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On conditions for univalence of some integral operators

Daniel Breaz* and Virgil Pescar†

Abstract

In this paper, we obtain new univalence conditions for the integral operators $F_{[|\delta|]}(z)$ and $G_{[|\delta|]}(z)$ of analytic functions defined in the open unit disk.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form $f(z) = z + a_2z^2 + ...$ which are analytic in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$. Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathcal{U}$.

Pescar [7], has obtained the following univalence criteria

1.1 Theorem. [7] *Let $\gamma \in \mathbb{C}$, $f \in \mathcal{S}$, $f(z) = z + a_2z^2 + ...$*
If

$$\left| \frac{zf'(z) - f(z)}{zf(z)} \right| \leq 1, \; \forall \, z \in \mathcal{U}$$

and

$$|\gamma| \leq \frac{1}{\max\limits_{|z| \leq 1} \left[\left(1 - |z|^2\right) \cdot |z| \cdot \frac{|z| + |a_2|}{1 + |a_2||z|} \right]},$$

then

$$F_\gamma(z) = \int_0^z \left(\frac{f(t)}{t} \right)^\gamma dt$$

*Department of Mathematics, "1 Decembrie 1918" University of Alba Iulia, Faculty of Science, 510009 Alba Iulia, Romania.
Email : `dbreaz@uab.ro`
†Department of Mathematics, "Transilvania" University of Braşov, Faculty of Mathematics and Computer Science, 500091 Braşov, Romania.
Email : `virgilpescar@unitbv.ro`

is in the class $\mathcal{S}$.

1.2 Theorem. [7] *Let* $\alpha, \beta, \gamma \in \mathbb{C}$, $f \in \mathcal{S}$, $f(z) = z + a_2 z^2 + \ldots$. *If*

$$\left| \frac{zf'(z) - f(z)}{zf(z)} \right| \leq 1, \ \forall\, z \in \mathcal{U},$$

$$\mathrm{Re}\beta \geq \mathrm{Re}\alpha > 0$$

and

$$|\gamma| \leq \frac{1}{\max\limits_{|z|\leq 1}\left[\frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha} \cdot |z| \cdot \frac{|z|+|a_2|}{1+|a_2||z|}\right]},$$

then

$$G_{\beta,\gamma}(z) = \left[\beta \int_0^z t^{\beta-1}\left(\frac{f(t)}{t}\right)^{\gamma} dt\right]^{\frac{1}{\beta}}$$

is in the class $\mathcal{S}$.

We define the next two integral operators

$$F_{[|\delta|]}(z) = \int_0^z \left(\frac{f_1(t)}{t}\right)^{\alpha_1} \cdot \ldots \cdot \left(\frac{f_{[|\delta|]}(t)}{t}\right)^{\alpha_{[|\delta|]}} dt,$$

where $\delta \in \mathbb{C}$, $|\delta| \notin [0,1)$, $\alpha_i \in \mathbb{C}$, $f_i \in \mathcal{A}$, $i = \overline{1,[|\delta|]}$, $\alpha_1 \cdot \ldots \cdot \alpha_{[|\delta|]} = \delta$ and

$$G_{[|\gamma|]}(z) = \left[\gamma \int_0^z t^{\gamma-1}\left(\frac{f_1(t)}{t}\right)^{\alpha_1} \cdot \ldots \cdot \left(\frac{f_{[|\gamma|]}(t)}{t}\right)^{\alpha_{[|\gamma|]}} dt\right]^{\frac{1}{\gamma}},$$

$\gamma \in \mathbb{C}$, $|\gamma| \notin [0,1)$, $\alpha_i \in \mathbb{C}$, $f_i \in \mathcal{A}$, $i = \overline{1,[|\gamma|]}$, $\alpha_1 \cdot \ldots \cdot \alpha_{[|\gamma|]} = \gamma$.

In this paper, we obtain new univalence conditions for the integral operators $F_{[|\delta|]}(z)$ and $G_{[|\delta|]}(z)$.

2. Preliminary results

In order to derive our main results, we have to recall here the following lemmas:

2.1 Lemma. [2] *If the function* f *is regular in unit disk* $\mathcal{U}$, $f(z) = z + a_2 z^2 + \ldots$ *and*

$$\left(1 - |z|^2\right)\left|\frac{zf''(z)}{f'(z)}\right| \leq 1,$$

for all $z \in \mathcal{U}$, *then the function* f *is univalent in* $\mathcal{U}$.

