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# Some identities and recurrences relations for the q-Bernoulli and q-Euler polynomials

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**Abstract**

In this article we prove some relations between two-variable $q$-Bernoulli polynomials and two-variable $q$-Euler polynomials. By using the equality $e_q(z)E_q(-z) = 1$, we give an identity for the two-variable $q$-Genocchi polynomials. Also, we obtain an identity for the two-variable $q$-Bernoulli polynomials. Furthermore, we prove two theorems which are analogues of the $q$-extension Srivastava-Pinter additional theorem.

**Keywords:** Bernoulli numbers and polynomials, Euler numbers and polynomials, Genocchi polynomials, generating functions, generalized Bernoulli polynomials, generalized Genocchi polynomials, $q$-Bernoulli polynomials, $q$-Euler polynomials, $q$-Genocchi polynomials.

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## 1. Introduction Definition and Notation

The classical Bernoulli polynomials $B_n(x)$ and the classical Euler polynomials $E_n(x)$ are usually defined by means of the following generating functions;

$$\frac{t}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \quad |t| < 2\pi$$

and

$$\frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}, \quad |t| < \pi,$$

respectively. The corresponding Bernoulli numbers $B_n$ and Euler numbers $E_n$ are given by

$$B_n := B_n(0) = (-1)^n B_n(1) = \left(2^{1-n}-1\right)^{-1} B_n\left(\frac{1}{2}\right)$$

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and

$$E_n := 2^n E_n\left(\frac{1}{2}\right), \quad n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},$$

respectively.

Many mathematicians investigated these polynomials in ([2]-[17]). They proved some theorems and gave some interesting recurrences relations. Firstly, Carlitz in [2] gave $q$-Bernoulli polynomials.

In this work we give some recurrences relations and properties for two-variable $q$-Bernoulli polynomials and $q$-Euler polynomials.

Throughout this paper, we make use of the following notations; $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{C}$ denotes the set of complex numbers and $q \in \mathbb{C}$ with $|q| < 1$. The $q$-basic numbers and $q$-factorials are defined ([2], [7]-[15]) by

$$\begin{aligned} [a]_q &= \frac{1-q^a}{1-q} = 1 + q + \ldots + q^{a-1}, \quad (q \neq 1), \\ [n]_q! &= [n]_q [n-1]_q \ldots [2]_q [1]_q, \end{aligned}$$

respectively, where $[0]_q! = 1$ and $n \in \mathbb{N}, a \in \mathbb{C}$.

The $q$-binomial formula is defined ([8], [14]) by

$$(x+y)_q^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} x^{n-k} y^k,$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is the $q$-binomial coefficient (or Gaussian binomial coefficient) given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_{n-k}(q;q)_k} = \frac{[n]_q!}{[n-k]_q! [k]_q!}.$$

The $q$-exponential functions are given ([1], [8], [12], [13]) by

$$e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} = \prod_{k=0}^{\infty} \frac{1}{(1-(1-q)q^k z)}, \quad 0 < |q| < 1, \quad |z| < \frac{1}{|1-q|}$$

and

$$E_q(z) = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{z^n}{[n]_q!} = \prod_{k=0}^{\infty} \left(1 + (1-q) q^k z\right), \quad 0 < |q| < 1, \quad z \in \mathbb{C}.$$

From the last equations, we can easliy see that $e_q(z) E_q(-z) = 1$.

The Jack-derivative $D_q$ is defined ([7], [10], [13], [14]) by

$$D_q f(z) = \frac{f(qz) - f(z)}{qz - z}, \quad 0 < |q| < 1, \quad 0 \neq z \in \mathbb{C}.$$

The derivative of the product of two functions and the derivative of the division of two functions are given by the following equations in [7], respectively.

$$\begin{aligned} D_q\left(\frac{f(z)}{g(z)}\right) &= \frac{g(qz) D_q f(z) - f(qz) D_q g(z)}{g(z) g(qz)}, \\ D_q(f(z) g(z)) &= f(qz) D_q g(z) + g(z) D_q f(z). \end{aligned} \tag{1.1}$$

Carlitz was the first to extend the classical Bernoulli numbers and polynomials, Euler numbers and polynomials ([2], [3]). Cheon in [5] gave explicit expansions for the classical Bernoulli polynomials and the classical Euler polynomials. Srivastava et al [16] proved some formulae for the Bernoulli polynomials and the Euler polynomials. Also, they gave the addition-formulae between the Bernoulli polynomials and the Euler polynomials. There are numerous recent investigations on the $q$-Bernoulli polynomials and $q$-Euler polynomials by many mathematicians, including as Cenkci et al [4], Choi et al [6], Kim ([8], [9]), Kim et al [10], Luo [11], Luo and Srivastava [12], Srivastava et al ([16], [17]), Tremblay et al [18] and Mahmudov ([13], [14]).

