Hacettepe Journal of Mathematics and Statistics
Volume 37 (1) (2008), 9 – 24

ON SEMI-CONTINUOUS MAPPINGS, EQUATIONS AND INCLUSIONS IN A BANACH SPACE

Kamal N. Soltanov*

Received 17 : 03 : 2008 : Accepted 05 : 05 : 2008

Abstract

We investigate the description of the image of a continuous mappings acting in a Banach space, and the solvability of equations and inclusions. The results obtained can be applied to the Cauchy problem for a nonlinear differential equation (or inclusion). In particular, a solvability theorem of the mixed problem for a nonlinear hyperbolic equation is proved, and one Nirenberg problem is studied.

Keywords: Banach space, Strictly convex norm, Multivalued mapping, Solvability and fixed-point theorems, Differential equations and inclusions

2000 AMS Classification: Primary 47H15, 46E35 Secondary 35G15

1. Introduction

In this article we study some questions concerning continuous (in general, multivalued) mappings acting in a reflexive Banach space, which together with its dual space, has a strictly convex norm (see, [6, 7] and their references). Here are considered with problems of the following types: when does a semi-continuous mapping possesses a fixed-point, what can be said about the image of subsets of the domain under a semi-continuous mapping, and when can the solvability of a nonlinear equation (or inclusion) given by such mappings be established?

It should be noted that questions of this type have been investigated under various conditions in many earlier works (see, [1 - 5, 9, 10, 13 - 14, 18 - 23] etc.). Here they are studied in a different way which is a generalization of [20] (see, also [21]), and are such that the results obtained can be used for the study of differential equations and inclusions in a Banach space, unlike the results mentioned above. Moreover, we prove here some fixed-point theorems, and also some general solvability theorems for nonlinear differential equations in a Banach space.

*Department of Mathematics, Faculty of Sciences, Hacettepe University, 06800 Beytepe, Ankara, Turkey. E-Mail : soltanov@hacettepe.edu.tr , sultan_kamal@hotmail.com

Let X, Y be reflexive Banach spaces and X^*, Y^* their respective duals. Also, let $f : D(f) \subseteq X \to Y$ be a multivalued mapping, i.e. f assigns to each point $x \in D(f)$ a subset $f(x)$ of Y, and G an open subset of $D(f)$. First we will discuss the geometrical meaning of the condition: for any $y^* \in S_1^{Y^*}(0) \equiv \left\{ y^* \in Y^* \bigm| \|y^*\|_{Y^*} = 1 \right\}$ there is $x \in G$, $x = x(y^*)$, such that $\langle f(x), y^* \rangle \cap \{\tau \,|\tau > 0\} \neq \emptyset$ (here $\langle \cdot, \cdot \rangle$ is a dual form with respect to the pair (Y, Y^*)). We let $Y \equiv X$ (since this case is easy to see), then we study the role of a local relation between the geometrical disposition of each point $x \in G \subseteq D(f)$ with its image $f(x) \subset f(G) \subseteq X$ through an investigation the image $f(G)$ of the mapping f. Consequently this allows us to study of the whole of the image. Clearly, for a $y \in Y$ the proof of the inclusion $y \in f(G)$ is equivalent to the proof of the statement: $\exists x \in G$ such that the inclusion $y \in f(x)$ holds. Therefore, what we will actually conduct here is the study of the solvability of an inclusion.

Furthermore, the following fixed-point theorem is proved here in a Banach space X, when X together with its dual space has a strictly convex norm .

1.1. Theorem. *Let $f : B_r^X(x_0) \to B_r^X(x_0)$ be a single-valued continuous mapping, i.e. $f \in C^0$, $B_r^X(x_0)$ a closed ball and the space X as above. Assume that for some function $\mu \in C^0$, $\mu : R_+^1 \longrightarrow R_+^1$ the inequality $\|f(x) - x_0\|_X \leq \mu\left(\|x - x_0\|_X\right)$ holds for any $x \in B_r^X(x_0)$, moreover the mapping $f_1 : f_1(x) \equiv x - f(x)$, $\forall x \in B_r^X(x_0)$ satisfies the inequality*

$$\langle f_1(x), \Im(x - x_0) \rangle \geq \nu\left(\|x - x_0\|_X\right) \|x - x_0\|_X \tag{1.1}$$

for almost any $x \in B_r^X(x_0)$, where $\nu(\tau) \equiv \tau - \mu(\tau)$, $\nu(r) \geq \delta > 0$ and $\Im$ is the duality mapping for (X, X^*). Then if $f\left(B_r^X(x_0)\right)$ is closed, f possesses a fixed point in $B_r^X(x_0)$.

As an application of the results obtained here we study the solvability of nonlinear differential-operator inclusions (in particular, equations of evolution) and of Nirenberg's problem concerning expanding mappings. Moreover we investigate the initial-boundary value problem for the following equation

$$\frac{\partial^2 u}{\partial t^2} - \sum_{i=1}^{d} D_i\left(|D_i u|^{p-2} D_i u\right) = h(t, x), \quad (t, x) \in Q,$$

and prove a theorem on dense solvability, where $Q \equiv (0, T) \times \Omega$, $T > 0$, $\Omega \subset R^d, d \geq 2$ is a domain with sufficiently smooth boundary $\partial\Omega$ and $D_i \equiv \frac{\partial}{\partial x_i}$, $i = \overline{1, d}$. It should be noted that, as was shown in [14, 18, 22] etc., the investigation of nonlinear hyperbolic problems under $d \geq 2$ gives rise to many difficulties.

2. On lower semi-continuous mappings on a Banach space

Let X , Y be reflexive Banach spaces and $f : D(f) \subseteq X \to Y$ be, in general, a lower semi-continuous multivalued mapping. Henceforth, we will denote a closed ball in the respective space X by $B_r^X(x_0)$, and its boundary by $S_r^X(x_0)$.

A property $P(x)$, where x runs over the points of a Banach space X, is said to be *fulfilled almost everywhere* if the set of points where it holds is an everywhere dense subset of X.

2.1. Theorem. *Let X be a reflexive Banach space which, together with its dual, has a strictly convex norm, and $f : D(f) \subseteq X \to X^*$. We assume that, on a closed ball $B_{r_0}^X(0) \subseteq D(f)$, $r_0 > 0$, the following conditions hold:*

i) *f is a single-valued continuous mapping (i.e. $f \in C^0$);*

ii) *There are continuous functions $\mu : R^1_+ \longrightarrow R^1_+$ and $\nu : R^1_+ \longrightarrow R^1$ nondecreasing for $\tau \geq \tau_0$, such that*

$$\|f(x) - f(0)\|_{X^*} \leq \mu\left(\|x\|_X\right), \tag{2.1}$$

holds for any $x \in B^X_{r_0}(0)$ and

$$\langle f(x) - f(0), x\rangle \geq \nu\left(\|x\|_X\right)\|x\|_X, \tag{2.2}$$

holds for almost all $x \in B^X_{r_0}(0)$, moreover $\nu(r_0) \geq \delta_0 > 0$.

Then $f\left(B^X_{r_0}(0)\right)$ contains an everywhere dense subset of

$$M^*_0 \equiv \left\{x^* \in X^* \mid 0 \leq \langle f(x) - x^*, x\rangle, \ \forall x \in S^X_{r_0}(0)\right\}. \tag{2.3}$$

2.2. Remark. In particular, a closed ball $B^{X^*}_\delta(f(0))$, $\delta = \delta_0 - \varepsilon > 0$, is contained in the set M^*_0, where $\varepsilon = \varepsilon(\delta_0) > 0$.