2.2 Lemma. [5] *Let* α *be a complex number,* $\mathrm{Re}\alpha > 0$ *and* $f(z) = z + a_2 z^2 + \ldots$ *be a regular function in* $\mathcal{U}$. *If*

$$\frac{1 - |z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1,$$

for all $z \in \mathcal{U}$, *then for any complex number* β, $\mathrm{Re}\beta \geq \mathrm{Re}\alpha$, *the function*

$$F_\beta(z) = \left[\beta \int_0^z t^{\beta-1} f'(t) dt\right]^{\frac{1}{\beta}}$$

is in the class $\mathcal{S}$.

2.3 Lemma. **[3]** *If the function g is regular in $\mathcal{U}$ and $|g(z)| < 1$ in $\mathcal{U}$, then for all $\xi \in \mathcal{U}$, the following inequalities hold*

$$\left|\frac{g(\xi) - g(z)}{1 - \overline{g(z)}g(\xi)}\right| \leq \left|\frac{\xi - z}{1 - \overline{z}\xi}\right| (2.1)$$

and

$$|g'(z)| \leq \frac{1 - |g(z)|^2}{1 - |z|^2},$$

the equalities hold in the case $g(z) = \epsilon\frac{z+u}{1+\overline{u}z}$, where $|\epsilon| = 1$ and $|u| < 1$.

2.4 Remark. **[3]** For $z = 0$, from inequality (2.1) we obtain for every $\xi \in \mathcal{U}$,

$$\left|\frac{g(\xi) - g(0)}{1 - \overline{g(0)}g(\xi)}\right| \leq |\xi|$$

and hence,

$$|g(\xi)| \leq \frac{|\xi| + |g(0)|}{1 + \overline{g(0)}g(\xi)}.$$

Considering $g(0) = a$ and $\xi = z$, then

$$|g(z)| \leq \frac{|z| + |a|}{1 + |a|\,|z|},$$

for all $z \in \mathcal{U}$.

3. Main results

3.1 Theorem. *Let $M > 1$, $\delta \in \mathbb{C}$, $|\delta| \notin [0, 1)$, $\alpha_i \in \mathbb{C}$, for $i = \overline{1, [|\delta|]}$ and $\alpha_1 \cdot ... \cdot \alpha_{[|\delta|]} = \delta$. If $f_i \in \mathcal{A}$, $f_i(z) = z + a_2^i z^2 + ...$, for $i = \overline{1, [|\delta|]}$ and*

$$\left|\frac{zf_i'(z) - f_i(z)}{zf_i(z)}\right| \leq 1,\ \forall i = \overline{1, [|\delta|]},\ z \in \mathcal{U}, (3.1)$$

$$\frac{|\alpha_1| + ... + |\alpha_{[|\delta|]}|}{|\alpha_1 \cdot ... \cdot \alpha_{[|\delta|]}|} \leq M, (3.2)$$

$$|\alpha_1 \cdot ... \cdot \alpha_{[|\delta|]}| \leq \frac{1}{M \max\limits_{|z|\leq 1}\left[\left(1 - |z|^2\right) \cdot |z| \cdot \frac{|z|+|c|}{1+|c||z|}\right]}, (3.3)$$

where

$$|c| = \frac{\left|\alpha_1 a_2^1 + ... + \alpha_{[|\delta|]} a_2^{[|\delta|]}\right|}{M\left|\alpha_1 \cdot ... \cdot \alpha_{[|\delta|]}\right|},$$

then

$$F_{[|\delta|]}(z) = \int_0^z \left(\frac{f_1(t)}{t}\right)^{\alpha_1} \cdot ... \cdot \left(\frac{f_{[|\delta|]}(t)}{t}\right)^{\alpha_{[|\delta|]}} dt$$

is in the class $\mathcal{S}$.

Proof. We have $f_i \in \mathcal{A}$, for all $i = \overline{1, [|\delta|]}$ and $\frac{f_i(z)}{z} \neq 0$, for all $i = \overline{1, [|\delta|]}$.

Let g be the function $g(z) = \left(\frac{f_1(z)}{z}\right)^{\alpha_1} \cdot ... \cdot \left(\frac{f_{[|\delta|]}(z)}{z}\right)^{\alpha_{[|\delta|]}}$, $z \in \mathcal{U}$. We have $g(0) = 1$.