Mahmudov defined and studied properties of the following generalized $q$-Bernoulli polynomials $\mathfrak{B}_{n,q}^{(\alpha)}(x,y)$ of order $\alpha$ and $q$-Euler polynomials $\mathfrak{E}_{n,q}^{(\alpha)}(x,y)$ of order $\alpha$ as follows ([13], [14]).

Let $q \in \mathbb{C}, \alpha \in \mathbb{N}$ and $0 < |q| < 1$. The $q$-Bernoulli numbers $\mathfrak{B}_{n,q}^{(\alpha)}$ and polynomials $\mathfrak{B}_{n,q}^{(\alpha)}(x,y)$ in $x, y$ of order $\alpha$ are defined by means of the generating functions:

$$\sum_{n=0}^{\infty} \mathfrak{B}_{n,q}^{(\alpha)} \frac{t^n}{[n]_q!} = \left( \frac{t}{e_q(t) - 1} \right)^{\alpha}, \qquad |t| < 2\pi \tag{1.2}$$

and

$$\sum_{n=0}^{\infty} \mathfrak{B}_{n,q}^{(\alpha)}(x,y) \frac{t^n}{[n]_q!} = \left( \frac{t}{e_q(t) - 1} \right)^{\alpha} e_q(xt) E_q(yt), \qquad |t| < 2\pi. \tag{1.3}$$

The $q$-Euler numbers $\mathfrak{E}_{n,q}^{(\alpha)}$ and polynomials $\mathfrak{E}_{n,q}^{(\alpha)}(x,y)$ in $x, y$ of order $\alpha$ are defined by means of the generating functions:

$$\sum_{n=0}^{\infty} \mathfrak{E}_{n,q}^{(\alpha)} \frac{t^n}{[n]_q!} = \left( \frac{2}{e_q(t) + 1} \right)^{\alpha}, \qquad |t| < \pi \tag{1.4}$$

and

$$\sum_{n=0}^{\infty} \mathfrak{E}_{n,q}^{(\alpha)}(x,y) \frac{t^n}{[n]_q!} = \left( \frac{2}{e_q(t) + 1} \right)^{\alpha} e_q(xt) E_q(yt), \qquad |t| < \pi. \tag{1.5}$$

The $q$-Genocchi numbers $\mathfrak{G}_{n,q}^{(\alpha)}$ and polynomials $\mathfrak{G}_{n,q}^{(\alpha)}(x,y)$ in $x, y$ of order $\alpha$ are defined by means of the generating functions:

$$\sum_{n=0}^{\infty} \mathfrak{G}_{n,q}^{(\alpha)} \frac{t^n}{[n]_q!} = \left( \frac{2t}{e_q(t) + 1} \right)^{\alpha}, \qquad |t| < \pi$$

and

$$\sum_{n=0}^{\infty} \mathfrak{G}_{n,q}^{(\alpha)}(x,y) \frac{t^n}{[n]_q!} = \left( \frac{2t}{e_q(t) + 1} \right)^{\alpha} e_q(xt) E_q(yt), \qquad |t| < \pi.$$

It is obvious that

$$\begin{aligned} \lim_{q \to 1^-} \mathfrak{B}_{n,q}^{(\alpha)}(x,y) &= B_n^{(\alpha)}(x+y), \\ \lim_{q \to 1^-} \mathfrak{E}_{n,q}^{(\alpha)}(x,y) &= E_n^{(\alpha)}(x+y), \\ \lim_{q \to 1^-} \mathfrak{G}_{n,q}^{(\alpha)}(x,y) &= G_n^{(\alpha)}(x+y) \end{aligned}$$

and

$$\begin{aligned} D_{q,x}^{(\alpha)} \mathfrak{B}_{n,q}(x,y) &= [n]_q \mathfrak{B}_{n-1,q}^{(\alpha)}(x,y), \quad D_{q,y} \mathfrak{B}_{n,q}^{(\alpha)}(x,y) = [n]_q \mathfrak{B}_{n-1,q}^{(\alpha)}(x,qy), \\ D_{q,t} e_q(xt) &= x e_q(xt), \quad D_{q,t} E_q(yt) = y E_q(qyt). \end{aligned}$$