2.3. Remark. It is well known [6, 7, 15] that in a reflexive Banach space we can define a norm which is equivalent to the usual norm of the space under consideration, and both the space and its dual - with the respective norms - will be strictly convex. Henceforth, without loss of generality, we will assume that all reflexive Banach spaces considered are endowed with such a norm.

Now we shall give the following general case of Theorem 2.1 for multivalued mapping, which is proved using Theorem 2.1.

2.4. Theorem. *Let $f : D(f) \subseteq X \to Y$ be a multivalued mapping and Y a reflexive Banach space. We assume that on the closed ball $B^X_{r_0}(x_0) \subseteq D(f)$, $r_0 > 0$, of centre x_0, the following conditions are fulfilled:*

α) *f is a lower semi-continuous mapping (i.e. $f \in C^0_{lsc}$) such that $f(x)$ is a convex closed set for any $x \in B^X_{r_0}(x_0)$ (if f is single-valued then $f \in C^0$);*

β) *There are functions $\mu, \nu \in C^0$ satisfying the condition ii) of Theorem 2.1 such that*

β_1) *For the Hausdorff distance $h^Y(f(x); f(x_0))$*†

$$h^Y(f(x); f(x_0)) \leq \mu\left(\|x - x_0\|_X\right) \tag{2.4}$$

holds for any $x \in B^X_{r_0}(x_0)$;

β_2) *There are $y_0 \in f(x_0)$ and a mapping $g : D(g) \subseteq X \to Y^*$, such that $g^{-1} \in C^0$, $g(x_0) = 0$, $\mathrm{cl}\, U_0 \equiv B^X_{r_0}(x_0)$, $\mathrm{cl}\,(g(U_0)) \equiv B^{Y^*}_1(0)$, $\mathrm{cl}\,(g(U_1)) \supseteq S^{Y^*}_1(0)$, and for any $y^* \in g(U_0) \subseteq B^{Y^*}_1(0)$, $x = g^{-1}(y^*) \in U_0$ $\langle y - y_0, y^*\rangle \geq \nu\left(\|x - x_0\|_X\right)$ holds for any $y \in f(x)$, where $U_0 \equiv B^X_{r_0}(x_0) \cap D(g)$, $U_1 \equiv S^X_{r_0}(x_0) \cap D(g) \subseteq g^{-1}\left(S^{Y^*}_1(0)\right)$. Moreover $\nu(r_0) \geq \delta_0 > 0$.*

Then $f(U(x_0))$ has a subset which is an everywhere dense in some connected subset $M_0 \subset Y$ with nonempty interior (i.e. a bodily set M_0). Furthermore, this subset has the form:

$$M_0 \equiv \left\{y \in Y \mid \langle y, y^*\rangle \leq \langle y_x, y^*\rangle, \ \forall y^* \in S^{Y^*}_1(0), \ \exists\, x \in S^X_{r_0}(x_0), \forall\, y_x \in f(x)\right\}. \tag{2.5}$$

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†We recall that the Hausdorff distance between the subsets G_1, G_2 of Y is

$$h^Y(G_1; G_2) \equiv \max\{\sup\{d^Y(G_1; y_2) \mid y_2 \in G_2\}, \ \sup\{d^Y(y_1; G_2) \mid y_1 \in G_1\}\}$$

where

$$d^Y(G_1; y_2) \equiv \inf\{d^Y(y_1; y_2) \mid y_1 \in G_1\} \equiv \inf\{\|y_1 - y_2\|_Y \mid y_1 \in G_1\}.$$

2.5. Note. A remark similar to Remark 2.2 holds in this case also, furthermore we will show later that the set M_0 (and M_0^*) is a convex body in Y (in X^*, respectively). It should be noted that Theorems 2.1 and 2.4 are solvability theorems, i.e. they imply that the equation $f(x) = y$, or inclusion $f(x) \ni y$, is solvable for any element y of an everywhere dense subset of the set M_0 (or M_0^*, respectively).

It is easy to see that the condition β_2) is more than enough for the conclusion of this theorem.

The proofs of these theorems are based on certain results which we now give. These results demonstrate that the conditions of Theorems 2.1 and 2.4 have a geometrical nature.

2.6. Lemma. *Let $f : D(f) \subseteq X \to Y$ be a continuous mapping (maybe lower or upper semi-continuous), G some subset of $D(f)$ and $\delta_0 > 0$ a number. If for almost every $y^* \in S_1^{Y^*}(0)$ there is $x = x(y^*) \in G$ such that $\langle y, y^* \rangle \geq \delta_0 > 0$ holds for some $y \in f(x)$, then the affine space $Y_{f(G)}$ generated by $f(G)$ contains at least an everywhere dense subset of Y. (In other words* clconv $f(G)$ *contains a convex body of Y.)*

2.7. Remark. If X and Y are a finite-dimensional spaces then $Y_{f(G)} \equiv Y$.

For the proof, it is enough to note that it follows from a similar result in [19][‡] (see, also [20, 21]). In fact, the fulfillment of Lemma 2.6 follows directly by the continuity of f from the following result.

2.8. Lemma. *Let $f : D(f) \subseteq X \to Y$ be a mapping (single or multi-valued, continuous or discontinuous), and $G \subseteq D(f)$ a subset. If for any $y^* \in S_1^{Y^*}(0)$ there is $x = x(y^*) \in G$ such that $\langle y, y^* \rangle > 0$ holds for a $y \in f(x)$, then the affine space $Y_{f(G)}$ generated by $f(G)$, so contains at least an everywhere dense subset of Y. (In other words,* cl conv $f(G)$ *contains a convex body of Y.)*

It should be noted that condition $\langle y, y^* \rangle \geq \delta_0 > 0$ of Lemma 2.6 is essential. In fact, even if f is a single-valued continuous mapping the conclusion of Lemma 2.6 may not be valid if this condition not is fulfilled. For example, let $X \equiv Y \equiv R^2$ be Euclidean spaces, $G \equiv B_1^{R^2}(0)$, and f be such that $f(x) \equiv f(x_1, x_2) \equiv (x_1, 0)$ for any $x \equiv (x_1, x_2) \in B_1^{R^2}(0)$. Then it is not difficult to see that the following expression holds:

$$\langle f(x), x \rangle \begin{cases} > 0 & \forall x \equiv (x_1, x_2),\ x_1 \neq 0, \\ = 0, & \forall x \equiv (0, x_2) \end{cases}$$

for all $x \in B_1^{R^2}(0)$. Thus, f is continuous and $\langle f(x), x \rangle > 0$ holds almost everywhere on the ball, but the conclusion of Lemma 2.6 is not valid.

2.9. Note. In the following results we can assume without loss of generality that $B_{r_0}^X(0) \subseteq D(f)$ and $f(0) = 0$ since X is a Banach space and we can choose $\widetilde{f}(x) \equiv f(x) - f(0)$ for any $x \in B_{r_0}^X(0)$.

2.10. Proposition. *Let $f : D(f) \subseteq X \to X^*$, $f(0) = 0$ and suppose that the conditions of Theorem 2.1 hold on the sphere $S_r^X(0) \subseteq D(f)$, $\nu(r) > \delta > 0$. Then $f\left(S_r^X(0)\right)$ belongs to a ring generated by a ball in the form: $B_{R_1}^{X^*}(0) \backslash B_{R_2}^{X^*}(0) \subset X^*$, i.e. $f\left(S_r^X(0)\right) \subseteq B_{R_1}^{X^*}(0) \backslash B_{R_2}^{X^*}(0)$, where $R_1 \equiv \mu(r)$, $R_2 \equiv \nu(r) - \varepsilon$, and $\varepsilon \equiv \varepsilon(\delta, r) > 0$ is some number.*

[‡]see, also Soltanov, K. N., *Fixed Point Theory and Applications*, 2007, v. Q2, ID 80987

Indeed from (2.1) it follows that $\|f(x)\|_{X^*} \leq R_1, \ \forall x \in S_r^X(0)$ is fulfilled, and from the continuity of the mapping f and (2.2) it follows that there is $\varepsilon \equiv \varepsilon(\delta, r) > 0$ such that $\langle f(x), x\rangle \geq (\nu(r) - \varepsilon)r$ holds for any $x \in S_r^X(0)$.