Consider the function

$$h(z) = \frac{1}{M\left|\alpha_1 \cdot ... \cdot \alpha_{[|\delta|]}\right|} \cdot \frac{F''_{[|\delta|]}(z)}{F'_{[|\delta|]}(z)}, z \in \mathcal{U}.$$

The function $h(z)$ has the form:

$$h(z) = \frac{1}{M\left|\alpha_1 \cdot ... \cdot \alpha_{[|\delta|]}\right|} \sum_{i=1}^{[|\delta|]} \alpha_i \frac{zf_i'(z) - f_i(z)}{zf_i(z)}.$$

Also,

$$h(0) = \frac{1}{M\left|\alpha_1 \cdot ... \cdot \alpha_{[|\delta|]}\right|} \sum_{i=1}^{[|\delta|]} \alpha_i a_2^i.$$

By using the relations (3.1) and (3.2) we obtain that $|h(z)| < 1$ and

$$|h(0)| = \frac{\left|\alpha_1 a_2^1 + ... + \alpha_{[|\delta|]} a_2^{[|\delta|]}\right|}{M\left|\alpha_1 \cdot ... \cdot \alpha_{[|\delta|]}\right|} = |c|\,.$$

Applying Remark 2.4 for the function h we obtain

$$\frac{1}{M\left|\alpha_1 \cdot ... \cdot \alpha_{[|\delta|]}\right|} \cdot \left|\frac{F''_{[|\delta|]}(z)}{F'_{[|\delta|]}(z)}\right| \leq \frac{|z| + |c|}{1 + |c|\,|z|}, \forall z \in \mathcal{U}$$

and

$$\left|\left(1 - |z|^2\right) \cdot z \cdot \frac{F''_{[|\delta|]}(z)}{F'_{[|\delta|]}(z)}\right| \leq M\left|\alpha_1 \cdot ... \cdot \alpha_{[|\delta|]}\right| \left(1 - |z|^2\right) \cdot |z| \cdot \frac{|z| + |c|}{1 + |c|\,|z|}, \forall z \in \mathcal{U}. (3.4)$$

Consider the function $H : [0,\ 1] \to \mathbb{R}$ defined by

$$H(x) = (1 - x^2)x\frac{x + |c|}{1 + |c|\,x};\ x = |z|\,.$$

We have

$$H\left(\frac{1}{2}\right) = \frac{3}{8} \cdot \frac{1 + 2\,|c|}{2 + |c|} > 0 \Rightarrow \max_{x \in [0,1]} H(x) > 0.$$

Using this result and from (3.4) we have:

$$\left|\left(1 - |z|^2\right) \cdot z \cdot \frac{F''_{[|\delta|]}(z)}{F'_{[|\delta|]}(z)}\right| \leq M\left|\alpha_1 \cdot ... \cdot \alpha_{[|\delta|]}\right| \cdot \max_{|z|<1} \left[\left(1 - |z|^2\right) \cdot |z| \cdot \frac{|z| + |c|}{1 + |c|\,|z|}\right], \forall z \in \mathcal{U}. (3.5)$$

Applying the condition (3.3) in the form (3.5) we obtain that

$$\left(1 - |z|^2\right) \cdot \left|z \cdot \frac{F''_{[|\delta|]}(z)}{F'_{[|\delta|]}(z)}\right| \leq 1, \forall z \in \mathcal{U},$$

and from Lemma 2.1 we obtain that $F_{[|\delta|]} \in \mathcal{S}$.

3.2 Theorem. *Let* $M > 1$, γ, $\delta \in \mathbb{C}$, $|\gamma| \notin [0,1)$, $\alpha_i \in \mathbb{C}$, *for* $i = \overline{1, [|\gamma|]}$, $\alpha_1 \cdot ... \cdot \alpha_n = [|\gamma|]$. *If* $f_i \in \mathcal{A}$, $f_i(z) = z + a_2^i z^2 + ...$, *for* $i = \overline{1, [|\gamma|]}$ *and*

$$\left|\frac{zf_i'(z) - f_i(z)}{zf_i(z)}\right| \leq 1,\ \forall i = \overline{1, [|\gamma|]},\ z \in \mathcal{U}, (3.6)$$

$$\frac{|\alpha_1| + ... + \left|\alpha_{[|\gamma|]}\right|}{\left|\alpha_1 \cdot ... \cdot \alpha_{[|\gamma|]}\right|} \leq M, (3.7)$$

