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BINARY DI-OPERATIONS AND SPACES OF REAL DIFUNCTIONS ON A TEXTURE

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Abstract

The authors consider the commutativity and associativity of binary di-operations on a texture and go on to study the space of real difunctions on a texture and the space of bicontinuous real difunctions on a ditopological texture space.

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1. Introduction

Let S be a non-empty set. We recall [1] that a *texturing* on S is a point separating, complete, completely distributive lattice $\mathcal{S}$ of subsets of S with respect to inclusion, which contains S, $\emptyset$, and for which meet $\bigwedge$ coincides with intersection $\bigcap$ and finite joins $\vee$ coincide with unions $\cup$. Textures first arose in connection with the representation of Hutton algebras and lattices of $\mathbb{L}$-fuzzy sets in a point-based setting [3], and have subsequently proved to be a fruitful setting for the investigation of complement-free concepts in mathematics. The sets

$$P_s = \bigcap\{A \in \mathcal{S} \mid s \in A\},\ Q_s = \bigvee\{P_u \mid u \in S,\ s \notin P_u\},\ s \in S,$$

are important in the study of textures, and the following facts concerning these so called p–sets and q–sets will be used extensively below.

1.1. Lemma. [5, Theorem 1.2]

(1) $s \notin A \implies A \subseteq Q_s \implies s \notin A^\flat$ *for all* $s \in S$, $A \in \mathcal{S}$.
(2) $A^\flat = \{s \mid A \not\subseteq Q_s\}$ *for all* $A \in \mathcal{S}$.
(3) *For* $A_i \in \mathcal{S}$, $i \in I$ *we have* $(\bigvee_{i\in I} A_i)^\flat = \bigcup_{i\in I} A_i^\flat$.
(4) *A is the smallest element of $\mathcal{S}$ containing $A^\flat$ for all* $A \in \mathcal{S}$.

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(5) *For* $A, B \in \mathcal{S}$, *if* $A \not\subseteq B$ *then there exists* $s \in S$ *with* $A \not\subseteq Q_s$ *and* $P_s \not\subseteq B$.
(6) $A = \bigcap\{Q_s \mid P_s \not\subseteq A\}$ *for all* $A \in \mathcal{S}$.
(7) $A = \bigvee\{P_s \mid A \not\subseteq Q_s\}$ *for all* $A \in \mathcal{S}$.

Here $A^\flat$ is defined by

$$A^\flat = \bigcap\Big\{\bigcup\{A_i \mid i \in I\} \mid \{A_i \mid i \in I\} \subseteq \mathcal{S},\ A = \bigvee\{A_i \mid i \in I\}\Big\}$$

and known as the *core* of $A \in \mathcal{S}$. The above lemma exposes an important formal duality in $(S, \mathcal{S})$, namely that between $\bigcap$ and $\bigvee$, Q_s and P_s, and $P_s \not\subseteq A$ and $A \not\subseteq Q_s$. Indeed, it is to emphasize this duality that we normally write $P_s \not\subseteq A$ in preference to $s \notin A$.

The simplest example of a texture is the *discrete texture* $(X, \mathcal{P}(X))$ *on* X, for which $P_x = \{x\}$ and $Q_x = X \setminus \{x\}$, $x \in X$. This texture is closed under set complementation, but this is certainly not the case in general. A texture that we will consider in the final section is the *real texture* $(\mathbb{R}, \mathcal{R})$, where $\mathbb{R}$ is the set of real numbers and

$$\mathcal{R} = \{(-\infty, r) \mid r \in \mathbb{R}\} \cup \{(-\infty, r] \mid r \in \mathbb{R}\} \cup \{\mathbb{R}, \emptyset\}.$$

For this texture $P_r = (-\infty, r]$ and $Q_r = (-\infty, r)$, $r \in \mathbb{R}$.

In the study of textures the ordinary notion of binary relation between sets is replaced by direlations, which are pairs consisting of a relation and corelation [5]. Defining a difunction as a special type of direlation, a theory is obtained that resembles in many respects that of ordinary binary relations and functions between sets. Our aim in this study is to continue this work by defining di-operations on textures and studying their commutativity and associativity. In particular a study of di-operations on $(\mathbb{R}, \mathcal{R})$ is presented and applied to the study of spaces of real (bicontinuous) difunctions, that is difunctions whose range is $(\mathbb{R}, \mathcal{R})$.

The reader is referred to [1–7] for background and motivation on textures. For the benefit of the reader we recall some basic definitions.

For textures $(S, \mathcal{S})$, $(T, \mathcal{T})$ we denote by $\mathcal{S} \otimes \mathcal{T}$ the product texturing of $S \times T$ [3]. Thus, $\mathcal{S} \otimes \mathcal{T}$ consists of arbitrary intersections of sets of the form $(A \times T) \cup (S \times B)$, $A \in \mathcal{S}$, $B \in \mathcal{T}$. For $s \in S$, P_s and Q_s will always denote the p-sets and q-sets for the texture $(S, \mathcal{S})$, while for $t \in T$, P_t and Q_t will denote the p-sets and q-sets for $(T, \mathcal{T})$. We reserve the notation $P_{(s,t)}$, $Q_{(s,t)}$, $s \in S$, $t \in T$, for the p-sets, q-sets in $(S \times T, \mathcal{S} \otimes \mathcal{T})$. On the other hand, $\overline{P}_{(s,t)}$ and $\overline{Q}_{(s,t)}$ will denote the p-sets and q-sets for the texture $(S \times T,\ \mathcal{P}(S) \otimes \mathcal{T})$. Hence (see [5]) we have $\overline{P}_{(s,t)} = \{s\} \times P_t$ and $\overline{Q}_{(s,t)} = [(S \setminus \{s\}) \times T] \cup [S \times Q_t]$. Likewise, $\overline{P}_{(t,s)}$ and $\overline{Q}_{(t,s)}$ are the p-sets and q-sets for $(T \times S,\ \mathcal{P}(T) \otimes \mathcal{S})$. It is easy to verify that $\overline{P}_{(s,t)} \not\subseteq \overline{Q}_{(s',t')} \iff s = s'$ and $P_t \not\subseteq Q_{t'}$. Again, we will use this fact, and its companion $\overline{P}_{(t,s)} \not\subseteq \overline{Q}_{(t',s')} \iff t = t'$ and $P_s \not\subseteq Q_{s'}$, without comment in what follows. Now let us recall:

