immediate estimator of W_{F_i}, denoted by $W_{\hat{F}_i}$, can be obtained by substituting $F_i(x)$ with $\hat{F}_i(x)$, $W_{\hat{F}_i}(t) = \int_{-\infty}^{t} \hat{F}_i(y)dy, \quad \forall t \in R$. Let

$$W_{\hat{F}}(t) = (W_{\hat{F}_1}(t), W_{\hat{F}_2}(t), \cdots, W_{\hat{F}_k}(t)), \qquad t \in R.$$

It is obvious that the vector $W_{\hat{F}}(t)$ need not satisfy inequality (2.6), even if the inequality holds. To get such estimators, we employ isotonic regression. Let $N_{rs} = \sum_{j=r}^{s} n_j$, and $Av_n[W_{\hat{F}}(t), r, s] = \sum_{j=r}^{s} n_j W_{\hat{F}_j}(t)/N_{rs}$ for $r \leq s$. Define the estimator of $W_{F_i}(t)$ by

(2.7) $W^*_{\hat{F}_i}(t) = \max_{r \leq i} \min_{s \geq i} Av_n[W_{\hat{F}}(t), r, s], \quad i = 1, \cdots, k.$

$Av_n[W_{\hat{F}}(t), r, s]$ is the weighted average of $W_{\hat{F}_r}(t), \cdots, W_{\hat{F}_s}(t)$, and for each t, $W^*_{\hat{F}_i}(t)$ is the isotonic regression estimator of $W_{\hat{F}_i}(t)$ with weights $\{n_1, \cdots, n_k\}$ (see Robertson et al.[16]).

Let $|| \cdot ||$ denote the sup norm. The following lemma gives the consistency of the estimators, and thus the reasonability to construct a test statistic with them.

2.2. Lemma. $P[\| W_{\hat{F}_i} - W_{F_i} \| \to 0, \quad n_i \to \infty, \ i = 1, \cdots, k] = 1.$

Furthermore, if inequality (2.6) holds, then

$$P[||W^*_{\hat{F}_i} - W_{F_i}|| \to 0, \quad n_i \to \infty, \ i = 1, 2, \cdots, k] = 1.$$

The first conclusion of Lemma 2.2 is a straightforward consequence of Glivenko-Cantelli Theorem in van der Vaart and Wellner [22], and the second one can be proved easily by combining the first one and the properties of isotonic regression (Robertson et al. [16]). We omit the proof (see also, for example, El Barmi and Marchev [9]).

3. Hypothesis Tests

In this section, we discuss the tests of hypotheses under second order stochastic dominance. The hypotheses are defined as

$$H_0: \ F_1 = F_2 = \cdots = F_k,$$

and

$$H_1: \ F_1 \geq_{SSD} F_2 \geq_{SSD} \cdots \geq_{SSD} F_k,$$

We first set the notation in Subsection 3.1, then study the tests of H_0 versus $H_1 - H_0$ in Subsection 3.2.

3.1. Notation and lemmas. Let $n = \sum_{i=1}^{k} n_i$,

$$\begin{aligned} a_{in} &= \frac{n_i}{n}, \\ Z_{in_i}(t) &= \sqrt{n_i}[W_{\hat{F}_i}(t) - W_{F_i}(t)], \\ Z^*_{in_i}(t) &= \sqrt{n_i}[W^*_{\hat{F}_i}(t) - W_{F_i}(t)], \ i = 1, 2, \cdots, k, \end{aligned}$$

and

$$A_{rsn} = \sum_{j=r}^{s} a_{jn}, \ 1 \leq r \leq s \leq k.$$

When limits

$$\lim_{n\to\infty} a_{in} = a_i > 0, \quad i = 1, 2, \cdots, k \tag{3.1}$$

exist, denote

$$A_{rs} = \lim_{n\to\infty} A_{rsn} = \sum_{j=r}^{s} a_j.$$

For standard Brownian bridge $B = (B(t))_{0\leq t\leq 1}$ and distribution function H on R, denote $B_H(x) = \int\limits_{-\infty}^{x} B(H(s))ds, \ x \in R$. If $\int x^2 H(dx) < \infty$, then $B_H = (B_H(x))_{x\in R}$ is a centered Gaussian process with covariance function

$$\rho_H(x, y) = \int\limits_{-\infty}^{x} \int\limits_{-\infty}^{y} (H(u \wedge v) - H(u)H(v))dudv, \quad x, y \in R.$$

See Berrendero and Carcamo [5].

In this paper, we use " $\xrightarrow{w}$ " to denote weak convergence (or convergence in distribution).

