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On M_1- and M_3-properties in the setting of ordered topological spaces

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Abstract

In 1961, J. G. Ceder [3] introduced and studied classes of topological spaces called M_i-spaces ($i = 1, 2, 3$) and established that *metrizable* $\Rightarrow M_1 \Rightarrow M_2 \Rightarrow M_3$. He then asked whether these implications are reversible. Gruenhage [5] and Junnila [8] independently showed that $M_3 \Rightarrow M_2$. In this paper, we investigate the M_1- and M_3- properties in the setting of ordered topological spaces. Among other results, we show that if $(X, \mathcal{T}, \leq)$ is an M_1 ordered topological C- and I-space then the bitopological space $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ is pairwise M_1. Here, $\mathcal{T}^\natural := \{U \in \mathcal{T} \mid U \text{ is an upper set}\}$ and $\mathcal{T}^\flat := \{L \in \mathcal{T} \mid L \text{ is a lower set}\}$.

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1. Introduction

It is a well-known fact that σ-locally finite collections are σ-closure-preserving (see [5] or [15]). Thus the characterization of metrizable spaces by Bing-Nagata-Smirnov [15, Theorem 23.9] in terms of σ-locally finite bases motivated Ceder to study spaces with σ-closure preserving bases. In his paper [3], Ceder gave examples of non-metrizable M_1-spaces and got researchers on their feet by asking whether the implications $M_1 \Rightarrow M_2 \Rightarrow M_3$ are reversible. See the definitions of these concepts at the bottom of the preliminaries section below. Many researchers have worked on this problem and have produced a number of partial results but, as far as we know, no general solution yet. In 1966, C. J. R. Borges [1] reviewed Ceder's work on M_3-spaces and improved some of his results, and he generally illustrated the importance of M_3-spaces and thus renamed them stratifiable spaces. In 1973, following Ceder's efforts [3, Theorem 7.6, p. 117], F. G. Slaughter, Jr established that if f is a closed continuous mapping from a metric space X onto a topological space Y then Y is an M_1-space [14].

2. Preliminaries

Following Priestley [13], we denote the intersection of all lower sets containing a subset S of an ordered set X by $d(S)$. Dually, the intersection of all upper sets containing S is denoted by $i(S)$. Then we say that an ordered topological space $(X, \mathcal{T}, \leq)$ is a C-space if $d(F)$ and $i(F)$ are closed whenever F is a closed subset of X. Similarly, $(X, \mathcal{T}, \leq)$ is called an I-space if $d(G)$ and $i(G)$ are open whenever G is an open subset of X. A collection $\mathcal{B}$ of subsets of a topological space $(X, \mathcal{T})$ is said to be $\mathcal{T}$-closure-preserving if for each subcollection $\mathcal{B}' \subseteq \mathcal{B}$, we have $\bigcup_{B\in\mathcal{B}'} \overline{B} = \overline{\bigcup_{B\in\mathcal{B}'} B}$.

For brevity, we are going to refer to bitopological spaces as bispaces. A bispace $(X, \mathcal{T}_1, \mathcal{T}_2)$ is said to be $\mathcal{T}_1$*-regular with respect to* $\mathcal{T}_2$ [§] if and only if for each point $x \in X$ and each $\mathcal{T}_1$-closed set F with $x \notin F$, there are a $\mathcal{T}_1$-open set U and a $\mathcal{T}_2$-open set V such that $x \in U$, $F \subseteq V$ and $U \cap V = \emptyset$. Similarly, a bispace $(X, \mathcal{T}_1, \mathcal{T}_2)$ is said to be $\mathcal{T}_2$*-regular with respect to* $\mathcal{T}_1$ if and only if for each point $x \in X$ and each $\mathcal{T}_2$-closed set F with $x \notin F$, there are a $\mathcal{T}_2$-open set U and a $\mathcal{T}_1$-open set V such that $x \in U$, $F \subseteq V$ and $U \cap V = \emptyset$. We say that a bispace $(X, \mathcal{T}_1, \mathcal{T}_2)$ is *pairwise regular* if and only if it is both $\mathcal{T}_1$-regular with respect to $\mathcal{T}_2$ and $\mathcal{T}_2$-regular with respect to $\mathcal{T}_1$. We define $\mathcal{T}^\natural$ and $\mathcal{T}^\flat$ like this: $\mathcal{T}^\natural := \{U \in \mathcal{T} \,|\, U \text{ is an upper set}\}$ and $\mathcal{T}^\flat := \{L \in \mathcal{T} \,|\, L \text{ is a lower set}\}$.