1.2. Definition. [5, Definition 2.1] Let $(S, \mathcal{S})$, $(T, \mathcal{T})$ be textures. Then

(1) $r \in \mathcal{P}(S) \otimes \mathcal{T}$ is called a *relation from* $(S, \mathcal{S})$ *to* $(T, \mathcal{T})$ if it satisfies
  R1 $r \not\subseteq \overline{Q}_{(s,t)}, P_{s'} \not\subseteq Q_s \implies r \not\subseteq \overline{Q}_{(s',t)}$.
  R2 $r \not\subseteq \overline{Q}_{(s,t)} \implies \exists s' \in S$ such that $P_s \not\subseteq Q_{s'}$ and $r \not\subseteq \overline{Q}_{(s',t)}$.
(2) $R \in \mathcal{P}(S) \otimes \mathcal{T}$ is called a *corelation from* $(S, \mathcal{S})$ *to* $(T, \mathcal{T})$ if it satisfies
  CR1 $\overline{P}_{(s,t)} \not\subseteq R, P_s \not\subseteq Q_{s'} \implies \overline{P}_{(s',t)} \not\subseteq R$.
  CR2 $\overline{P}_{(s,t)} \not\subseteq R \implies \exists s' \in S$ such that $P_{s'} \not\subseteq Q_s$ and $\overline{P}_{(s',t)} \not\subseteq R$.
(3) A pair (r, R), where r is a relation and R a corelation from $(S, \mathcal{S})$ to $(T, \mathcal{T})$ is called a *direlation from* $(S, \mathcal{S})$ *to* $(T, \mathcal{T})$.

Normally, relations will be denoted by lower case and corelations by upper case letters, as in the above definition.

For a general texture $(S, \mathcal{S})$ we define

$$i = i_S = \bigvee\{\overline{P}_{(s,s)} \mid s \in S\} \text{ and } I = I_S = \bigcap\{\overline{Q}_{(s,s)} \mid s \in S\}.$$

If we note that $i \not\subseteq \overline{Q}_{(s,t)} \iff P_s \not\subseteq Q_t$ and $\overline{P}_{(s,t)} \not\subseteq I \iff P_t \not\subseteq Q_s$ then it is trivial to verify that i is a relation and I a corelation from $(S, \mathcal{S})$ to $(S, \mathcal{S})$. We refer to (i, I) as the *identity direlation* on $(S, \mathcal{S})$.

If (r, R) is a direlation from $(S, \mathcal{S})$ to $(T, \mathcal{T})$, the *inverse* $(r, R)^{\leftarrow} = (R^{\leftarrow}, r^{\leftarrow})$ of (r, R) is the direlation from $(T, \mathcal{T})$ to $(S, \mathcal{S})$ defined by

$$r^{\leftarrow} = \bigcap\{\overline{Q}_{(t,s)} \mid r \not\subseteq \overline{Q}_{(s,t)}\}$$
$$R^{\leftarrow} = \bigvee\{\overline{P}_{(t,s)} \mid \overline{P}_{(s,t)} \not\subseteq R\}$$

An important concept for direlations, which we will use extensively in this paper, is that of composition. We recall the following:

1.3. Definition. [5, Definition 2.13] Let $(S, \mathcal{S})$, $(T, \mathcal{T})$, $(U, \mathcal{U})$ be textures.

(1) If p is a relation from $(S, \mathcal{S})$ to $(T, \mathcal{T})$ and q a relation from $(T, \mathcal{T})$ to $(U, \mathcal{U})$ then their *composition* is the relation $q \circ p$ from $(S, \mathcal{S})$ to $(U, \mathcal{U})$ defined by

$$q \circ p = \bigvee\{\overline{P}_{(s,u)} \mid \exists\, t \in T \text{ with } p \not\subseteq \overline{Q}_{(s,t)} \text{ and } q \not\subseteq \overline{Q}_{(t,u)}\}.$$

(2) If P is a corelation from $(S, \mathcal{S})$ to $(T, \mathcal{T})$ and Q a corelation from $(T, \mathcal{T})$ to $(U, \mathcal{U})$ then their *composition* is the corelation $Q \circ P$ from $(S, \mathcal{S})$ to $(U, \mathcal{U})$ defined by

$$Q \circ P = \bigcap\{\overline{Q}_{(s,u)} \mid \exists\, t \in T \text{ with } \overline{P}_{(s,t)} \not\subseteq P \text{ and } \overline{P}_{(t,u)} \not\subseteq Q\}.$$

(3) With p, q; P, Q as above, the *composition* of the direlations (p, P), (q, Q) is the direlation

$$(q, Q) \circ (p, P) = (q \circ p, Q \circ P).$$

It is shown in [5] that the operation of taking the composition of direlations is associative, and that the identity direlations are identities for this operation.

The notion of difunction is derived from that of direlation as follows.