The following result is helpful to derive the asymptotic distributions of test statistics. Its proof is similar to that of Lemma 1 in Baringhaus and Grübel [4].

3.1. Lemma. *Let $f_n, g_n (n \in \mathbb{N}), g, h$ be continuous real functions on $K = [-\infty, \infty]$ such that $f_n = g_n + c_n h$, where $(c_n)_{n\in\mathbb{N}}$ is a sequence of non-negative real numbers with $\lim\limits_{n\to\infty} c_n = \infty$. Assume further that $h \leq 0$, $A = \{h = 0\} \neq \emptyset$, and g_n converges uniformly to g. Then*

$$\lim_{n\to\infty} \sup_{t\in K} f_n(t) = \sup_{t\in A} g(t).$$

3.2. Test of H_0 versus $H_1 - H_0$. In this subsection, we consider the problem of testing H_0 versus $H_1 - H_0$. To this end, define test statistic T_n by

$$T_n = \sqrt{n} \sup_{t \in R} (W^*_{\hat{F}_k}(t) - W^*_{\hat{F}_1}(t)).$$

It is easy to see from Lemma 2.2 that when the alternative hypotheses holds, $W^*_{\hat{F}_k}$ and $W^*_{\hat{F}_1}$ would have different limits, thus T_n would take large values with large probability. To obtain the properties of T_n more explicitly, we next study its asymptotic distribution.

3.2. Theorem. *If for all F_is have finite second moments, then*

$$(Z_{1n_1}(t), Z_{2n_2}(t), \cdots, Z_{kn_k}(t))' \xrightarrow{w} (B_{F_1}(t), B_{F_2}(t), \cdots, B_{F_k}(t))', \quad \forall t \in R,$$

as $\min n_i \to \infty$.

The theorem is an easy result of empirical process theory (van der Vaart and Wellner [22], see also Theorem 1 in Baringhaus and Grübel [4]). From Theorem 3.2, it may be shown the following theorem.

Let $S_i = \{j : W_{F_j}(t) = W_{F_i}(t), \forall t \in R, j = 1, \cdots, k\}$, c_i and d_i be the left and right endpoints of the support of $F_i, i = 1, 2, \cdots, k$. The following condition will be employed to give the asymptotic distribution of $Z^*_{in_i}$s.

$$\inf_{c_i + \eta \le t \le d_i - \eta} [W_{F_j}(t) - W_{F_i}(t)] > 0, \tag{3.2}$$

for some $\eta > 0$ and all $j > S_i$, $i = 1, 2, \cdots, k$, where $\inf_{\emptyset}(.) = \infty$, and $j > S_i$ means $j > l$ for all $l \in S_i$.

3.3. Theorem. *Suppose all the k distributions have finite second moments, and (3.1) and (3.2) hold. Then under H_1 it holds that*

$$(Z^*_{1n_1}(t), Z^*_{2n_2}(t), \cdots, Z^*_{kn_k}(t))' \xrightarrow{w} (Z^*_1(t), Z^*_2(t), \cdots, Z^*_k(t))', \quad t \in R$$

as $n \to \infty$, where

$$Z^*_i(t) = \sqrt{a_i} \max_{r \le i, r \in S_i} \min_{i \le s, s \in S_i} \frac{\sum_{\{r \le j \le s\}} \sqrt{a_j} B_{F_j}(t)}{A_{rs}}.$$

We omit its proof, which is similar to that of Theorem 4 in El Barmi and Mukerjee [10]. Based on the conclusion, we may obtain the asymptotic distribution of the test statistic.

3.4. Theorem. *Suppose the conditions of Theorem 3.3 are satisfied. Then under H_0 it holds that*

$$T_n \xrightarrow{w} T = \sup_{t \in R} \{ \max_{r \le k} \frac{\sum_{j=r}^{k} \sqrt{a_j} B_{F_j}(t)}{A_{rk}} - \min_{s \ge 1} \frac{\sum_{j=1}^{s} \sqrt{a_j} B_{F_j}(t)}{A_{1s}} \}.$$

Proof. Define stochastic processes $V_n(t) = \sqrt{n}(W^*_{\hat{F}_k}(t) - W^*_{\hat{F}_1}(t))$, then

$$\begin{aligned}
V_n(t) &= \sqrt{n}(W^*_{\hat{F}_k}(t) - W_{F_k}(t)) - \sqrt{n}(W^*_{\hat{F}_1}(t) - W_{F_1}(t)) \\
&\quad + \sqrt{n}(W_{F_k}(t) - W_{F_1}(t)) \\
&= \sqrt{\tfrac{n}{n_k}} \sqrt{n_k}(W^*_{\hat{F}_k}(t) - W_{F_k}(t)) - \sqrt{\tfrac{n}{n_1}} \sqrt{n_1}(W^*_{\hat{F}_1}(t) - W_{F_1}(t)) \\
&\quad + \sqrt{n}(W_{F_k}(t) - W_{F_1}(t)) \\
&= \sqrt{\tfrac{n}{n_k}} Z^*_{kn_k}(t) - \sqrt{\tfrac{n}{n_1}} Z^*_{1n_1}(t) + \sqrt{n}(W_{F_k}(t) - W_{F_1}(t)).
\end{aligned} \tag{3.3}$$