2.11. Corollary. *Let the conditions of Proposition 2.10 hold. Then $f(x)$ belongs to a set $K_x^* \equiv B_{R_1}^{X^*}(0) \cap \left(X^*_{L^*_x}\right)^+$ for any $x \in S_r^X(0)$, where $\left(X^*_{L^*_x}\right)^+$ is a non-negative half-space of X^* defined by a hyperplane $L_x^* \equiv \{x^* \in X^* \mid \langle x^*, x\rangle = (\nu(r) - \varepsilon)r, \ \varepsilon > 0\}$, [11]. Moreover, $f(x)$ belongs to a conic section $K_{R_1}^* \equiv B_{R_1}^{X^*}(0) \cap K_0^*(x)$, here $K_0^*(x)$ is a cone of X^* with generatrix $S_{R_1}^{X^*}(0) \cap L_x^*$, and vertex in the zero of X^*.*

Indeed, from the Proposition it follows that $f(x) \in B_{R_1}^{X^*}(0)$ and $f(x) \in \left(X^*_{L^*_x}\right)^+$. The remainder of the proof is obvious.

2.12. Proposition. *Let $f : D(f) \subseteq X \to X^*$, $f(0) = 0$, and let the conditions of Theorem 2.1 hold on $B_{r_0}^X(0) \subseteq D(f)$. Then for any $\varepsilon > 0$ and $R : 0 < R < R_0 \leq \mu(r_0)$ there exists a sphere $S_r^X(0) \subset B_{r_0}^X(0)$ and a neighbourhood $U_\delta(x_1) \cap S_r^X(0)$ for a $x_1 \in S_r^X(0)$ such that $f\left(U_\delta(x_1) \cap S_r^X(0)\right)$ is contained in a ring generated by the ball: $B_{R+\varepsilon}^{X^*}(0) \setminus B_{R-\varepsilon}^{X^*}(0)$. (It is easy to see that if there is $x_0 \in S_r^X(0)$ such that $\|f(x_0)\|_{X^*} = \mu(r_0)$ then we can choose $R_0 = \mu(r_0)$).*

Proof. Since $\left\{\|f(x)\|_{X^*} : x \in B_{r_0}^X(0)\right\}$ is a bounded set and X a reflexive space we may define $R_0 = \sup\{\|f(x)\|_{X^*} : x \in B_{r_0}^X(0)\}$. Then for any $R : 0 < R < R_0 \leq \mu(r_0)$, there exists an element $x_1 \in B_{r_0}^X(0)$ such that $\|f(x_1)\|_{X^*} = R$ and $x_1 \in S_r^X(0)$ for a number $r : 0 < r \leq r_0$. The proof may now be completed easily by applying the conditions of the Proposition. □

Now, we will give a generalization of [20, Lemma 2] (see, also [21]), which corresponds to the case considered here. For the proof of this result we will use concepts similar to those in [20, 21] (see, also [6, 7, 23]).

2.13. Lemma. *Let the conditions of Theorem 2.1 be fulfilled, $f(0) = 0$ and $r : 0 < r \leq r_0$ be such that $\nu(r) \geq \delta_r > 0$. Then $f\left(S_r^X(0)\right)$ includes a subset which is everywhere dense in the boundary of a neighbourhood of the zero of X^*, which contains the ball $B_{\nu(r)-\varepsilon}^{X^*}(0)$, where $\varepsilon : \delta_r > \varepsilon > 0$ is some number.*

For the proof we will note that it is conducted analogously to the proof of the similar Lemma from [20] (see, also [21]), using the properties of reflexive Banach spaces [6 - 9, 12, 15], homotopy theory [11], and continuity of the mapping f . However in this case we must take into account the fact that inequality (2.2) holds only almost everywhere, therefore here we use the essential condition $\nu(r_0) \geq \delta_0 > 0$ (really, $\nu(r) \geq \delta_r > 0$) and the above-mentioned concept of non-closed hypersurface from [20, 21].[§]

[§]It should be noted that here we use the following concept. A connected component of the boundary $\partial\Omega$ of $\Omega \subset X$, where Ω is a connected bounded closed neighborhood of zero in X, is called a closed hypersurface. If Γ is a closed hypersurface in X then the space X has the form $X \equiv X_0 \cup \Gamma \cup X_1$, where $\mathrm{cl}\, X_0 \cap X_1 = \mathrm{cl}\, X_1 \cap X_0 = \varnothing$. Clearly a closed hypersurface can be defined as the image of the unit sphere under a certain single-valued continuous mapping. A hypersurface $\Gamma \subset X$ is called non-closed if any point of X can be connected with the zero of X. Consequently if Γ is a non-closed hypersurface then the space X is not as the form mentioned above.

We conduct the proof by assuming the contrary. So if we assume that $f\left(S_r^X(0)\right)$ is a non-closed hypersurface and belongs to the boundary ∂M^* of a star-like closed absorbing neighborhood M^* ($\mathrm{cl}\, f\left(S_r^X(0)\right) \subset \partial M^*$), then it is shown that such an assumption contradicts the conditions of this Lemma.

2.14. Proposition. *Under the conditions of Lemma 2.13 the image of every "ray" of the form* $\left\{\lambda x : x \in S_1^X(0) \subset X,\ 0 \le \lambda \le r_0\right\}$ *under the mapping* f *is in the form*

$$\left\{f(\lambda x) : x \in S_1^X(0),\ 0 \le \lambda \le r_0\right\},$$

which satisfies the following inequalities for almost all $x \in S_1^X(0)$*:*

$$\|f(\lambda x)\|_{X^*} \le \mu(\lambda),\ \nu(\lambda)\lambda \le \langle f(\lambda x), \lambda x\rangle .$$

The proof is obvious, and is omitted.

2.15. Lemma. *The set* M_0^* *(*$M_0 \subset Y$*) defined by formula* (2.1) *(respectively, by formula* (2.5)*) is a convex body of* X^* *(respectively, of* Y*).*

Proof. The convexity of M_0^* (M_0) is obvious from the definition. It is enough to show that if $y_0 \in M_0^*$ is a relative interior point of M_0^* (i.e. $y_0 \in riM_0^*$ [8]) then there is a neighbourhood $V(y_0) \subset X^*$ that belongs to M_0^*. For simplicity we only consider the case of Theorem 2.1. In fact, if for each $y \in M_0^*$ the inequality $\langle y, x\rangle \le \langle f(x), x\rangle$ holds for any $x \in S_{r_0}^X(0)$ then for a $y_0 \in riM_0^*$ there exists $\sigma > 0$ such that the inequality $\langle y_0, x\rangle + \sigma \le \langle f(x), x\rangle$ holds.

Let $V_\delta(0)$ be a neighbourhood of zero of X^*, where $\delta \equiv \delta(\sigma, r_0) > 0$ is some number. Then we can represent every element $y \in V_\delta(y_0)$ in the form: $y_0 + \widetilde{y}$, $\widetilde{y} \in V_\delta(0)$, i.e. $V_\delta(y_0) \equiv y_0 + V_\delta(0)$. Consequently for the proof of the embedding $V_\delta(y_0) \subset M_0^*$ it is sufficient to show that $y_0 + \widetilde{y} \in M_0^*$ for any $\widetilde{y} \in V_\delta(0)$. As we can choose $\|\widetilde{y}\|_{X^*} < \delta(\sigma, r_0) \le \frac{\sigma}{r_0}$ and $x \in S_{r_0}^X(0)$ this statement follows from the following sequence of inequalities:

$$\begin{aligned}\langle y, x\rangle &\equiv \langle y_0 + \widetilde{y}, x\rangle \\ &\le \langle y_0, x\rangle + \|\widetilde{y}\|_{X^*}\|x\|_X \\ &\le \langle y_0, x\rangle + \delta(\sigma, r_0)\, r_0 \\ &\le \langle y_0, x\rangle + \sigma \\ &\le \langle f(x), x\rangle .\end{aligned}$$

□

PROOFS OF THEOREMS 2.1 AND 2.4.