1.4. Definition. [5, Definition 2.22] Let (f, F) be a direlation from $(S, \mathcal{S})$ to $(T, \mathcal{T})$. Then (f, F) is called a *difunction from* $(S, \mathcal{S})$ *to* $(T, \mathcal{T})$ if it satisfies the following two conditions.

DF1 For $s, s' \in S$, $P_s \not\subseteq Q_{s'} \implies \exists\, t \in T$ with $f \not\subseteq \overline{Q}_{(s,t)}$ and $\overline{P}_{(s',t)} \not\subseteq F$.

DF2 For $t, t' \in T$ and $s \in S$, $f \not\subseteq \overline{Q}_{(s,t)}$ and $\overline{P}_{(s,t')} \not\subseteq F \implies P_{t'} \not\subseteq Q_t$.

Difunctions are preserved under composition. It is easy to see that the identity direlation (i_S, I_S) on $(S, \mathcal{S})$ is in fact a difunction from $(S, \mathcal{S})$ to $(S, \mathcal{S})$. In this context we refer to (i_S, I_S) as the *identity difunction on* $(S, \mathcal{S})$.

Let (f, F) be a difunction from $(S, \mathcal{S})$ to $(T, \mathcal{T})$, and $B \in \mathcal{T}$. Then the *inverse image* $f^{\leftarrow}(B)$ and the *inverse co-image* $F^{\leftarrow}(B)$ of B are given by the formulae

(1) $f^{\leftarrow}(B) = \bigvee\{P_s \mid \forall\, t,\ f \not\subseteq \overline{Q}_{(s,t)} \implies P_t \subseteq B\} \in \mathcal{S}$, and

(2) $F^{\leftarrow}(B) = \bigcap\{Q_s \mid \forall\, t,\ \overline{P}_{(s,t)} \not\subseteq F \implies B \subseteq Q_t\} \in \mathcal{S}$,

respectively [5, Lemma 2.8] It is shown in [5] that for difunctions these sets coincide for all $B \in \mathcal{T}$ and that these inverses preserve arbitrary intersections and joins.

We conclude by recalling the notion of ditopology. A *dichotomous topology*, or *ditopology* for short, on a texture $(S, \mathcal{S})$ is a pair (τ, κ) of subsets of $\mathcal{S}$, where the set of *open sets* τ satisfies

(1) $S, \emptyset \in \tau$,
(2) $G_1, G_2 \in \tau \implies G_1 \cap G_2 \in \tau$ and
(3) $G_i \in \tau,\ i \in I \implies \bigvee_i G_i \in \tau$,

and the set of *closed sets* κ satisfies

(1) $S, \emptyset \in \kappa$,
(2) $K_1, K_2 \in \kappa \implies K_1 \cup K_2 \in \kappa$ and
(3) $K_i \in \kappa,\ i \in I \implies \bigcap K_i \in \kappa$.

Hence a ditopology is essentially a "topology" for which there is no *a priori* relation between the open and closed sets. The reader is referred to [1,5–8] for some results on ditopological texture spaces and their relation with fuzzy topologies.

A subset β of τ is called a *base of* τ if every set in τ can be written as a join of sets in β, while a subset β of κ is a *base of* κ if every set in κ can be written as an intersection of sets in β.

For the real texture $(\mathbb{R}, \mathcal{R})$ mentioned above, we may define a natural ditopology (θ, ϕ), called the *usual ditopology on* $(\mathbb{R}, \mathcal{R})$, by

$$\theta = \{(-\infty, s) \mid s \in \mathbb{R}\} \cup \{\mathbb{R}, \emptyset\},\ \ \phi = \{(-\infty, s] \mid s \in \mathbb{R}\} \cup \{\mathbb{R}, \emptyset\}.$$

Continuity of difunctions is the subject of the following definition.

1.5. Definition. [6, Definition 2.2] Let $(S_k, \mathcal{S}_k, \tau_k, \kappa_k)$, $k = 1, 2$, be ditopological texture spaces and (f, F) a difunction from $(S_1, \mathcal{S}_1)$ to $(S_2, \mathcal{S}_2)$. Then

(1) (f, F) is *continuous* if $G \in \tau_2 \implies F^{\leftarrow}(G) \in \tau_1$.
(2) (f, F) is *cocontinuous* if $K \in \kappa_2 \implies f^{\leftarrow}(K) \in \kappa_1$.
(3) (f, F) is *bicontinuous* if it is continuous and cocontinuous.

The reader is referred to [8] for terms related to lattice theory that are not defined here.

2. The commutativity and associativity direlations

In this section we introduce two direlations which will play an important role in the study of di-operations on a texture.

2.1. Definition. Let $(S, \mathcal{S})$ be a texture.

(1) The direlation (c, C) on $(S \times S, \mathcal{S} \otimes \mathcal{S})$ defined by

$$c = \bigvee\{\overline{P}_{((s_1,s_2),(s_2,s_1))} \mid s_1, s_2 \in S\}$$
$$C = \bigcap\{\overline{Q}_{((s_1,s_2),(s_2,s_1))} \mid s_1, s_2 \in S^\flat\}$$

is called the *commutativity direlation on* $(S, \mathcal{S})$.

(2) The direlation (a, A) from $(S \times (S \times S), \mathcal{S} \otimes (\mathcal{S} \otimes \mathcal{S}))$ to $((S \times S) \times S, (\mathcal{S} \otimes \mathcal{S}) \otimes \mathcal{S})$ defined by

$$a = \bigvee\{\overline{P}_{((s_1,(s_2,s_3)),((s_1,s_2),s_3))} \mid s_1, s_2, s_3 \in S\}$$
$$A = \bigcap\{\overline{Q}_{((s_1,(s_2,s_3)),((s_1,s_2),s_3))} \mid s_1, s_2, s_3 \in S^\flat\}$$

is called the *associativity direlation on* $(S, \mathcal{S})$.