## 2. Main Theorems

In this section, we give some relations for $q$-Bernoulli polynomials $\mathfrak{B}_{n,q}^{(\alpha)}(x,y)$ and $q$-Euler polynomials $\mathfrak{E}_{n,q}^{(\alpha)}(x,y)$. By applying the derivative operator to $q$-Bernoulli polynomials and $q$-Euler polynomials, we have recurrences relations for these polynomials.

**2.1. Proposition.** *The generalized $q$-Bernoulli polynomials satisfy the following relation.*

$$\sum_{l=0}^{n}\begin{bmatrix} n \\ l \end{bmatrix}_q \mathfrak{B}_{n-l,q}^{(\alpha)}(x,y)-\mathfrak{B}_{n,q}^{(\alpha)}(x,y)=[n]_q\,\mathfrak{B}_{n-1,q}^{(\alpha-1)}(x,y). \tag{2.1}$$

*Proof.* From (1.3), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{(\alpha)}(x,y)\frac{t^n}{[n]_q!} &= \left(\frac{t}{e_q(t)-1}\right)^{\alpha} e_q(xt)\,E_q(yt)\\ \sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{(\alpha)}(x,y)\frac{t^n}{[n]_q!}(e_q(t)-1) &= t\left(\frac{t}{e_q(t)-1}\right)^{\alpha-1} e_q(xt)\,E_q(yt)\\ \sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{(\alpha)}(x,y)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}\frac{t^n}{[n]_q!}-\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{(\alpha)}(x,y)\frac{t^n}{[n]_q!} &= t\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{(\alpha)}(x,y)\frac{t^n}{[n]_q!}.\end{aligned}$$

By using Cauchy product and comparing the coefficient of $\frac{t^n}{[n]_q!}$ we have (2.1). ∎

The following equations can be obtained easily from (1.2)-(1.5).

$$\mathfrak{B}_{n,q}^{(\alpha-\beta)}(x,y) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{B}_{k,q}^{(\alpha-\beta)}(0,0)\,(x+y)_q^{n-k}, \tag{2.2}$$

$$\mathfrak{B}_{n,q}^{(\alpha-\beta)}(x,y) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{B}_{k,q}^{(\alpha)}(x,0)\,\mathfrak{B}_{n-k,q}^{(-\beta)}(0,y), \tag{2.3}$$

$$(x+y)_q^n = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{E}_{n-k,q}^{(\alpha)}(x,y)\,\mathfrak{E}_{k,q}^{(-\alpha)}(0,0), \tag{2.4}$$

$$2\mathfrak{E}_{n,q}^{(\alpha-1)}(x,y) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{E}_{n-k,q}^{(\alpha)}(x,y)+\mathfrak{E}_{n,q}^{(\alpha)}(x,y), \tag{2.5}$$

where $\alpha,\beta\in\mathbb{N}$.

**2.2. Theorem.** *The generalized $q$-Bernoulli polynomials satisfy the following recurrence relation.*

$$\begin{aligned}\mathfrak{B}_{n+1,q}(x,y) &= \mathfrak{B}_{n,q}(x,y)+[n+1]_q\left\{qy\mathfrak{B}_{n,q}(qx,y)+qx\mathfrak{B}_{n,q}(x,y)\right\}\\ &\quad-\sum_{k=0}^{n+1}\begin{bmatrix} n+1 \\ k \end{bmatrix}_q \mathfrak{B}_{k,q}(x,y)\,q^k\mathfrak{B}_{n+1-k,q}(1,0).\end{aligned} \tag{2.6}$$

*Proof.* In (1.3), for $\alpha=1$, we take the $q$-Jackson derivative of the generalized $q$-Bernoulli polynomials $\mathfrak{B}_{n,q}(x,y)$ according to $t$, then we have

$$\sum_{n=0}^{\infty}D_{q,t}\mathfrak{B}_{n,q}(x,y)\frac{t^n}{[n]_q!}=D_{q,t}\left(\frac{te_q(xt)\,E_q(yt)}{e_q(t)-1}\right).$$