Let $\mathcal{J}$ be the Euclidean topology on the unit interval $[0, 1]$, carrying its usual order. A bispace $(X, \mathcal{T}_1, \mathcal{T}_2)$ is *pairwise completely regular* if and only if for each $x \in X$ and each $\mathcal{T}_1$-closed set F with $x \notin F$, there exists a bicontinuous function $f : (X, \mathcal{T}_1, \mathcal{T}_2) \to ([0, 1], \mathcal{J}^\natural, \mathcal{J}^\flat)$ such that $f(x) = 1$ and $f(F) = \{0\}$; and for each $\mathcal{T}_2$-closed set Q with $x \notin Q$, there exists a bicontinuous function $g : (X, \mathcal{T}_1, \mathcal{T}_2) \to ([0, 1], \mathcal{J}^\natural, \mathcal{J}^\flat)$ such that $g(x) = 0$ and $g(Q) = \{1\}$ (see [9]).

Furthermore, recall that a topological space X is called an M_1*-space* if it is regular and has a σ-closure preserving base. In a bispace setting we follow Gutierrez and Romaguera [6] and say that a bispace $(X, \mathcal{T}_1, \mathcal{T}_2)$ is $\mathcal{T}_1$-M_1 *with respect to* $\mathcal{T}_2$ if and only if it is $\mathcal{T}_1$-regular with respect to $\mathcal{T}_2$ and there exists a base of $\mathcal{T}_1$ which is $\mathcal{T}_2$-σ-closure preserving. A $\mathcal{T}_2$-M_1 *with respect to* $\mathcal{T}_1$ bispace is defined similarly. Then a bispace $(X, \mathcal{T}_1, \mathcal{T}_2)$ is said to be *pairwise* M_1 if and only if it is both $\mathcal{T}_1$-M_1 with respect to $\mathcal{T}_2$ and $\mathcal{T}_2$-M_1 with respect to $\mathcal{T}_1$.

[§]Alternatively, some authors say that in the bispace $(X, \mathcal{T}_1, \mathcal{T}_2)$, the topology $\mathcal{T}_1$ is *regular with respect to* $\mathcal{T}_2$ whenever the condition given above holds.

We also need the notion of stratifiability. A bispace $(X, \mathcal{T}_1, \mathcal{T}_2)$ is said to be *pairwise semi-stratifiable* if and only if for each $\mathcal{T}_i$-closed set $F \subseteq X$ there exists a sequence of $\mathcal{T}_j$-open sets $(F_n)_{n\in\mathbb{N}}$ satisfying the following two conditions ($i, j \in \{1, 2\}$ and $i \neq j$): (i) If $F \subseteq K$ then $F_n \subseteq K_n$ for all $n \in \mathbb{N}$; (ii) $F = \bigcap_{n=1}^{\infty} F_n$. If, in addition, we also have (iii) $F = \bigcap_{n=1}^{\infty} cl_{\mathcal{T}_i} F_n$, then $(X, \mathcal{T}_1, \mathcal{T}_2)$ is said to be *pairwise stratifiable*. It has been established that a bispace is pairwise M_3 if and only if it is pairwise stratifiable [6, Proposition 1(b)]. Hence the terms pairwise stratifiable and pairwise M_3 shall be used exchangeably below.

3. Closure-Preserving Collections

In this section we prove some facts about closure-preserving collections which are interesting in their own right, and we will apply them in the next section. As usual, $\overline{A}$ and $cl_{\mathcal{T}^\natural} A$ denote the closure of A in $(X, \mathcal{T})$, and in $\mathcal{T}^\natural$ respectively.

1. Lemma. If $(X, \mathcal{T}, \leq)$ is an ordered topological C-space and $A \subseteq X$ then $cl_{\mathcal{T}^\natural} A = d(\overline{A}) = d(\overline{d(A)})$.