It is easy to verify that (c, C) and (a, A) are indeed direlations. In fact, (c, C) is the bijective difunction from $(S \times S, \mathcal{S} \otimes \mathcal{S})$ to $(S \times S, \mathcal{S} \otimes \mathcal{S})$ corresponding, in the sense of ([7], Lemma 3.8), to the textural isomorphism [1] $\varphi : S \times S \to S \times S$, $(s_1, s_2) \mapsto (s_2, s_1)$. Likewise, (a, A) is the bijective difunction corresponding to the isomorphism $\psi : S \times (S \times S) \to (S \times S) \times S$, $(s_1, (s_2, s_3)) \mapsto ((s_1, s_2), s_3)$.

2.2. Lemma. *Let (c, C) be the commutativity direlation on $(S, \mathcal{S})$. Then $c \circ c = i_{S \times S}$ and $C \circ C = I_{S \times S}$.*

Proof. To prove the first result it is sufficient to show that $c \circ c \not\subseteq \overline{Q}_{((s_1,s_2),(t_1,t_2))}$ if and only if $P_{(s_1,s_2)} \not\subseteq Q_{(t_1,t_2)}$.

$\Longrightarrow$. We have $\overline{P}_{((s_1,s_2),(t_1',t_2'))} \not\subseteq \overline{Q}_{((s_1,s_2),(t_1,t_2))}$ so that for some $(u_1, u_2) \in S \times S$, $c \not\subseteq \overline{Q}_{((s_1,s_2),(u_1,u_2))}$, $c \not\subseteq \overline{Q}_{((u_1,u_2),(t_1',t_2'))}$. From here, $P_{(s_2,s_1)} \not\subseteq Q_{(u_1,u_2)}$ and $P_{(u_2,u_1)} \not\subseteq Q_{(t_1',t_2')}$. Hence, $P_{s_2} \not\subseteq Q_{u_1}$, $P_{u_1} \not\subseteq Q_{t_2'}$ which gives $P_{s_2} \not\subseteq Q_{t_2'}$, and $P_{s_1} \not\subseteq Q_{u_2}$, $P_{u_2} \not\subseteq Q_{t_1'}$, which gives $P_{s_1} \not\subseteq Q_{t_1'}$. On the other hand $P_{(t_1',t_2')} \not\subseteq Q_{(t_1,t_2)}$, and we deduce $P_{(s_1,s_2)} \not\subseteq Q_{(t_1,t_2)}$, as required.

$\Longleftarrow$. From $P_{(s_1,s_2)} \not\subseteq Q_{(t_1,t_2)}$ we have $P_{s_k} \not\subseteq Q_{t_k}$, so we may take $u_k \in S$ with $P_{s_k} \not\subseteq Q_{u_k}$, $P_{u_k} \not\subseteq Q_{t_k}$, $k = 1, 2$. Also we may take $u_k' \in S$ satisfying $P_{s_k} \not\subseteq Q_{u_k'}$, $P_{u_k'} \not\subseteq Q_{u_k}$, $k = 1, 2$. We see that $c \not\subseteq \overline{Q}_{((s_1,s_2),(u_2',u_1'))}$ and $c \not\subseteq \overline{Q}_{((u_2',u_1'),(u_1,u_2))}$, whence $\overline{P}_{((s_1,s_2),(u_1,u_2))} \subseteq c \circ c$. But $P_{(u_1,u_2)} \not\subseteq Q_{(t_1,t_2)}$ which gives $c \circ c \not\subseteq \overline{Q}_{((s_1,s_2),(t_1,t_2))}$, as required.

The proof of $C \circ C = I_{S \times S}$ is dual to the above. □

2.3. Lemma. *Let (a, A) be the associativity direlation on $(S, \mathcal{S})$ and $(a, A)^{\leftarrow} = (A^{\leftarrow}, a^{\leftarrow})$ its inverse. Then*

(1) $A^{\leftarrow} = \bigvee \{\overline{P}_{(((s_1,s_2),s_3),(s_1,(s_2,s_3)))} \mid s_1, s_2, s_3 \in S\}$.
(2) $a^{\leftarrow} = \bigcap \{\overline{Q}_{(((s_1,s_2),s_3),(s_1,(s_2,s_3)))} \mid s_1, s_2, s_3 \in S^{\flat}\}$.
(3) $a \circ A^{\leftarrow} = i_{(S \times S) \times S}$ *and* $A \circ a^{\leftarrow} = I_{(S \times S) \times S}$.
(4) $A^{\leftarrow} \circ a = i_{S \times (S \times S)}$ *and* $a^{\leftarrow} \circ A = I_{S \times (S \times S)}$.

Proof. (1) Denote the right hand side by r and suppose first that $A^{\leftarrow} \not\subseteq r$. Then we have $\overline{P}_{((s_1,(s_2,s_3)),((t_1,t_2),t_3))} \not\subseteq A$ with $\overline{P}_{(((t_1,t_2),t_3),(s_1,(s_2,s_3)))} \not\subseteq r$. By the definition of A we have $P_{((t_1,t_2),t_3)} \not\subseteq Q_{((s_1,s_2),s_3)}$, whence $P_{t_k} \not\subseteq Q_{s_k}$ for $k = 1, 2, 3$. Hence $P_{s_k} \subseteq P_{t_k}$, $k = 1, 2, 3$, so $\overline{P}_{(((t_1,t_2),t_3),(s_1,(s_2,s_3)))} \subseteq \overline{P}_{(((t_1,t_2),t_3),(t_1,(t_2,t_3)))} \subseteq r$, which is a contradiction.