By using the equalities (1.1) in the last expression we have

$$\sum_{n=0}^{\infty} D_{q,t}\mathfrak{B}_{n,q}(x,y)\frac{t^n}{[n]_q!} = \frac{(e_q(qxt)-1)\,D_{q,t}\left[te_q(xt)\,E_q(yt)\right] - qte_q(qxt)\,E_q(qyt)\,D_{q,t}\left[e_q(t)-1\right]}{(e_q(t)-1)(e_q(qt)-1)},$$

$$\sum_{n=0}^{\infty}\frac{1}{[n+1]_q}\mathfrak{B}_{n+1,q}(x,y)\frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty} qy\mathfrak{B}_{n,q}(qx,qy) + qx\mathfrak{B}_{n,q}(x,y)\frac{t^n}{[n]_q!}$$
$$+\frac{1}{[n+1]_q}\left\{\mathfrak{B}_{n,q}(x,y) + \sum_{k=0}^{n+1}\left[\begin{array}{c} n+1 \\ k \end{array}\right]_q \mathfrak{B}_{k,q}(x,y)\,q^k\mathfrak{B}_{n+1-k,q}(1,0)\frac{t^n}{[n]_q!}\right\}.$$

Comparing the coefficient of $\frac{t^n}{[n]_q!}$ we obtain (2.6). ∎

**2.3. Theorem.** *The generalized q-Euler polynomials $\mathfrak{E}_{n,q}(x,y)$ satisfy the following relation.*

$$\mathfrak{E}_{n+1,q}(x,y) = [n+1]_q \times\left\{y\mathfrak{E}_{n,q}(qx,qy) + x\mathfrak{E}_{n,q}(x,y) - \frac{1}{4}\sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_q \mathfrak{E}_{k,q}(x,y)\,q^k\mathfrak{E}_{n-k,q}(1,0)\right\}$$

*Proof.* In (1.5), for $\alpha = 1$, by using the equalities (1.1), the proof can be obtained. ∎

**2.4. Theorem.** *There is the following relation.*

(2.7)
$$\mathfrak{B}_{n,q}^{(\alpha)}(x,y) = \frac{m^{-n}}{[n+1]_q}\sum_{k=0}^{n+1}\left[\begin{array}{c} n+1 \\ k \end{array}\right]_q\left\{\left[\mathfrak{B}_{k,q}^{(\alpha)}\left(\frac{1}{m},0\right) - \mathfrak{B}_{k,q}^{(\alpha)}(0,0)\right]\mathfrak{B}_{n+1-k,q}(x,y)\,m^k\right\}.$$

*Proof.* From (1.2), we have

$$\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{(\alpha)}\frac{t^n}{[n]_q!} = \left(\frac{t}{e_q(t)-1}\right)^{\alpha}\frac{e_q\left(\frac{t}{m}\right)-1}{\frac{t}{m}}\frac{\frac{t}{m}}{e_q\left(\frac{t}{m}\right)-1}$$
$$= \frac{m}{t}\left\{\left(\frac{t}{e_q(t)-1}\right)^{\alpha} e_q\left(\frac{t}{m}\right)\frac{\frac{t}{m}}{e_q\left(\frac{t}{m}\right)-1} - \left(\frac{t}{e_q(t)-1}\right)^{\alpha}\frac{\frac{t}{m}}{e_q\left(\frac{t}{m}\right)-1}\right\}$$
$$= \frac{m}{t}\sum_{n=0}^{\infty}\left[\mathfrak{B}_{n,q}^{(\alpha)}\left(\frac{1}{m},0\right) - \mathfrak{B}_{n,q}^{(\alpha)}(0,0)\right]\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{(\alpha)}(0,0)\frac{t^n}{m^n[n]_q!}.$$

Using the Cauchy product and comparing the coefficient of $\frac{t^n}{[n]_q!}$ we obtain (2.7). ∎

**2.5. Theorem.** *The generalized q-Euler numbers $\mathfrak{E}_{n,q}^{(\alpha)}(0,0)$ satisfy the following relation.*

$$\mathfrak{E}_{n,q}^{(\alpha)} = \frac{1}{2[n+1]_q}\sum_{k=0}^{n+1}\left[\begin{array}{c} n+1 \\ k \end{array}\right]_q\left\{\left[\mathfrak{E}_{k,q}^{(\alpha)}\left(\frac{1}{m},0\right) + \mathfrak{E}_{k,q}^{(\alpha)}(0,0)\right]\mathfrak{G}_{n+1-k,q}(0,0)\,m^{k-n}\right\}.$$

## 3. Some Relations Between the q-Bernoulli Polynomials and q-Euler Polynomials

In this section, we prove an interesting relationship between the $q$-Bernoulli polynomials $\mathfrak{B}_{n,q}^{(\alpha)}(x,y)$ of order $\alpha$ and $q$-Euler polynomials $\mathfrak{E}_{n,q}^{(\alpha)}(x,y)$ of order $\alpha$.