Under H_0, the third term on the right-hand side is just zero. By Theorem 3.3 and Slutsky theorem, we obtain

$$\begin{aligned}\sqrt{\tfrac{n}{n_k}}Z^*_{k,n_k}(t) - \sqrt{\tfrac{n}{n_1}}Z^*_{1,n_1}(t) &\xrightarrow{w} \sqrt{1/a_k}Z^*_k(t) - \sqrt{1/a_1}Z^*_1(t)\\ &= \max_{r\le k}\frac{\sum_{j=r}^{k}\sqrt{a_j}B_{F_j}(t)}{A_{rk}} - \min_{s\ge 1}\frac{\sum_{j=1}^{s}\sqrt{a_j}B_{F_j}(t)}{A_{1s}}.\end{aligned}$$

By Lemma 3.1 and continuous mapping theorem, we have

$$T_n \xrightarrow{w} T \quad = \quad \sup_{t\in R}\{\max_{r\le k}\frac{\sum_{j=r}^{k}\sqrt{a_j}B_{F_j}(t)}{A_{rk}} - \min_{s\ge 1}\frac{\sum_{j=1}^{s}\sqrt{a_j}B_{F_j}(t)}{A_{1s}}\}. \quad \square$$

3.5. Theorem. *Suppose that $H_1 - H_0$ does hold. Then $P(T_n \to \infty) = 1$.*

Proof. The first two terms on the right-hand side of (3.3) are stochastically bounded. If $H_1 - H_0$ does hold, then there is at least one i which satisfies $W_{F_i}(t) < W_{F_{i+1}}(t)$ for all t in some non-empty interval $(a, b) \subset R$. As $\sqrt{n} \to \infty$, we obtain

$$\sup_{t\in R} V_n(t) \to \infty$$

with probability one. $\square$

Theorem 3.4 gives the null asymptotic distribution of T_n, thus the feasibility of the test theoretically. Theorem 3.5 reveals that the proposed test is consistent.

4. Bootstrap Procedure

To use the statistic T_n to make a decision in practice, we require the p-value of the test statistic. Although the asymptotic distribution of T_n under the null hypothesis is given, however, it is very complicated, and depends on the underlying unknown distributions F_i, thus is difficult to be used directly to compute the critical value. In this section, we give a bootstrap method to compute an approximated p-value for T_n.

4.1. Asymptotic behavior of Bootstrap statistic. Recall that $\hat{F}_i$ are the empirical distribution functions associated with the samples $X_{i1}, \cdots, X_{in_i}$ from $F_i, i = 1, \cdots, k$. These random variables are the initial segments of k infinite sequences $(X_{ij})_{j\in\mathbb{N}}$ of random variables defined on some background probability space $(\Omega, \mathcal{A}, P)$; the almost sure statements below refer to P. Given the initial segments, let $\hat{\zeta}_{n,1} \cdots, \hat{\zeta}_{n,n}$ be a sample of size n from the (random) distribution function

(4.1) $H_n = \frac{n_1}{n}\hat{F}_1 + \frac{n_2}{n}\hat{F}_2 + \cdots + \frac{n_k}{n}\hat{F}_k.$

Let

$$\begin{aligned}\hat{F}_{n,n_i}(x) &= \frac{1}{n_i}\sum_{j=n_1+\cdots+n_{i-1}+1}^{n_1+\cdots+n_i} I_{[\hat{\zeta}_{n,j},\infty)}(x),\\ W^*_{\hat{F}_{n,n_i}}(x) &= \max_{r\le i}\min_{s\ge i} Av_n[W_{\hat{F}_{n,n_i}}(x), r, s],\\ \hat{Z}_{n,n_i} &= \sqrt{n_i}(W_{\hat{F}_{n,n_i}} - W_{H_n}),\\ \hat{Z}^*_{n,n_i} &= \sqrt{n_i}(W^*_{\hat{F}_{n,n_i}} - W_{H_n}), \quad i = 1, \cdots, k,\end{aligned}$$

and define the bootstrap version of T_n by

(4.2) $\hat{T}_n = \sup_{t\in R}\sqrt{n}(W^*_{\hat{F}_{n,n_k}}(t) - W^*_{\hat{F}_{n,n_1}}(t))$

The following theorem shows that, with probability 1, the limit distribution of $\hat{T}_n$ is the same as that of T_n.