Conversely, if $r \not\subseteq A^{\leftarrow}$ we have $\overline{P}_{(((s_1,s_2),s_3),(s_1,(s_2,s_3)))} \not\subseteq \overline{Q}_{(((s_1,s_2),s_3),(t_1,(t_2,t_3)))}$ satisfying $\overline{P}_{(((s_1,s_2),s_3),(t_1,(t_2,t_3)))} \not\subseteq A^{\leftarrow}$. Then $\overline{P}_{((t_1,(t_2,t_3)),((s_1,s_2),s_3))} \not\subseteq A$ since $P_{s_k} \not\subseteq Q_{t_k}$, $k = 1, 2, 3$, and we obtain the contradiction $\overline{P}_{(((s_1,s_2),s_3),(t_1,(t_2,t_3)))} \subseteq A^{\leftarrow}$ by [5, Lemma 2.4 (1)].

(2) Dual to (1).

(3) We need only show $a \circ A^{\leftarrow} = i$, since then $A \circ a^{\leftarrow} = (a \circ A^{\leftarrow})^{\leftarrow} = i^{\leftarrow} = I$.

Suppose first that $a \circ A^{\leftarrow} \not\subseteq i$. Then we have $s_k, t_k \in S$, $k = 1, 2, 3$ so that $a \circ A^{\leftarrow} \not\subseteq \overline{Q}_{(((s_1,s_2),s_3),((t_1,t_2),t_3))}$ and $\overline{P}_{(((s_1,s_2),s_3),((t_1,t_2),t_3)))} \not\subseteq i$. Now we have $t_k' \in S$, $k = 1, 2, 3$, so that $\overline{P}_{(((s_1,s_2),s_3),((t_1',t_2'),t_3'))} \not\subseteq \overline{Q}_{(((s_1,s_2),s_3),((t_1,t_2),t_3))}$ and $u_k \in S$, $k = 1, 2, 3$ with $A^{\leftarrow} \not\subseteq \overline{Q}_{(((s_1,s_2),s_3),(u_1,(u_2,u_3)))}$ and $a \not\subseteq \overline{Q}_{(((u_1,(u_2,u_3)),((t_1',t_2'),t_3'))}$. By (1) and the definition of a we deduce $P_{s_k} \not\subseteq Q_{u_k}$, $P_{u_k} \not\subseteq Q_{t_k'}$ for $k = 1, 2, 3$. Also, $P_{t_k'} \not\subseteq Q_{t_k}$, so $P_{s_k} \not\subseteq Q_{t_k}$ and hence $P_{t_k} \subseteq P_{s_k}$, $k = 1, 2, 3$. But now $\overline{P}_{(((s_1,s_2),s_3),((t_1,t_2),t_3))} \subseteq \overline{P}_{(((s_1,s_2),s_3),((s_1,s_2),s_3))} \subseteq i$, which is a contradiction. This shows that $a \circ A^{\leftarrow} \subseteq i$.

Now suppose that $i \not\subseteq a\circ A^{\leftarrow}$. Then we have $s_k, t_k \in S$ for which $i \not\subseteq \overline{Q}_{(((s_1,s_2),s_3),((t_1,t_2),t_3))}$ and $\overline{P}_{(((s_1,s_2),s_3),((t_1,t_2),t_3))} \not\subseteq a \circ A^{\leftarrow}$. The first gives us $P_{s_k} \not\subseteq Q_{t_k}$, $k = 1,2,3$. Choose $u_k \in S$, $k = 1,2,3$ satisfying $P_{s_k} \not\subseteq Q_{u_k}$ and $P_{u_k} \not\subseteq Q_{t_k}$. Then by (1), $A^{\leftarrow} \not\subseteq \overline{Q}_{(((s_1,s_2),s_3),(u_1,(u_2,u_3)))}$ and by the definition of a, $a \not\subseteq \overline{Q}_{((u_1,(u_2,u_3)),((t_1,t_2),t_3))}$. Hence $\overline{P}_{(((s_1,s_2),s_3),((t_1,t_2),t_3))} \subseteq a \circ A^{\leftarrow}$, which is a contradiction. This verifies that $i \subseteq a \circ A^{\leftarrow}$, as required.

(4) Dual to (3). □

3. Commutativity and associativity of di-operations

Let us begin by making precise the notion of di-operation on a texture $(S, \mathcal{S})$.

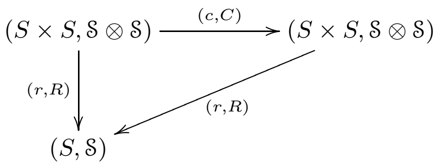
3.1. Definition. Let $(S, \mathcal{S})$ be a texture. Then a difunction $(\square, \Box)$ from $(S \times S, \mathcal{S} \otimes \mathcal{S})$ to $(S, \mathcal{S})$ is called a *(binary) di-operation* on $(S, \mathcal{S})$.

In this section we define the commutativity and associativity of di-operations in terms of the commutativity and associativity direlations. However, the definitions do not rely on the fact that a di-operation is a difunction and so we will define these concepts for general direlations from $(S \times S, \mathcal{S} \otimes \mathcal{S})$ to $(S, \mathcal{S})$.

3.2. Definition. Let (r, R) be a direlation from $(S \times S, \mathcal{S} \otimes \mathcal{S})$ to $(S, \mathcal{S})$. Then

(1) r is *commutative* if $r = r \circ c$.
(2) R is *commutative* if $R = R \circ C$.
(3) (r, R) is *commutative* if r and R are commutative. In particular a binary di-operation $(\square, \Box)$ on $(S, \mathcal{S})$ is commutative if it is commutative as a direlation from $(S \times S, \mathcal{S} \otimes \mathcal{S})$ to $(S, \mathcal{S})$.