Proof. Let A be a subset of an ordered topological C-space $(X, \mathcal{T}, \leq)$. Then $d(\overline{A})$ is closed. Since $A \subseteq \overline{A} \subseteq d(\overline{A})$, $A \subseteq d(\overline{A})$. Then we have

$$d(\overline{A}) \subseteq d(\overline{d(A)}) \subseteq cl_{\mathcal{T}^\natural} (cl_{\mathcal{T}^\natural}(cl_{\mathcal{T}^\natural}(A))) = cl_{\mathcal{T}^\natural}(A) \subseteq cl_{\mathcal{T}^\natural}(d(\overline{A})) = d(\overline{A}),$$

the last equality because $d(\overline{A})$ is a closed lower set given that X is a C-space. Therefore the result holds. □

A similar argument proves the following:

2. Lemma. If $(X, \mathcal{T}, \leq)$ is an ordered topological C-space and $A \subseteq X$ then $cl_{\mathcal{T}^\flat} A = i(\overline{A}) = i(\overline{i(A)})$. □

1. Proposition. If $(X, \mathcal{T}, \leq)$ is an ordered topological C- and I-space and $\mathcal{B}$ is an open and closure-preserving collection in $(X, \mathcal{T})$ then $\mathcal{B}_d = \{d(B) \mid B \in \mathcal{B}\}$ is an open collection in $(X, \mathcal{T}^\flat)$ which is closure-preserving in $(X, \mathcal{T}^\natural)$.

Proof. Suppose $(X, \mathcal{T}, \leq)$ is an ordered topological C- and I-space. Let $\mathcal{B}$ be an open and closure-preserving collection in $(X, \mathcal{T})$. Since X is an I-space, $d(B)$ is an open lower set for each $B \in \mathcal{B}$. Hence $\mathcal{B}_d$ is open in $(X, \mathcal{T}^\flat)$. It remains to show that $\mathcal{B}_d$ is closure-preserving in $(X, \mathcal{T}^\natural)$. Note that the operator d commutes with set union. Let $\mathcal{B}' \subseteq \mathcal{B}$. Then by Lemma 1, we get

$$cl_{\mathcal{T}^\natural} \left(\bigcup_{B\in\mathcal{B}'} d(B) \right) = d \left(\overline{\bigcup_{B\in\mathcal{B}'} d(B)} \right) = d \left(\overline{d(\bigcup_{B\in\mathcal{B}'} B)} \right) = d \left(\overline{\bigcup_{B\in\mathcal{B}'} B} \right)$$

$$= d \left(\bigcup_{B\in\mathcal{B}'} \overline{B} \right) = \bigcup_{B\in\mathcal{B}'} d(\overline{B}) = \bigcup_{B\in\mathcal{B}'} d(\overline{d(B)}) = \bigcup_{B\in\mathcal{B}'} cl_{\mathcal{T}^\natural} d(B). \text{ So,}$$

$cl_{\mathcal{T}^\natural}(\bigcup_{B\in\mathcal{B}'} d(B)) = \bigcup_{B\in\mathcal{B}'} cl_{\mathcal{T}^\natural} d(B)$. Hence $\mathcal{B}_d$ is closure-preserving in $(X, \mathcal{T}^\natural)$. □

Similarly, the following result emerges.

2. Proposition. If $(X, \mathcal{T}, \leq)$ is an ordered topological C- and I-space and $\mathcal{B}$ is an open and closure-preserving collection in $(X, \mathcal{T})$ then $\mathcal{B}_i = \{i(B) \mid B \in \mathcal{B}\}$ is an open collection in $(X, \mathcal{T}^\natural)$ which is closure-preserving in $(X, \mathcal{T}^\flat)$. □

4. On Pairwise M_1- versus Pairwise M_3- (Stratifiable) Bispaces

In 1986, A. Gutierrez and S. Romaguera [6] introduced the concepts of pairwise M_i-spaces into the theory of bispaces as a generalization of Ceder's M_i-spaces (i=1,2,3). We recall the following nice result.

3. Proposition. ([10]) If $(X, \mathcal{T}, \leq)$ is a stratifiable ordered topological C-space then the bispace $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ is pairwise stratifiable. □

The reader is referred to [7] for the definition and basic properties of monotonically normal spaces. For these in the bispace setting, see [12]. It is known that a (bi) space is (pairwise) stratifiable if and only if it is (pairwise) semi-stratifiable and (pairwise) monotonically normal. K. Li and F. Lin showed that one can relax the assumption of the above proposition and obtain:

4. Proposition. ([11]) If $(X, \mathcal{T}, \leq)$ is a monotonically normal ordered topological C-space then the bispace $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ is pairwise monotonically normal.