**3.1. Theorem.** *There is the following relation between the q-Euler polynomials and q-Bernoulli polynomials.*

(3.1)

$$\mathfrak{B}_{n,q}^{(\alpha)}(x,y)=\frac{1}{2}\sum_{k=0}^{n}\left[\begin{array}{c}n\\k\end{array}\right]_q\left\{\sum_{r=0}^{p}\left[\begin{array}{c}p\\r\end{array}\right]_q\mathfrak{B}_{r,q}^{(\alpha)}(x,0)\,m^{r-n}+\mathfrak{B}_{n-k,q}^{(\alpha)}(x,0)\,m^{-k}\right\}\mathfrak{E}_{k,q}(0,my).$$

*Proof.* From (1.3), we have

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{(\alpha)}(x,y)\frac{t^n}{[n]_q!}&=\frac{2}{e_q(t)+1}E_q\left(my\frac{t}{m}\right)\frac{e_q\left(\frac{t}{m}\right)+1}{2}\left(\frac{t}{e_q(t)-1}\right)^{\alpha}e_q(xt)\\
&=\frac{1}{2}\frac{2}{e_q\left(\frac{t}{m}\right)+1}E_q\left(my\frac{t}{m}\right)e_q\left(\frac{t}{m}\right)\left(\frac{t}{e_q(t)-1}\right)^{\alpha}e_q(xt)\\
&\quad+\frac{1}{2}\frac{2}{e_q\left(\frac{t}{m}\right)+1}E_q\left(my\frac{t}{m}\right)\left(\frac{t}{e_q\left(\frac{t}{m}\right)-1}\right)^{\alpha}e_q(xt)\\
&=\frac{1}{2}\sum_{n=0}^{\infty}\mathfrak{E}_{n,q}(0,my)\frac{t^n}{m^n[n]_q!}\sum_{n=0}^{\infty}\frac{t^n}{m^n[n]_q!}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{(\alpha)}(x,0)\frac{t^n}{[n]_q!}\\
&\quad+\frac{1}{2}\sum_{n=0}^{\infty}\mathfrak{E}_{n,q}(0,my)\frac{t^n}{m^n[n]_q!}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{(\alpha)}(x,0)\frac{t^n}{[n]_q!}\\
&=\frac{1}{2}\left[\sum_{n=0}^{\infty}\mathfrak{E}_{n,q}(0,my)\frac{t^n}{m^n[n]_q!}\right]\\
&\quad\times\left[\sum_{p=0}^{\infty}\sum_{r=0}^{p}\left[\begin{array}{c}p\\r\end{array}\right]_q\mathfrak{B}_{r,q}^{(\alpha)}(x,0)\,m^{r-p}\frac{t^p}{[p]_q!}+\sum_{p=0}^{\infty}\mathfrak{B}_{p,q}^{(\alpha)}(x,0)\frac{t^p}{[p]_q!}\right].
\end{aligned}$$

Comparing the coefficient of $\frac{t^n}{[n]_q!}$ we obtain

$$\mathfrak{B}_{n,q}^{(\alpha)}(x,y)=\frac{1}{2}\sum_{k=0}^{n}\left[\begin{array}{c}n\\k\end{array}\right]_q\left\{\sum_{r=0}^{p}\left[\begin{array}{c}p\\r\end{array}\right]_q mathfrakB_{r,q}^{(\alpha)}(x,0)\,m^{r-n}+\mathfrak{B}_{n-k,q}^{(\alpha)}(x,0)\,m^{-k}\right\}\mathfrak{E}_{k,q}(0,my).$$