Now we can now give the proofs of the theorems.

Proof. (Theorem 2.1). From Lemma 2.13 and Proposition 2.12 it follows that $f\left(B_{r_0}^X(0)\right)$ contains a subset everywhere dense in some body of X^*. We will denote by M_1^* the closure (i.e. $\mathrm{cl}\, f\left(B_{r_0}^X(0)\right) \equiv M_1^*$) of this subset and we will show the inclusion $M_0^* \subseteq M_1^*$.

We can assume, without loss of generality, that $f(0) = 0$. (If $f(0) \ne 0$ then we can consider instead the mapping f_1 defined by $f_1(x) = f(x) - f(0)$ on $B_{r_0}^X(0)$.) From Proposition 2.12 it follows that $f\left(S_{r_0}^X(0)\right)$ belongs into $B_{R_1^0+\varepsilon}^{X^*}(0) \diagdown B_{R_2^0-\varepsilon}^{X^*}(0)$, and consequently the boundary ∂M_0^* containing this image also belongs to this set. Thus we obtain that the ball $B_{\nu(r_0)}^{X^*}(0)$ is contained in M_1^* and in M_0^*, because for every $x^* \in M_0^*$ the inequality $\langle x^*, x\rangle \le \langle f(x), x\rangle$ holds for all $x \in S_{r_0}^X(0)$ by virtue of the definition.

From here we obtain also the following inequality:

$$\langle x^*, x\rangle \le \langle f(x), x\rangle,\ \forall\, x \in S_{r_0}^X(0)$$

for every $x^* \in \partial M_0^*$. Furthermore for an element $x_1^* \in \partial M_1^*$ collinear to $x^* \in \partial M_0^*$ the inequality $\|x^*\|_{X^*} \le \|x_1^*\|_{X^*}$ holds (i.e. $x_1^* \equiv \alpha x^*$ for some number α). Further, we obtain that $f\left(S_{r_0}^X(0)\right)$ belongs to M_1^*, consequently $M_0^* \subseteq M_1^*$ holds, since there exists a $\lambda_{r_0} : 0 < \lambda_{r_0} \le 1$ (and may be $\lambda_{r_0} < 1$) such that

$$\sup\left\{\|f(\lambda x)\|_{X^*} : x \in S_{r_0}^X(0),\ 0 \le \lambda \le 1\right\} = \|f(\lambda_{r_0} x)\|_{X^*}.$$

Hence it follows that $f\left(B_{r_0}^X(0)\right)$ contains a subset that is dense in the set M_0^*. Thus Theorem 2.1 is completely proved. □

We now prove Theorem 2.4 using the proof of Theorem 2.1.

Proof. (Theorem 2.4). For the proof we will use the well-known Michael Selection Theorem [17]. This will enable us to obtain Theorem 2.4 from Theorem 2.1. In fact, by virtue of this theorem we have the following fact: if f is a lower semi-continuous mapping then there exists a continuous mapping f_0 which is the selection mapping for f. Moreover, from the condition on the mapping g we have that $g^{-1} : D(g) \subseteq Y^* \longrightarrow X$ is a continuous mapping. Therefore we can consider the mapping $f_0 \circ g^{-1}$ acting from Y^* into Y and for which the conditions of Theorem 2.1 are fulfilled. Then Theorem 2.1 is valid for the mapping $f_0 \circ g^{-1}$, from which the statement of Theorem 2.4 immediately follows. □

The following results immediately follow from Theorems 2.1 and 2.4.

2.16. Corollary. *Under the conditions of Theorem 2.4 (Theorem 2.1) if*

$$f(U(x_0)) \quad \left(f\left(B_{r_0}^X(0)\right)\right)$$

is a closed subset of Y (X^*) *then* $f(U(x_0))$ $\left(f\left(B_{r_0}^X(0)\right)\right)$ *is a bodily subset of* Y *(respectively, of* X^**). Furthermore* $M \subseteq f(U(x_0))$ *(respectively,* $M_0^* \subseteq f\left(B_{r_0}^X(0)\right)$*) holds.*

Now we will consider the duality mapping $\Im : X \longrightarrow X^*$ for the pair (X, X^*), i.e. $X \xrightarrow{\Im} X^*$, see [6, 7, 15, 23] and the references therein. In the case considered the duality mapping is bijective and - together with its inverse mapping - strictly monotone, surjective, odd, demicontinuous, bounded and coercive since X is a reflexive Banach space which together with its dual space X^* has a strictly convex norm. Hence we have for any $x \in X$ there is $x^* \in X^*$ such that

$$\langle x, x^* \rangle \equiv \langle x^*, x \rangle = \|x\|_X \cdot \|x^*\|_{X^*},$$

and in particular for any $x \in X$ we have $x \longleftrightarrow x^* = \Im(x)$, i.e. it is an equivalence relation. In addition, there is a strongly monotone increasing continuous function $\Phi : \Re_+ \longrightarrow \Re_+$, $\Phi(0) = 0$, $\Phi(\tau) \nearrow +\infty$ when $\tau \nearrow +\infty$ such that $\Im(\tau x) = \Phi(\tau)\, x^*$ for any $x \in S_1^X$ and $x^* \in S_1^{X^*}$, where $\langle x, x^* \rangle \equiv 1$ and $\tau \in \Re_+$, [15].

2.17. Corollary. (**Fixed point Theorem**) *Let* X *be a Banach space as above. Assume that a mapping* f*, acting in* X*, on a ball* $B_r^X(x_0)$ *satisfies conditions* α*,* β_1 *of Theorem 2.4 and* $f\left(B_r^X(x_0)\right) \subseteq B_r^X(x_0)$*, furthermore the mapping* f_1 *with the duality mapping* $\Im$ *satisfies the condition* β_2 *of Theorem 2.4 on the ball* $B_r^X(x_0)$*, where* $f_1(x) \equiv x - f(x)$ *for all* $x \in B_r^X(x_0)$*. If* $f_1\left(B_r^X(x_0)\right)$ *is closed in* X *then* f *possess a fixed point in* $B_r^X(x_0)$*.*

In particular, if the function μ *is such that* $r - \mu(r) \ge \delta > 0$ *and* f_1 *with the duality mapping* $\Im$ *satisfies the condition* β_2 *with* $\nu(\tau) \equiv \tau - \mu(\tau)$ *for* $\tau \in [0, r]$*, then* f *possess a fixed point in* $B_r^X(x_0)$ *if* $f_1\left(B_r^X(x_0)\right)$ *is closed in* X*.*

2.18. Remark. From the proof of Theorem 2.1 is easy to see that in the formulations of Theorems 2.1 and 2.4 we can use a closed star-absorbing neighborhood of the zero of X and of Y^*, respectively, instead of a closed ball.

Now we present some further results that follow immediately from Theorem 2.4.

2.19. Corollary. *Let the conditions of Theorem 2.4 be fulfilled. In addition let f be a single-valued continuous proper mapping [5] and g a continuous open mapping on a closed convex body $G \subset X$. Then the equation*

$$f(x) = y$$

is solvable for any $y \in Y$ $(Y \equiv Y^{**})$ satisfying the inequality $\langle y, g(x)\rangle \le \langle f(x), g(x)\rangle$ for all $x \in \partial G$.