We are now ready to present the following observation.

1. Theorem. If $(X, \mathcal{T}, \leq)$ is an M_1 ordered topological C- and I-space then the bispace $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ is pairwise M_1.

Proof. Let $(X, \mathcal{T}, \leq)$ be an M_1 ordered topological C- and I-space. We first show that $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ is pairwise regular. Let $x \in X$, and $F \subseteq X$ be a closed lower set such that $x \notin F$. Then $G := X \setminus F$ is a open neighbourhood of x. Since $(X, \mathcal{T}, \leq)$ is M_1, it is regular. Hence there exists an open neighbourhood H of x such that $\overline{H} \subseteq G$. Since X is an I-space, the upper set $U = i(H)$ is an open neighbourhood of x. By Lemma 2, we have
$$\overline{U} = \overline{i(H)} \subseteq i(\overline{i(H)}) = i(\overline{H}) \subseteq i(G) = G.$$
Since X is a C-space, $i(\overline{H})$ is closed and hence $V := X \setminus i(\overline{H})$ is an open lower set containing F and $U \cap V = \emptyset$. Thus the bispace $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ is $\mathcal{T}^\natural$-regular with respect to $\mathcal{T}^\flat$. Similarly, one can easily show that $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ is $\mathcal{T}^\flat$-regular with respect to $\mathcal{T}^\natural$ and hence pairwise regular. Since $(X, \mathcal{T}, \leq)$ is an M_1-space, $\mathcal{T}$ has a σ-closure-preserving base, say $\mathcal{B}$. Let $\mathcal{B} = \bigcup_{n\in\mathbb{N}} \mathcal{B}_n$ where each $\mathcal{B}_n$ is a $\mathcal{T}$-closure-preserving subcollection of $\mathcal{B}$. Now we need to produce σ-closure-preserving bases for $\mathcal{T}^\natural$ and $\mathcal{T}^\flat$. Let $\mathcal{D}_n = \{d(B) \mid B \in \mathcal{B}_n\}$ and put $\mathcal{D} = \bigcup_{n\in\mathbb{N}} \mathcal{D}_n$. Then $\mathcal{D}$ is a base for $\mathcal{T}^\flat$ which is, by Proposition 1, σ-closure-preserving in $\mathcal{T}^\natural$. Similarly, let $\mathcal{I}_n = \{i(B) \mid B \in \mathcal{B}_n\}$ and $\mathcal{I} = \bigcup_{n\in\mathbb{N}} \mathcal{I}_n$. Then $\mathcal{I}$ is a base for $\mathcal{T}^\natural$ which is, by Proposition 2, σ-closure-preserving in $\mathcal{T}^\flat$. Hence the bispace $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ is $\mathcal{T}^\natural$-M_1 with respect to $\mathcal{T}^\flat$ and $\mathcal{T}^\flat$-M_1 with respect to $\mathcal{T}^\natural$. Therefore $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ is pairwise M_1. □.

Since every pairwise M_1-bispace is pairwise stratifiable [6], we get:

1. Corollary. If $(X, \mathcal{T}, \leq)$ is an M_1 ordered topological C- and I-space then the bispace $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ is pairwise stratifiable. □

Finally, we briefly turn our minds to the following result involving countability. Recall that a bispace $(X, \mathcal{T}_1, \mathcal{T}_2)$ is doubly first countable if both topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ are first countable (see for instance J. Deak [4]).

5. Proposition. ([10]) If $(X, \mathcal{T}, \leq)$ is a first countable ordered topological I-space then $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ is doubly first countable. □

Since any metric space is first countable and stratifiable, the following fact follows immediately and it fits in here.

2. Corollary. If $(X, \mathcal{T}, \leq)$ is a metrizable ordered topological C- and I-space then $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ is pairwise M_1 (and thus pairwise stratifiable) and doubly first countable. □

Remark. As mentioned in the introduction above, F. G. Slaughter, Jr showed that if f is a closed continuous mapping from a metric space X onto the space Y, then Y is an M_1-space [14, Theorem 6]. It is therefore natural to wonder whether, in the same vein, the assumption of the above theorem can be relaxed without destroying the theorem in the sense that the bispace $(X, \mathcal{T}^\natural, \mathcal{T}^\flat)$ is pairwise M_1 whenever $(X, \mathcal{T}, \leq)$ is an M_1 ordered topological C-space.

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