∎

**3.2. Theorem.** *There is the following relation between the q-Bernoulli polynomials and q-Euler polynomials.*

(3.2)

$$\begin{aligned}
\mathfrak{E}_{n,q}^{(\alpha)}(x,y)&=\frac{m}{[n+1]_q}\sum_{k=0}^{n+1}\left[\begin{array}{c}n+1\\k\end{array}\right]_q\\
&\quad\times\left\{\sum_{r=0}^{n+1-k}\left[\begin{array}{c}n+1-k\\r\end{array}\right]_q\mathfrak{E}_{r,q}^{(\alpha)}(x,0)\,m^{r-n-1}-\mathfrak{E}_{n+1-k,q}^{(\alpha)}(x,0)\,m^{-k}\right\}\mathfrak{B}_{k,q}(0,my).
\end{aligned}$$

*Proof.* From (1.5), we write

$$\sum_{n=0}^{\infty}\mathfrak{E}_{n,q}^{(\alpha)}(x,y)\frac{t^n}{[n]_q!}=\frac{\frac{t}{m}}{e_q\left(\frac{t}{m}\right)-1}E_q\left(my\frac{t}{m}\right)\frac{e_q\left(\frac{t}{m}\right)-1}{\frac{t}{m}}\left(\frac{2}{e_q(t)+1}\right)^{\alpha}e_q(xt)$$

$$=\frac{m}{t}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}(0,my)\frac{t^n}{m^n[n]_q!}\sum_{n=0}^{\infty}\frac{t^n}{m^n[n]_q!}\sum_{n=0}^{\infty}\mathfrak{E}_{n,q}^{(\alpha)}(x,0)\frac{t^n}{[n]_q!}$$

$$-\frac{m}{t}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}(0,my)\frac{t^n}{m^n[n]_q!}\sum_{n=0}^{\infty}\mathfrak{E}_{n,q}^{(\alpha)}(x,0)\frac{t^n}{[n]_q!}$$

$$=\frac{m}{t}\sum_{k=0}^{\infty}\mathfrak{B}_{k,q}(0,my)\frac{t^k}{m^k[k]_q!}\left\{\sum_{p=0}^{\infty}\sum_{r=0}^{p}\left[\begin{array}{c}p\\r\end{array}\right]_q\mathfrak{E}_{r,q}^{(\alpha)}(x,0)m^{r-p}-\mathfrak{E}_{r,q}^{(\alpha)}(x,0)\right\}\frac{t^p}{[p]_q!}.$$

Using the Cauchy product and comparing the the coefficient of $\frac{t^n}{[n]_q!}$ we obtain (3.2). ∎

**3.3. Corollary.** *The following relations holds*

(3.3)

$$\mathfrak{B}_{n,q}^{(\alpha)}=\frac{1}{[n+1]_q}\sum_{k=0}^{n+1}\left[\begin{array}{c}n+1\\k\end{array}\right]_q m^{k-n}\left\{\mathfrak{B}_{k,q}^{(\alpha)}\left(\frac{1}{m},0\right)+\mathfrak{B}_{k,q}^{(\alpha)}(0,0)\right\}\mathfrak{E}_{n+1-k,q}(0,0)$$

*and*

(3.4)

$$\mathfrak{E}_{n,q}^{(\alpha)}=\frac{1}{[n+1]_q}\sum_{k=0}^{n+1}\left[\begin{array}{c}n+1\\k\end{array}\right]_q m^{k-n}\left\{\mathfrak{E}_{k,q}^{(\alpha)}\left(\frac{1}{m},0\right)-\mathfrak{E}_{k,q}^{(\alpha)}(0,0)\right\}\mathfrak{B}_{n+1-k,q}^{(\alpha)}(0,0).$$

**3.4. Corollary.** *From (3.3) and (3.4), we have*

$$\left\{\mathfrak{B}_{k,q}^{(\alpha)}\left(\frac{1}{m},0\right)+\mathfrak{B}_{k,q}^{(\alpha)}(0,0)\right\}\mathfrak{E}_{n+1-k,q}(0,0)\mathfrak{E}_{n,q}^{(\alpha)}(0,0)$$
$$=\left\{\mathfrak{E}_{k,q}^{(\alpha)}\left(\frac{1}{m},0\right)-\mathfrak{E}_{k,q}^{(\alpha)}(0,0)\right\}\mathfrak{B}_{n+1-k,q}^{(\alpha)}(0,0)\mathfrak{B}_{n,q}^{(\alpha)}(0,0).$$

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