This result is a solvability theorem (see [21] etc.).

2.20. Note. It is clear that if the conditions of Corollary 2.19 are fulfilled on X and $\nu(\tau)$ is such that $\frac{\nu(\|x\|_X)}{\|g(x)\|_{Y^*}} \nearrow \infty$ for $\|x\|_X \nearrow \infty$, then the equation $f(x) = y$ is solvable for any $y \in Y$, i.e. f is a surjection.

3. Applications of the main result.

1. Our first main result is an application to Nirenberg's problem on expanding mappings¶.

3.1. Corollary. *Let X be a reflexive Banach space and $f : X \longrightarrow X$ a mapping such that*

(1) *$f \in C^0$ (i.e. f is a single-valued continuous mapping);*
(2) *f is an "expanding" mappings in the sense that the inequality $\|f(x) - f(y)\|_X \ge c\|x-y\|_X$ holds for all $x, y \in X$, where $c > 0$ is a number;*
(3) *For any $x^* \in S_1^{X^*}(0)$ and $r > 0$ there is $x(x^*) \in S_r^X(0)$ such that*
$$\langle f(x(x^*)) - f(0), x^*\rangle \ge \nu(\|x\|_X)$$
holds, where the function $\nu \in C^0$ satisfies the conditions of Theorem 2.1. Then $f(X) \equiv \Im f = X$.

Proof. To begin we note that from Condition (2) we have $\|f(x) - f(0)\|_X \ge c\|x\|_X$. Consequently, $\|f(x)\|_X \nearrow \infty$ under $\|x\|_X \nearrow \infty$. Without loss of generality we assume that $f(0) = 0$. From Condition (3) we have that the affine space generated by $f\left(S_r^X(0)\right)$ (i.e. $Aff_{f\left(S_r^X(0)\right)}$) is an everywhere dense subspace of X by virtue of Lemma 2.6. Moreover, in the conditions (1) and (3) we have that the set $f\left(S_r^X(0)\right)$ is at least everywhere dense in the boundary of a closed neighbourhood of zero of X by Lemma 2.13 for each $r > 0$ such that $\nu(r) \ge \delta > 0$. Further from this and Condition (2) we see that

$$\mathrm{cl}\left\{\frac{f(x)}{\|f(x)\|_X} \;\Big|\; x \in S_r^X(0)\; r > 0\right\} \supseteq S_1^X(0)$$

holds for any $r > \tau_0$ for some $\tau_0 \ge 0$. Clearly the relation between x^* and $x(x^*)$ in Condition (3) can be reexpressed using this result. Hence there is a continuous mapping $g : X \longrightarrow X^*$ which, together with f, fulfills the conditions of Theorem 2.4 on each ball $B_r^X(0)$ of X. In particular the inequality

$$\langle f(x), g(x)\rangle \ge \nu_1\left(\|x\|_X\right)\|x\|_X \tag{3.1}$$

holds almost everywhere on each ball $B_r^X(0)$ which with the continuous function $\nu_1 : R_+^1 \longrightarrow R^1$ fulfills the conditions of Theorem 2.1. Moreover, if we define a function g by

$$g(x) \equiv J^{-1}\left(\frac{f(x)}{\|f(x)\|_X}\right),$$

¶Morel J. M., Steinlein H., *On a problem of Nirenberg concerning expanding maps*, J. Math. Anal. **59** (1), 1984.

then we have $g : S_r^X(0) \longrightarrow S_1^{X^*}(0)$ for any $r > 0$, $\mathrm{cl}\, g\left(S_r^X(0)\right) \equiv S_1^{X^*}(0)$ and so Condition (3) is fulfilled for the mapping g. Thus we see from Condition (1) and previous arguments that the hypotheses of the corollary imply the conditions of Theorem 2.4 on the space X. Then we can apply Theorem 2.4, Corollary 2.19 and Note 2.20 to end the proof since $f(X)$ is closed from Conditions (2) and (1). $\square$

3.2. Note. In Nirenberg's problem the third condition was that $f(X)$ contains an open subset of X. We now explain the relation between this condition and Condition (3). Using the same reasoning as above we can assume that this open set is an open neighbourhood of the zero $U(0)$, and then there is a ball $B_\varepsilon^X(0) \subset U(0)$. Clearly in this case there is a continuous mapping $g : X \longrightarrow X^*$ and a continuous function $\nu : R_+^1 \longrightarrow R^1$ that satisfy the conditions of Theorem 2.4 (and also the inequalities (3.1)) on this ball by virtue of the conditions (1) and (2) of the problem. But from here it does not follow that we can extend the mapping g outside $U(0)$ while maintaining the same properties in the infinite-dimensional Banach spaces. For example, we can consider an operator defined in the following way. Let $X \equiv H$ be a separable Hilbert space with orthonormal basis $\{e_i\}_1^\infty$, and φ a real function of the form $\varphi(\tau) \equiv r - \tau$ if $\tau \leq r < \frac{1}{2}$, and $\varphi(\tau) \equiv 0$ if $\tau \geq r$. Moreover let $L : H \longrightarrow H$ be a translation operator such that

$$Lx \equiv L\left(\sum_{i\geq 1} x_i e_i\right) \equiv \sum_{i\geq 2} x_{i-1} e_i, \ \forall x \in H.$$

Consequently L translate H onto a subspace. Then we consider the operator $f : H \longrightarrow H$ and $f(x) \equiv \varphi\left(\|x\|_H\right) x + Lx$.

This suggests that Condition (3) might be in its weakest form for the Corollary to hold.

2. Let X and Y be Banach spaces, $f(t,\cdot) : D(f(t,\cdot)) \subseteq X \longrightarrow Y$ a mapping. We will consider the following Cauchy problem:

$$\frac{dx}{dt} + f(t,x) \ni y(t), \ y(t) \in L_q(0,T;Y), \ x(0) = 0, \tag{3.2}$$

where $L_q(0,T;Y)$ is the space of vector functions $y(\cdot) : [0,T] \longrightarrow Y$, $(q > 1)$ and $x(\cdot) : [0,T] \longrightarrow X$. (For simplicity we assume that $x(0) = x_0 = 0$.)

Assume that X is a reflexive Banach space as in Section 1, $Y \equiv X^*$ and $D(f) \equiv X$. Moreover, we assume there is a Hilbert space H such that the inclusions $X \subseteq H \subseteq X^*$ are dense. Finally, we assume that the following condition is fulfilled:

iii) $f : L_p(0,T;X) \longrightarrow L_q(0,T;X^*)$ is a bounded multivalued lower semi-continuous mapping such that $f(\cdot,x)$ is a convex closed set for any $x \in L_p(0,T;X)$. (The mapping $f : D(f) \subseteq X \longrightarrow Y$ is called *bounded* if the image $f(G)$ of each subset $G \subset D(f)$ bounded in X is a bounded subset of Y), and there are functions $\mu, \nu \in C^0$ satisfying the condition ii) of Theorem 2.1 such that

$$\sup\left\{\|y\|_{L_q(X^*)} \mid y \in f(x(t))\right\} \leq \mu\left(\|x\|_{L_p(X)}\right), \ \forall\, x \in L_p(0,T;X),$$

$$\int_0^T \langle y(t), x(t)\rangle\, dt \geq \nu\left(\|x\|_{L_p(X)}\right) \|x\|_{L_p(X)}, \text{ for any } y(\cdot) \in f(\cdot,x)$$

holds for any $x(t) \in L_p(0,T;X)$. Here $\langle\cdot,\cdot\rangle$ is a dual form for (X,X^*) and $p \geq 2$, $q = p'$.