In this definition (c, C) is the commutativity direlation on $(S, \mathcal{S})$. The above definition gives rise to the following commutative diagram.



The definition of associativity requires a notion of product for direlations. This is detailed in the next lemma.

3.3. Lemma. *Let $(S_k, \mathcal{S}_k)$, $(T_k, \mathcal{T}_k)$ be textures and (r_k, R_k) direlations from $(S_k, \mathcal{S}_k)$ to $(T_k, \mathcal{T}_k)$, $k = 1, 2$. Then*

(1) $r_1 \times r_2 = \bigvee\{\overline{P}_{((s_1,s_2),(t_1,t_2))} \mid r_k \not\subseteq \overline{Q}_{(s_k,t_k)},\ k = 1,2\}$ *is a relation from* $(S_1 \times S_2, \mathcal{S}_1 \otimes \mathcal{S}_2)$ *to* $(T_1 \times T_2, \mathcal{T}_1 \otimes \mathcal{T}_2)$.
(2) $R_1 \times R_2 = \bigcap\{\overline{Q}_{((s_1,s_2),(t_1,t_2))} \mid \overline{P}_{(s_k,t_k)} \not\subseteq R_k,\ k = 1,2\}$ *is a corelation from* $(S_1 \times S_2, \mathcal{S}_1 \otimes \mathcal{S}_2)$ *to* $(T_1 \times T_2, \mathcal{T}_1 \otimes \mathcal{T}_2)$.
(3) $(r_1, R_1) \times (r_2, R_2) = (r_1 \times r_2, R_1 \times R_2)$ *is a direlation from* $(S_1 \times S_2, \mathcal{S}_1 \otimes \mathcal{S}_2)$ *to* $(T_1 \times T_2, \mathcal{T}_1 \otimes \mathcal{T}_2)$. *In particular, if* (r_k, R_k), $k = 1, 2$ *are difunctions then* $(r_1, R_1) \times (r_2, R_2)$ *is a difunction.*

Proof. Straightforward. □

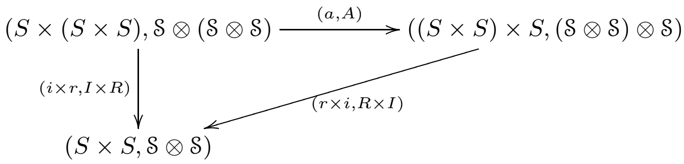
Now we may give:

3.4. Definition. Let (r, R) be a direlation from $(S \times S, \mathcal{S} \otimes \mathcal{S})$ to $(S, \mathcal{S})$. Then

(1) r is called *associative* if $r \circ (i \times r) = r \circ (r \times i) \circ a$.

(2) R is called *associative* if $R \circ (I \times R) = R \circ (R \times I) \circ A$.
(3) (r, R) is called *associative* if r and R are associative. In particular a binary di-operation $(\square,\Box)$ on $(S, \mathcal{S})$ is associative if it is associative as a direlation from $(S \times S, \mathcal{S} \otimes \mathcal{S})$ to $(S, \mathcal{S})$.

In this definition (a, A) is the associativity direlation and (i, I) the identity direlation on $(S, \mathcal{S})$. The associativity of (r, R) may be illustrated by the following commutative diagram.



It will be useful to have point-based characterizations of commutativity and associativity, and these are the subject of the following two theorems.

3.5. Theorem. *Let (r, R) be a direlation from $(S \times S, \mathcal{S} \otimes \mathcal{S})$ to $(S, \mathcal{S})$.*
(1) *r is commutative if and only if*

$$r \not\subseteq \overline{Q}_{((s_1,s_2),s)} \iff r \not\subseteq \overline{Q}_{((s_2,s_1),s)} \ \forall\, s_1, s_2, s \in S.$$

(2) *R is commutative if and only if*

$$\overline{P}_{((s_1,s_2),s)} \not\subseteq R \iff \overline{P}_{((s_2,s_1),s)} \not\subseteq R \ \forall\, s_1, s_2, s \in S.$$

Proof. Straightforward. □

3.6. Theorem. *Let (r, R) be a direlation from $(S \times S, \mathcal{S} \otimes \mathcal{S})$ to $(S, \mathcal{S})$.*
(1) *r is associative if and only if the following are equivalent $\forall\, s_1, s_2, s_3, w \in S$.*
  (i) *There exists $u \in S$ with $r \not\subseteq \overline{Q}_{((s_1,s_2),u)}$ and $r \not\subseteq \overline{Q}_{((u,s_3),w)}$.*
  (ii) *There exists $v \in S$ with $r \not\subseteq \overline{Q}_{((s_2,s_3),v)}$ and $r \not\subseteq \overline{Q}_{((s_1,v),w)}$.*
(2) *R is associative if and only if the following are equivalent $\forall\, s_1, s_2, s_3, w \in S$.*
  (i) *There exists $u \in S$ with $\overline{P}_{((s_1,s_2),u)} \not\subseteq R$ and $\overline{P}_{((u,s_3),w)} \not\subseteq R$.*
  (ii) *There exists $v \in S$ with $\overline{P}_{((s_2,s_3),v)} \not\subseteq R$ and $\overline{P}_{((s_1,v),w)} \not\subseteq R$.*

Proof. We outline the proof of (1), leaving the proof of the dual result (2) to the reader.