$$\mathrm{Re}\gamma \geq \mathrm{Re}\delta > 0,$$

$$\left|\alpha_1 \cdot ... \cdot \alpha_{[|\gamma|]}\right| \leq \frac{1}{M \max\limits_{|z|\leq 1}\left[\left(1-|z|^2\right)\cdot |z| \cdot \frac{|z|+|c|}{1+|c||z|}\right]}, (3.8)$$

where

$$|c| = \frac{\left|\alpha_1 a_2^1 + ... + \alpha_{[|\gamma|]} a_2^{[|\gamma|]}\right|}{M\left|\alpha_1 \cdot ... \cdot \alpha_{[|\gamma|]}\right|},$$

then

$$G_{[|\gamma|]}(z) = \left[\gamma \int\limits_0^z t^{\gamma-1}\left(\frac{f_1(t)}{t}\right)^{\alpha_1} \cdot ... \cdot \left(\frac{f_{[|\gamma|]}(t)}{t}\right)^{\alpha_{[|\gamma|]}} dt\right]^{\frac{1}{\gamma}}$$

is in the class $\mathcal{S}$.

Proof. We consider the function

$$h(z) = \int\limits_0^z \left(\frac{f_1(t)}{t}\right)^{\alpha_1} \cdot ... \cdot \left(\frac{f_{[|\gamma|]}(t)}{t}\right)^{\alpha_{[|\gamma|]}} dt.$$

Define the function

$$p(z) = \frac{1}{M\left|\alpha_1 \cdot ... \cdot \alpha_{[|\gamma|]}\right|} \cdot \frac{h''(z)}{h'(z)},\ z \in \mathcal{U}.$$

The function $p(z)$ has the form:

$$p(z) = \frac{1}{M\left|\alpha_1 \cdot ... \cdot \alpha_{[|\gamma|]}\right|} \sum_{i=1}^{[|\gamma|]} \alpha_i \frac{zf_i'(z) - f_i(z)}{zf_i(z)}.$$

By using the relations (3.6) and (3.7) we obtain $|p(z)| < 1$ and

$$|p(0)| = \frac{\left|\alpha_1 a_2^1 + ... + \alpha_{[|\gamma|]} a_2^{[|\gamma|]}\right|}{M\left|\alpha_1 \cdot ... \cdot \alpha_{[|\gamma|]}\right|} = |c|.$$

Applying Remark 2.4 for the function h we obtain

$$\frac{1}{M\left|\alpha_1 \cdot ... \cdot \alpha_{[|\gamma|]}\right|} \cdot \left|\frac{h''(z)}{h'(z)}\right| \leq \frac{|z| + |c|}{1 + |c|\,|z|}, \forall z \in \mathcal{U}$$

and

$$\left|\frac{1-|z|^{2\mathrm{Re}\delta}}{\mathrm{Re}\delta} \cdot z \cdot \frac{h''(z)}{h'(z)}\right| \leq M\left|\alpha_1 \cdot ... \cdot \alpha_{[|\gamma|]}\right| \frac{1-|z|^{2\mathrm{Re}\delta}}{\mathrm{Re}\delta} \cdot |z| \cdot \frac{|z|+|c|}{1+|c|\,|z|}, \forall z \in \mathcal{U}. (3.9)$$

Consider the function $Q : [0,\ 1] \to \mathbb{R}$ defined by

$$Q(x) = \frac{1 - x^{2\mathrm{Re}\delta}}{\mathrm{Re}\delta} \cdot x \cdot \frac{x + |c|}{1 + |c|\,x};\ x = |z|.$$

We have $Q\left(\frac{1}{2}\right) > 0 \Rightarrow \max\limits_{x\in[0,1]} Q(x) > 0$.

Using this result in (3.9), we have:

$$\frac{1-|z|^{2\text{Re}\delta}}{\text{Re}\delta}\left|\frac{zh''(z)}{h'(z)}\right| \le M\left|\alpha_1 \cdot ... \cdot \alpha_{[|\gamma|]}\right| \cdot \max_{|z|<1}\left[\frac{1-|z|^{2\text{Re}\delta}}{\text{Re}\delta}\cdot|z|\cdot\frac{|z|+|c|}{1+|c|\,|z|}\right], \forall z \in \mathcal{U}. (3.10)$$

Applying the condition (3.8) in the relation (3.10), we obtain that

$$\frac{1-|z|^{2\text{Re}\delta}}{\text{Re}\delta}\left|\frac{zh''(z)}{h'(z)}\right| \le 1, \forall z \in \mathcal{U}$$

and from Lemma 2.2, we obtain that $G_{[|\gamma|]} \in \mathcal{S}$.

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