We consider the operator $\Lambda \equiv \left\{\frac{d}{dt}, \gamma_0\right\} : W_p^1(0,T;X) \longrightarrow L_p(0,T;X) \times H$, and a similar operator from $W_p^1(0,T;X^*)$, where $W_p^1(0,T;X)$ is the vector Sobolev space [16] and $p > 1$ a number. In the case considered Λ is equivalent to its second conjugate Λ^{**}, as was shown in [16]. In fact, it is easy to see that in this case the operator $\frac{d}{dt}$ acts as an operator $\Lambda \equiv \left\{\frac{d}{dt}, \gamma_0\right\}$ in the form $\Lambda : V_0 \longrightarrow L_q(0,T;X^*) \times H$, where $V_0 \equiv W_q^1(0,T;X^*) \cap L_p(0,T;X) \cap \{x(t) \mid \gamma_0 x(t) = 0\}$. Then using the transposition method [16] we obtain that Λ has a second conjugate which is equivalent to Λ, as these spaces are reflexive [7 – 9].

Now, applying Theorem 2.4 to problem (3.2) we obtain the following result.

3.3. Theorem. *Let all the conditions of this section be fulfilled for problem (3.2), and $\nu(\tau) \nearrow +\infty$ under $\tau \nearrow +\infty$. Then for any $y(t) \in M \subseteq L_q(0,T;X^*)$ Problem (3.2) is solvable in V_0, where M is an everywhere dense subset of $L_q(0,T;X^*)$.*

Proof. For the proof it is enough to conduct the following arguments. From the conditions of Theorem 5.1 we obtain a mapping $F : L_p(0,T;X) \longrightarrow \left(W_{p,0}^1(0,T;X)\right)^*$ (where $W_{p,0}^1(0,T;X) \equiv W_p^1(0,T;X) \cap \{x(t) \mid \gamma_0 x(t) = 0\}$) generated by Problem (3.2) that satisfies all the conditions of Theorem 2.4 with the unity operator onto $W_{p,0}^1(0,T;X)$. In fact, under the conditions of this Theorem there exists a continuous function $\mu_1 : R_+^1 \longrightarrow R_+^1$ $(\mu \in C^0)$ such that

$$\begin{aligned}\sup\left\{\|y\|_{W_q^{-1}(X^*)} \mid y \in F(x)\right\} &\leq \|x\|_{L_q(X^*)} + \mu\left(\|x\|_{L_p(X)}\right)\\ &\leq \mu_1\left(\|x\|_{L_p(X)}\right),\end{aligned} \tag{3.3}$$

holds for any $x \in L_p(0,T;X)$, and

$$\begin{aligned}\inf\left\{\int_0^T \langle y,x\rangle(t)\,dt \;\Big|\; y \in F(x)\right\} \equiv \int_0^T \left\langle \frac{dx}{dt}, x\right\rangle(t)\,dt +&\\ \inf\left\{\int_0^T \langle y,x\rangle(t)\,dt \;\Big|\; y \in f(\cdot,x)\right\}&\\ \geq \nu_1\left(\|x\|_{L_p(X)}\right)\|x\|_{L_p(X)},\ \forall\, x \in V_0&\end{aligned} \tag{3.4}$$

holds for any $x \in V_0$, by virtue of Condition (iii) and as $\int_0^t \left\langle \frac{dx}{ds}, x\right\rangle(s)\,ds \geq 0$ a.e., $t \in (0,T)$ holds. Consequently, Condition (ii) of Theorem 2.4 holds for the mapping $F : L_p(0,T;X) \longrightarrow \left(W_{p,0}^1(0,T;X)\right)^*$ by (3.3) and (3.4). So, we can apply Theorem 2.4 to the problem considered because $F : L_p(0,T;X) \longrightarrow \left(W_{p,0}^1(0,T;X)\right)^*$ is continuous. Hence applying Theorem 2.4 we obtain that there exists an everywhere dense subset M in $\left(W_{p,0}^1(0,T;X)\right)^*$ such that problem (3.2) is solvable for any $y(t) \in M$.

Further we can make use of the argument at the beginning of this section applied to the space $W_{p,T}^1(0,T;X) \equiv W_p^1(0,T;X) \cap \{x(t) \mid \gamma_T x(t) = 0\}$. (It should be noted that this space is everywhere dense in each of spaces $L_p(0,T;X)$, $L_q(0,T;X^*)$ and $\left(W_{p,0}^1(0,T;X)\right)^*$ under conditions of this Theorem). Thus all that remains is to note that we consider the equation for an $y(t) \in L_q(0,T;X^*)$, and $f : L_p(0,T;X) \longrightarrow L_q(0,T;X^*)$ is a bounded continuous mapping, therefore Problem (3.2) is solvable in V_0 for any $y(t) \in M_1 \subset L_q(0,T;X^*)$ by virtue of Theorem 2.4, where the set M_1 is everywhere dense in $L_q(0,T;X^*)$. (In the case when $f(t,\cdot)$ is a linear mapping this result may be compared with analogous results using the transposition method [16]). $\square$

It should be noted that the case $x(0) = x_0 \not\equiv 0$ can be studied in the same way with some modifications, as in the linear case [16].

4. Some sufficient conditions on completeness of the image

Now we consider some sufficient conditions on the completeness of the image of a mapping.

4.1. Lemma. *Let X be a Banach space such as above, $f : X \longrightarrow X^*$ a monotone mapping satisfying the conditions of Theorem 2.1, and $r \geq \tau_1$ some number. Then $f(G)$ is a bounded closed subset containing a ball $B_{r_1}^{X^*}(0)$ for every bounded closed convex body $G \subset X$ such that $B_r^X(0) \subset G$, where $r_1 = r_1(r) \geq \delta_1 > 0$.*

Proof. Let G be a bounded closed convex body of X containing a ball $B_r^X(0)$ with radius $r \geq \tau_1$. From Theorem 2.1 it follows that $f(G)$ contains a subset which is dense in a convex closed body of X^*. Consequently we can choose a sequence $\{y_m\}_{m=1}^{\infty}$ from $f(G)$ which is fundamental in X^*. It is clear that there is an element $y_0 \in X^*$ such that a $y_m \longrightarrow y_0$ in X^*.

We can choose a weakly convergent subsequence $\{x_{m_k}\}_{m_k=1}^{\infty}$ from $f^{-1}\left(\{y_m\}_{m=1}^{\infty}\right) \cap G$ in the following form

$$\left\{x_{m_k} \mid x_{m_k} \in f^{-1}(y_{m_k}) \cap G,\ y_{m_k} \in \{y_m\}_{m=1}^{\infty}\right\}$$

by using the reflexivity of X and the boundedness of G.

It now follows that $x_{m_k} \rightharpoonup x_0 \in G$ weakly in X as $m_k \nearrow \infty$; and $y_{m_k} \rightharpoonup y_0$ weakly in X^* as $m_k \nearrow \infty$ (and strongly in X^* by virtue of the assumption).

Clearly we can write the second relation also in the following form: $f(x_{m_k}) \rightharpoonup y_0$ weakly in X^* (and strongly by virtue of the assumption) as $m_k \nearrow \infty$.