Suppose first that $r \circ (r \times i) \circ a \subseteq r \circ (i \times r)$. Given $s_1, s_2, s_3, w \in S$, suppose we have $u \in S$ satisfying $r \not\subseteq \overline{Q}_{((s_1,s_2),u)}$ and $r \not\subseteq \overline{Q}_{((u,s_3),w)}$. It may be verified that $r \circ (r \times i) \circ a \not\subseteq \overline{Q}_{((s_1,(s_2,s_3)),w)}$. Hence, by hypothesis, $r \circ (i \times r) \not\subseteq \overline{Q}_{((s_1,(s_2,s_3)),w)}$. Now we have $w' \in S$ with $P_{w'} \not\subseteq Q_w$ and $t', v' \in S$ satisfying $i \times r \not\subseteq \overline{Q}_{((s_1,(s_2,s_3)),(t',v'))}$ and $r \not\subseteq \overline{Q}_{((t',v'),w')}$. Hence for some $t, v \in S$ with $P_{(t,v)} \not\subseteq Q_{(t',v')}$ we have $i \not\subseteq \overline{Q}_{(s_1,t)}$ and $r \not\subseteq \overline{Q}_{((s_2,s_3),v)}$. We see that $P_{(s_1,v)} \not\subseteq Q_{(t',v')}$, whence $r \not\subseteq \overline{Q}_{((s_1,v),w')}$ by *R1*, and so $r \not\subseteq \overline{Q}_{((s_1,v),w)}$ since $Q_w \subseteq Q_{w'}$. This verifies (ii), and we have established that $(i) \implies (ii)$.

Conversely, suppose that $(i) \implies (ii)$ but that $r \circ (r \times i) \circ a \not\subseteq r \circ (i \times r)$. We have $s_1, s_2, s_3, w \in S$ with $\overline{P}_{((s_1,(s_2,s_3)),w)} \not\subseteq r \circ (i \times r)$, so that $a \not\subseteq \overline{Q}_{((s_1,(s_2,s_3)),((s'_1,s'_2),s'_3))}$, $r \times i \not\subseteq \overline{Q}_{(((s'_1,s'_2),s'_3),(u',t'))}$ and $r \not\subseteq \overline{Q}_{((u',t'),w)}$ for some $s'_1, s'_2, s'_3, u', t' \in S$. We deduce $P_{s_k} \not\subseteq Q_{s'_k}$, $k = 1, 2, 3$, so we may write $r \times i \not\subseteq \overline{Q}_{(((s'_1,s_2),s_3),(u',t'))}$ by definition, and then we have $u, t \in S$ with $P_{(u,t)} \not\subseteq Q_{(u',t')}$, $r \not\subseteq \overline{Q}_{((s'_1,s_2),u)}$ and $i \not\subseteq \overline{Q}_{(s_3,t)}$. This gives $P_{(u,s_3)} \not\subseteq Q_{(u',t')}$, whence $r \not\subseteq \overline{Q}_{((u,s_3),w)}$ by *R1*. By hypothesis we now have $v \in S$

satisfying $r \not\subseteq \overline{Q}_{((s_2,s_3),v)}$ and $r \not\subseteq \overline{Q}_{((s_1',v),w)}$. Now we have $v' \in S$ with $P_{v'} \not\subseteq Q_v$ and $r \not\subseteq \overline{Q}_{((s_2,s_3),v')}$. Also we may take $s_1'' \in S$ satisfying $P_{s_1} \not\subseteq Q_{s_1''}$ and $P_{s_1''} \not\subseteq Q_{s_1'}$, whence $i \not\subseteq \overline{Q}_{(s_1,s_1'')}$. This now gives $\overline{P}_{((s_1,(s_2,s_3)),(s_1'',v'))} \subseteq i \times r$ and so $i \times r \not\subseteq \overline{Q}_{((s_1,(s_2,s_3)),(s_1',v))}$. With $r \not\subseteq \overline{Q}_{((s_1',v),w)}$ this gives the contradiction $\overline{P}_{((s_1,(s_2,s_3)),w)} \subseteq r \circ (i \times r)$, and we have established $r \circ (r \times i) \circ a \subseteq r \circ (i \times r)$.

Using Lemma 2.3 (1) we may verify in the same way that $(ii) \implies (i)$ is equivalent to $r \circ (i \times r) \circ A^{\leftarrow} \subseteq r \circ (r \times i)$, and this is equivalent to $r \circ (i \times r) \subseteq r \circ (r \times i) \circ a$ by Lemma 2.3 (4). This completes the proof of (1). □

4. Operations on direlations and difunctions

Now let (r_k, R_k), $k = 1,2$ be direlations from $(S, \mathcal{S})$ to $(T, \mathcal{T})$ and suppose $(\square, \boxdot)$ is a binary di-operation on $(T, \mathcal{T})$. We wish to apply $(\square, \boxdot)$ to obtain a new direlation from $(S, \mathcal{S})$ to $(T, \mathcal{T})$. We begin by defining a direlation $(r_1, R_1) \cdot (r_2, R_2)$ from $(S, \mathcal{S})$ to $(T \times T, \mathcal{T} \otimes \mathcal{T})$.