4.1. Theorem. *Suppose that the conditions of Theorem 3.4 hold, then with probability one,*

(4.3) $\hat{T}_n \xrightarrow{w} \sup_{t\in R}\{\max_{r\le k} \frac{\sum_{j=r}^{k} \sqrt{a_j} B_{F_j}(t)}{A_{rk}} - \min_{s\ge 1} \frac{\sum_{j=1}^{s} \sqrt{a_j} B_{F_j}(t)}{A_{1s}}\}$

where $a_i, A_{rk}, A_{1s}, i, r, s = 1, \cdots, k$ are the same as in Theorem 3.4.

Proof. Let $\mathcal{F} = \{\phi_t(x) = (x-t)_- : t \in R\}$, and $\hat{U}^{F_i}_{n,n_i} = (\hat{U}^{F_i}_{n_i}(\phi))_{\phi\in\mathcal{F}}$,

$$\hat{U}^{F_i}_{n,n_i}(\phi) := \sqrt{n_i}(\int \phi d\hat{F}_{n,n_i} - \int \phi dH_n)$$

be the empirical processes associated with the k parts of the resamples. See van der Vaart and Wellner [22], with probability one, we have

(4.4) $\hat{U}^{F_i}_{n,n_i} \xrightarrow{w} B_{F_i}, \quad i = 1\cdots, k.$

In analogy to (3.2), we now define the stochastic processes $\hat{V}_n(t)$ by

$$\begin{aligned}
\hat{V}_n(t) &= \sqrt{n}(W^*_{\hat{F}_{n,n_k}}(t) - W^*_{\hat{F}_{n,n_1}}(t)) \\
&= \sqrt{n}[(W^*_{\hat{F}_{n,n_k}}(t) - \int \phi_t dH_n) - (W^*_{\hat{F}_{n,n_1}}(t) - \int \phi_t dH_n)] \\
&= \sqrt{\frac{n}{n_k}}\sqrt{n_k}(W^*_{\hat{F}_{n,n_k}}(t) - W_{H_n}(t)) - \sqrt{\frac{n}{n_1}}\sqrt{n_1}(W^*_{\hat{F}_{n,n_1}}(t) - W_{H_n}(t)) \\
&= \sqrt{\frac{n}{n_k}}\hat{Z}^*_{n,n_k}(t) - \sqrt{\frac{n}{n_1}}\hat{Z}^*_{n,n_1}(t)
\end{aligned}$$

Note that $\hat{Z}^*_{n,n_k}(t)$ may be obtained from the isotonic regression of $\hat{U}^{F_i}_{n,n_i}(\phi_t), i = 1, \cdots, k$. By continuous mapping theorem, the conditional independence of the subsamples and (4.4), we obtain

$$\hat{V}_n(t) \xrightarrow{w} \max_{r\le k} \frac{\sum_{j=r}^{k} \sqrt{a_j} B_{F_j}(t)}{A_{rk}} - \min_{s\ge 1} \frac{\sum_{j=1}^{s} \sqrt{a_j} B_{F_j}(t)}{A_{1s}}$$

with probability one. This leads to that

$$\hat{T}_n = \sup_{t\in R} \hat{V}_n(t) \xrightarrow{w} \sup_{t\in R}\{\max_{r\le k} \frac{\sum_{j=r}^{k} \sqrt{a_j} B_{F_j}(t)}{A_{rk}} - \min_{s\ge 1} \frac{\sum_{j=1}^{s} \sqrt{a_j} B_{F_j}(t)}{A_{1s}}\}$$

with probability one, by continuous mapping theorem and Lemma 3.1. □

4.2. Determination of the p-value. To apply the test in practice, we propose a bootstrap approximation to the p-value of the test as follows.