Now, for the proof of the equality $f(x_0) = y_0$ we will use monotonicity of the mapping $f : X \longrightarrow X^*$. As known [3, 9, 15, 23], for any $x, \widetilde{x} \in X$ the inequality $\langle f(x) - f(\widetilde{x}), x - \widetilde{x} \rangle \geq 0$ holds by definition. Then under $m_k \nearrow \infty$ we have

$$0 \leq \langle f(x_{m_k}) - f(x), x_{m_k} - x \rangle = \langle f(x_{m_k}), x_{m_k} - x \rangle - \langle f(x), x_{m_k} - x \rangle \longrightarrow$$

$$\langle y_0, x_0 - x \rangle - \langle f(x), x_0 - x \rangle = \langle y_0 - f(x), x_0 - x \rangle .$$

Thus we obtain the following inequality:

$$\langle y_0 - f(x), x_0 - x \rangle \geq 0,\ \forall\, x \in X,$$

which proves that $y_0 = f(x_0)$. Consequently, $f(G)$ is a closed subset in X^*. □

4.2. Remark. The assertion of Lemma 4.1 remains true if f is a lower semi-continuous maximal monotone bounded operator satisfying the remaining conditions of Theorem 5.1 and having a convex closed image for any x.

4.3. Lemma. *Let X, Y be Banach spaces as above, and suppose the mapping $f : D(f) \subseteq X \longrightarrow Y$ has a weakly closed graph and that for any bounded subset of Y its pre-image under mapping f is a bounded subset of X. Then f is a weakly closed mapping (compare a similar lemma from [18]).*

If the mapping $f : X \longrightarrow X^*$ from the previous section satisfies some additional conditions, then we can prove the completeness of its image. For example, the following statements are true.

4.4. Corollary. *Let all conditions of Theorem 5.1 be fulfilled. We assume that*

(iv) $f : L_p(0, T; X) \longrightarrow L_q(0, T; X^*)$ *is a weakly closed operator.*

Then problem (3.2) is solvable for any $y(t) \in L_q(0,T;X^*)$, *i.e. the mapping* $\digamma$ *is surjective.*

Proof. For the proof it is sufficient to show that the image of every bounded closed convex subset of V is a closed subset of $L_q(0,T;X^*)$. This property follows from Lemma 4.3 by virtue of the conditions of Theorem 5.1 and the inequalities (3.3) and (3.4). $\square$

4.5. Corollary. *Let the conditions of Theorem 5.1 be fulfilled and* $H = [X, X^*]_{\frac{1}{2}}$. *Moreover, assume one of the following conditions is satisfied:*

(v) $f : L_p(0,T;X) \longrightarrow L_q(0,T;X^*)$, $p > 1$, $q = p'$ *is a monotone operator,* [15, 23];

(vi) $f : L_p(0,T;X) \longrightarrow L_q(0,T;X^*)$, $p > 1$, $q = p'$ *is a pseudo-monotone operator,* [15, 23] *and references therein.*

Then problem (3.2) (with the equation $\frac{dx}{dt} + f(t,x) = y(t)$*) is solvable in* $V(Q)$ *for any* $y(t) \in L_q(0,T;X^*)$.

Proof. Let Condition (v) be fulfilled. It is easy to see that $\digamma : V(Q) \longrightarrow L_q(0,T;X^*)$ is a continuous mapping. Moreover, $\digamma$ is a weakly closed operator (this follows from the linearity of the first operator and from conditions on the mapping $f : L_p(0,T;X) \longrightarrow L_q(0,T;X^*)$). The image of $\digamma$ is a closed subset (this is obtained using the monotonicity of the mapping f and the reflexivity of the space). Furthermore, Problem (3.2) is solvable for any $y(t) \in M \subset L_q(0,T;X^*)$ and $\overline{M}^{L_q(0,T;X^*)} \equiv L_q(0,T;X^*)$ by virtue of Theorem 5.1. Consequently $\overline{\digamma(V(Q))}^{L_q(0,T;X^*)} = L_q(0,T;X^*)$.

Thus to complete the proof it remains to show that $\digamma(V(Q))$ is weakly closed. In fact let $\{y_m(t)\}_{m=1}^{\infty} \subset \Im\digamma \subseteq L_q(0,T;X^*)$ be a fundamental sequence in the space $L_q(0,T;X^*)$ and $\lim\limits_{m\to\infty} y_m(t) = y(t) \in L_q(0,T;X^*)$. From the conditions and Inequality (3.4) it follows that $\digamma^{-1}\left(\{y_m(t)\}_{m=1}^{\infty}\right) \subset V(Q)$, and it is a bounded subset of $V(Q)$. Consequently, we can select a weakly convergent subsequence $\{x_{m_k}(t)\}_{k=1}^{\infty} \subset V$ from $\digamma^{-1}\left(\{y_m(t)\}_{m=1}^{\infty}\right)$ such that $x_{m_k}(t) \in \digamma^{-1}(y_{m_k}(t))$, $k = 1, 2, \ldots$. Then we have

$$\digamma(x_{m_k}(t)) \equiv \frac{dx_{m_k}}{dt} + f(x_{m_k}(t)) = y_{m_k}(t)$$

and

$$\frac{dx_{m_k}}{dt} \overset{L_q(X^*)}{-\!-\!\rightharpoonup} \frac{dx}{dt}; \quad f(x_{m_k}(t)) \overset{L_q(X^*)}{-\!-\!\rightharpoonup} \chi(t) \text{ - weakly.}$$

From here we obtain - using monotonicity - (we assume that $x_{m_k} \equiv x_k$)

$$\begin{aligned} 0 &\le \int_0^T \langle f(x_k) - f(x_0), x_k - x_0 \rangle \, dt \\ &= \int_0^T \langle y_k - f(x_0), x_k - x_0 \rangle \, dt + \int_0^T \left\langle \frac{dx_k}{dt}, x_0 \right\rangle dt - \int_0^T \left\langle \frac{dx_k}{dt}, x_k \right\rangle dt, \ \forall x_0 \in V. \end{aligned} \tag{4.1}$$

It is clear that we may pass to the limit as $k \nearrow \infty$ in the first and second terms of (4.1), therefore all that remains is to consider the last term. For this we have

$$\int_0^T \left\langle \frac{dx_k}{dt}, x_k \right\rangle dt = \frac{1}{2} \int_0^T \frac{d}{dt} \|x_k\|_H^2 \, dt = \frac{1}{2} \|x_k\|_H^2(T). \tag{4.2}$$

From the conditions we can see that for a fixed $T > 0$ the inclusion $x_k(T) \in H$ holds for any $k = 1, 2, \ldots$, and $\{x_k(T)\}$ belongs to a bounded subset of H. Hence we obtain that the sequence $\{x_k(T)\}$ weakly converges (maybe after passing to a subsequence). Consequently the following relations hold

$$
\begin{aligned}
&x_k(T) \rightharpoonup x(T) \text{ weakly in } H;\\
&\|x\|_H(T) \le \lim \|x_k\|_H(T);\\
&\frac{1}{2}\|x\|_H^2(T) = \int_0^T \left\langle \frac{dx}{dt}, x \right\rangle dt.
\end{aligned}
$$

Further, taking into account these relations and using (4.2) after passing to the limit in (4.1) as $k : k \nearrow \infty$, we obtain

$$
\begin{aligned}
0 &\le \lim_{k\to\infty} \int_0^T \langle f(x_k) - f(x_0), x_k - x_0 \rangle dt\\
&= \lim_{k\to\infty} \int_0^T \left\langle y_k - \frac{dx_k}{dt} - f(x_0), x_k - x_0 \right\rangle dt\\
&\le \int_0^T \langle \chi(t) - f(x_0), x - x_0 \rangle dt\\
&\Longrightarrow f(x_k) \overset{L_q(X^*)}{\rightharpoonup} y(t) \text{ weakly, i.e. } \chi(t) \equiv y(t).
\end{aligned}
$$

The proof of the other case may be carried out analogously (see [21][‖]), where it is necessary to make use of the pseudo-monotonicity of the mapping. □

5. On the solvability of the mixed problem for nonlinear hyperbolic equations.

Now we will study the solvability of the following problem:

$$
F(u) \equiv \frac{\partial^2 u}{\partial t^2} - \sum_{i=1}^{n} D_i\left(|D_i u|^{p-2} D_i u\right) = h(t,x), \ (t,x) \in Q, \ p \ge 2. \tag{5.1}
$$

$$
\begin{aligned}
&u(0,x) = 0, \ \frac{\partial u}{\partial t}(0,x) = 0, \ x \in \Omega \subset R^n, \ n \ge 2;\\
&u(t,x)\mid_\Gamma = 0, \ (t,x) \in \Gamma \equiv [0,T] \times \partial\Omega.
\end{aligned} \tag{5.2}
$$

[‖]Also: Soltanov K.N. Russian Ac.Sci. Dokl.Math. (1992) 45, 3.