Step 1. Compute test statistic T_n from the original samples $X_{i1}, \cdots, X_{in_i}, i = 1, \cdots, k$;

Step 2. Let $\hat{\zeta}_{n,1}\cdots, \hat{\zeta}_{n,n}$ be a bootstrap sample of size n from the pooled empirical distribution function $H_n = \frac{n_1}{n}\hat{F}_1 + \frac{n_2}{n}\hat{F}_2 + \cdots + \frac{n_k}{n}\hat{F}_k$, where $\hat{F}_i$ is the empirical distribution function associated with the sample $X_{i1}, \cdots, X_{in_i}, i = 1, \cdots, k$. Divide this bootstrap sample into k parts $\hat{\zeta}_{n,n_1+\cdots+n_{i-1}+1}, \cdots, \hat{\zeta}_{n,n_1+\cdots+n_i}, i = 1, \cdots, k$. Use these k parts to compute a bootstrap version of the test statistic $\hat{T}_n$ by (4.2);

Step 3. Repeat step 2 a large number B of times, yielding B bootstrap test statistics $\hat{T}^{(b)}_n, b = 1, \cdots, B$;

Step 4. The p-value of the proposed test is given by $p = \frac{Card\{b: \hat{T}_n^{(b)} > T_n, b=1,\cdots,B\}}{B}$. We reject H_0 at a given level α when $p < \alpha$. Theorem 4.1, Theorem 3.4 and Theorem 3.5 ensure that the true level of the proposed test would be closed to the nominal significant level under H_0, and the power (the rejection probability) should be high under $H_1 - H_0$ when the sample sizes are enough large. The simulation in next section will confirm the intuition statements.

5. Simulation Study

To investigate the properties of the tests, we carried out a simulation study for $k = 3$. The empirical rejection rates of T_n in 1000 replications are recorded for various scenarios. For each of the scenarios, the number of resampling is taken as 1000; the sample sizes of the three distributions are taken as the same, and they are set at 100, 200 in different simulations for evaluating the effect of sample size.

Table 1 reports the simulation results for scenarios for which H_0 is true. Two different significance levels are considered. In Table 2, the empirical rejection rates of the test are given for the scenarios for which $H_1 - H_0$ is true. The significance level is taken as $\alpha = 0.05$.

TABLE 1: Empirical rejection rates of the test under H_0

Distributions	$n_1 = n_2 = n_3 = 100$		$n_1 = n_2 = n_3 = 200$	
$F_1 = F_2 = F_3$	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.01$	$\alpha = 0.05$
U(0,1)	0.012	0.060	0.015	0.051
Exp(1)	0.013	0.057	0.010	0.056
N(0,1)	0.012	0.062	0.010	0.048
$\chi^2(2)$	0.016	0.037	0.012	0.047
Beta(2,2)	0.007	0.055	0.009	0.052

TABLE 2: Empirical rejection rates of the test under $H_1 - H_0$

Distributions			$n_1 = n_2 = n_3 = 100$	$n_1 = n_2 = n_3 = 200$
F_1	F_2	F_3	$\hat{p}$	$\hat{p}$
Uni(0,1.1)	Uni(0,1)	Uni(0,1)	0.732	0.946
Uni(0,1.1)	Uni(0,1.1)	Uni(0,1)	0.721	0.985
Exp(1)	Exp(1)	Exp(1.1)	0.213	0.236
Exp(1)	Exp(1.1)	Exp(1.1)	0.134	0.291
Exp(1)	Exp(1.1)	Exp(1.2)	0.324	0.569
N(0.1,1)	N(0,1)	N(0,1)	0.191	0.275
N(0.1,1)	N(0.1,1)	N(0,1)	0.174	0.253
N(0.5,1)	N(0.25,1)	N(0,1)	0.976	1.000
Uni(0,1)	Uni(0,1)	Beta(2,2)	0.478	0.853
Uni(0,1)	Beta(2,2)	Beta(2,2)	0.629	0.860

From Table 1, we see that the simulated size of the proposed test is reasonable and gets closer to α with the sample size n increasing. For fixed sample sizes, the performance of the test vary slightly with the population distributions.

Furthermore, from Table 2, we could have the following observations.

(1) With the increasing of sample sizes, the power (empirical rejection rate) of the proposed test increases fast.

(2) The power is related to how the probability distributions going against the null hypothesis. It is lower when the differences of the population distributions are slight, and goes higher when the differences become significant. In addition, the test looks more

sensitive to the differences of the distributions with bounded supports than that with unbounded supports.

6. Concluding Remarks

In this article, we present an extension of Baringhaus and Grübel [4] through isotonic regression, and give a test for the homogeneity of multiple populations against the second stochastic dominance ordering. The method can be used also to test the null hypothesis of second stochastic dominance ordering, even umbrella ordering in the sense of second stochastic dominance ordering, with an appropriate estimators of the distributions under umbrella ordering restriction.

Bootstrap method is employed to give the p-value of the proposed test. Generally, the approximated p-value is good for limited sample sizes. However, when the null hypothesis is not the homogeneity of the distributions, based on our simulations those are not presented here, the obtained p-value may not enough accurate (conservative in general), and how to improve the approximation should be studied further.

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