So, let $\Omega \subset R^n$, $n \geq 2$, be a domain as in the previous section. We introduce the following spaces and mappings:

$$V_1(Q) \equiv L_p\Big(0,T;\overset{0}{W}{}_p^1(\Omega)\Big) \cap \underset{0}{W}{}_q^1\,(0,T;L_q(\Omega)),$$

$$V_2(Q) \equiv \Big\{u(t,x) \mid u(t,x) \in L_p(0,T;\overset{0}{W}{}_p^1(\Omega)), \frac{\partial^2 u}{\partial t^2} \in L_q(0,T;W_q^{-1}(\Omega)),$$
$$u(0,x)=0, \frac{\partial u}{\partial t}(0,x)=0\Big\},$$

$$\widetilde{V}_0(Q) \equiv \Big\{u(t,x) \mid u(t,x) \& \frac{\partial u}{\partial t} \in L_p(0,T;\overset{0}{W}{}_p^1(\Omega)), u(0,x)=0\Big\}$$

$$F_1,\ F_1(u) \equiv \frac{\partial^2 u}{\partial t^2} + f(u);\ f,\ f(u) \equiv -\sum_{i=1}^{n} D_i(|D_i u|^{p-2} D_i u),$$

$$F_1 : V_1(Q) \longrightarrow W_q^{-1}(0,T;L_q(\Omega)) + L_q\left(0,T;W_q^{-1}(\Omega)\right),$$

$$F_1 : V_2(Q) \longrightarrow L_q\left(0,T;W_q^{-1}(\Omega)\right);$$

$$f : L_p\left(0,T;\overset{0}{W}{}_p^1(\Omega)\right) \longrightarrow L_q\left(0,T;W_q^{-1}(\Omega)\right).$$

We will show that $F_1 : V_2(Q) \longrightarrow L_q\left(0,T;W_q^{-1}(\Omega)\right)$ satisfies the conditions of Theorem 2.1. It not is difficult to see that the following equality holds for any function $u(t,x) \in V_2(Q) \cap \widetilde{V}_0(Q)$:

$$\begin{aligned}
\int_0^t \left\langle F_1(u), \frac{\partial u}{\partial s}\right\rangle ds &\equiv \int_0^t \left\langle \frac{\partial^2 u}{\partial s^2}, \frac{\partial u}{\partial s}\right\rangle ds - \int_0^t \left\langle \sum_{i=1}^{n} D_i\left(|D_i u|^{p-2} D_i u\right), \frac{\partial u}{\partial s}\right\rangle ds \\
&= \frac{1}{2}\int_0^t \frac{\partial}{\partial s}\left\|\frac{\partial u}{\partial s}\right\|_H^2 ds + \int_0^t \sum_{i=1}^{n}\left\langle |D_i u|^{p-2} D_i u, \frac{\partial D_i u}{\partial s}\right\rangle ds \\
&= \frac{1}{2}\int_0^t \frac{\partial}{\partial s}\left\|\frac{\partial u}{\partial s}\right\|_H^2 ds \\
&\quad + \frac{1}{p}\sum_{i=1}^{n}\int_0^t \frac{\partial}{\partial s}|D_i u|^p\, ds = \frac{1}{2}\left\|\frac{\partial u}{\partial t}\right\|_H^2 (t) \\
&\quad + \frac{1}{p}\|Du\|_{L_p}^p(t),\ \forall t \in [0,T].
\end{aligned} \tag{5.3}$$

Furthermore the mapping $F_1 : V_1(Q) \longrightarrow W_q^{-1}(0,T;L_q(\Omega)) + L_q\left(0,T;W_q^{-1}(\Omega)\right)$ is a bounded since the following inequality is true

$$\begin{aligned}
&\|F_1(u)\|_{W_q^{-1}\left(0,T;L_q(\Omega)\right)+\ L_q\left(0,T;W_q^{-1}(\Omega)\right)} \\
&\quad \leq \left\|\frac{\partial^2 u}{\partial t^2}\right\|_{W_q^{-1}\left(0,T;L_q(\Omega)\right)} + \left\|\sum_{i=1}^{n} D_i\left(|D_i u|^{p-2} D_i u\right)\right\|_{L_q\left(0,T;W_q^{-1}(\Omega)\right)} \\
&\quad \leq C\left(\left\|\frac{\partial u}{\partial t}\right\|_{L_q\left(0,T;L_q(\Omega)\right)} + \|u\|^{p-1}_{L_p\left(0,T;\overset{0}{W}{}_p^1(\Omega)\right)}\right).
\end{aligned}$$

Thus we have verified that the problem in question satisfies all conditions of Theorem 2.1 by virtue of the continuity and boundedness of the mapping $F_1 : V_1(Q) \longrightarrow W_q^{-1}(0,T;L_q(\Omega)) + L_q\left(0,T;W_q^{-1}(\Omega)\right)$, and because $V_2(Q) \cap \widetilde{V}_0(Q)$ is everywhere dense in $V_1(Q)$ and in $L_p\big(0,T;\overset{0}{W}{}^1_p(\Omega)\big)$. Consequently, using Theorem 2.1 we deduce that Problem (4.1) - (4.2) is solvable for any $h(t,x) \in M$, where M contains an everywhere dense subset of $W_q^{-1}(0,T;L_q(\Omega)) + L_q\left(0,T;W_q^{-1}(\Omega)\right)$.

Furthermore, useing Equation (4.1) once more we obtain that if $h(t,x) \in M \subseteq L_q\left(0,T;W_q^{-1}(\Omega)\right)$ then a solution $u(t,x)$ belongs to $V_2(Q)$. From this we can conclude that any solution $u(t,x)$ belongs to a bounded subset of $\underset{0}{W}{}^1_\infty(0,T;H) \cap L^\infty\left(0,T;\overset{0}{W}{}^1_p(\Omega)\right)$. Thus we have obtained the following solvability theorem.

5.1. Theorem. *Let all conditions considered in this section for the problem (4.1) – (4.2) be fulfilled, and let $V_1(Q)$ be the space defined above. Then Problem (4.1) – (4.2) is solvable in $V_1(Q)$ for any $h(t,x) \in M$, where M contains an everywhere dense subset of $W_q^{-1}(0,T;L_q(\Omega)) + L_q\left(0,T;W_q^{-1}(\Omega)\right)$.*

6. Problems

Here we note some questions arising from the above-mentioned results.

6.1. Problem. Let the inequality (1.1) hold for any $x \in B_r^X(x_0)$. It is clear that, if $X \equiv R^n$ then this theorem differs from Brouwer's fixed-point theorem only in the condition $\nu(r) \geq \delta > 0$. Can we change the condition $\nu(r) \geq \delta > 0$ of this theorem with the condition $\nu(r) > 0$?

6.2. Problem. In order to prove the solvability of Problem (4.1) – (4.2) for any $h(t,x) \in L_q\left(0,T;W_q^{-1}(\Omega)\right)$ is it is enough to show $F(V_1)$ is a closed subset of $L_q\left(0,T;W_q^{-1}(\Omega)\right)$ in some sense?

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