4.1. Lemma. *Let (r_k, R_k), $k = 1,2$ be direlations from $(S, \mathcal{S})$ to $(T, \mathcal{T})$. Then*

(1) $r_1 \cdot r_2 = \bigvee\{\overline{P}_{(s,(t_1,t_2))} \mid \exists u \in S$ *with* $P_s \not\subseteq Q_u,\ r_1 \not\subseteq \overline{Q}_{(u,t_1)},\ r_2 \not\subseteq \overline{Q}_{(u,t_2)}\}$ *is a relation from $(S, \mathcal{S})$ to $(T \times T, \mathcal{T} \otimes \mathcal{T})$.*

(2) $R_1 \cdot R_2 = \bigcap\{\overline{Q}_{(s,(t_1,t_2))} \mid \exists u \in S$ *with* $P_u \not\subseteq Q_s,\ \overline{P}_{(u,t_1)} \not\subseteq R_1,\ \overline{P}_{(u,t_2)} \not\subseteq R_2\}$ *is a corelation from $(S, \mathcal{S})$ to $(T \times T, \mathcal{T} \otimes \mathcal{T})$.*

(3) *If (f_k, F_k), $k = 1,2$ are difunctions from $(S, \mathcal{S})$ to $(T, \mathcal{T})$ then the direlation $(f_1, F_1) \cdot (f_2, F_2) = (f_1 \cdot f_2, F_1 \cdot F_2)$ is a difunction from $(S, \mathcal{S})$ to $(T \times T, \mathcal{T} \otimes \mathcal{T})$.*

Proof. Straightforward. □

It will be noted that $(f_1, F_1) \cdot (f_2, F_2)$ is a special case of the difuncton $\langle (f_j, F_j) \rangle$ considered in [6, Theorem 3.10].

4.2. Proposition. *Let (r_k, R_k), $k = 1,2$ be direlations from $(S, \mathcal{S})$ to $(T, \mathcal{T})$ and (c, C) the commutativity direlation on $(T, \mathcal{T})$. Then*

$$(r_2, R_2) \cdot (r_1, R_1) = (c, C) \circ ((r_1, R_1) \cdot (r_2, R_2)).$$

Proof. We establish $r_2 \cdot r_1 = c \circ (r_1 \cdot r_2)$, leaving the dual result $R_2 \cdot R_1 = C \circ (R_1 \cdot R_2)$ to the reader.

First suppose that $r_2 \cdot r_1 \not\subseteq c \circ (r_1 \cdot r_2)$. Then we have $s \in S$, $t_1, t_2 \in T$ with $r_2 \cdot r_1 \not\subseteq \overline{Q}_{(s,(t_1,t_2))}$ and $\overline{P}_{(s,(t_1,t_2))} \not\subseteq c \circ (r_1 \cdot r_2)$. Now we may take $t_1', t_2' \in T$ satisfying $\overline{P}_{(s,(t_1',t_2'))} \not\subseteq \overline{Q}_{(s,(t_1,t_2))}$ for which we have $u \in S$ with $P_s \not\subseteq Q_u$, $r_2 \not\subseteq \overline{Q}_{(u,t_1')}$ and $r_1 \not\subseteq \overline{Q}_{(u,t_2')}$. Finally, take $v_1, v_2 \in T$ satisfying $r_2 \not\subseteq \overline{Q}_{(u,v_1)}$, $P_{v_1} \not\subseteq Q_{t_1'}$ and $r_1 \not\subseteq \overline{Q}_{(u,v_2)}$, $P_{v_2} \not\subseteq Q_{t_2'}$. Now $\overline{P}_{(s,(v_2,v_1))} \subseteq r_1 \cdot r_2$ and so $r_1 \cdot r_2 \not\subseteq \overline{Q}_{(s,(t_2',t_1'))}$. On the other hand, $\overline{P}_{((t_2',t_1'),(t_1',t_2'))} \not\subseteq \overline{Q}_{((t_2',t_1'),(t_1,t_2))}$ and so $c \not\subseteq \overline{Q}_{((t_2',t_1'),(t_1,t_2))}$, which implies that $\overline{P}_{(s,(t_1,t_2))} \subseteq c \circ (r_1 \cdot r_2)$. This contradiction establishes $r_2 \cdot r_1 \subseteq c \circ (r_1 \cdot r_2)$.

To obtain the reverse inclusion, interchange r_1 and r_2 in the above, and compose each side with c. According to [5, Proposition 2.17], we have

$$c \circ (r_1 \cdot r_2) \subseteq c \circ (c \circ (r_2 \cdot r_1)) = (c \circ c) \circ (r_2 \cdot r_1).$$

But $c \circ c = i_{T \times T}$ by Lemma 2.2, and we obtain $c \circ (r_1 \cdot r_2) \subseteq r_2 \cdot r_1$ as required. □

4.3. Proposition. *Let (r_k, R_k), $k = 1,2,3$ be direlations from $(S, \mathcal{S})$ to $(T, \mathcal{T})$ and (a, A) the associativity direlation on $(T, \mathcal{T})$. Then*

$$((r_1, R_1) \cdot (r_2, R_2)) \cdot (r_3, R_3) = (a, A) \circ [(r_1, R_1) \cdot ((r_2, R_2) \cdot (r_3, R_3))].$$

Proof. We verify $(r_1 \cdot r_2) \cdot r_3 = a \circ (r_1 \cdot (r_2 \cdot r_3))$, leaving the proof of the dual result $(R_1 \cdot R_2) \cdot R_3 = A \circ (R_1 \cdot (R_2 \cdot R_3))$ to the reader.

First suppose $(r_1 \cdot r_2) \cdot r_3 \not\subseteq a \circ (r_1 \cdot (r_2 \cdot r_3))$. Then we have $s \in S$, $t'_k \in T$, $k = 1,2,3$ with $(r_1 \cdot r_2) \cdot r_3 \not\subseteq \overline{Q}_{(s,((t'_1,t'_2),t'_3))}$ and $\overline{P}_{(s,((t'_1,t'_2),t'_3))} \not\subseteq a \circ (r_1 \cdot (r_2 \cdot r_3))$. Now we have $t_k \in T$, $k = 1,2,3$ with $\overline{P}_{(s,((t_1,t_2),t_3))} \not\subseteq \overline{Q}_{(s,((t'_1,t'_2),t'_3))}$ for which there exists $u \in S$ satisfying $P_s \not\subseteq Q_u$, $r_1 \cdot r_2 \not\subseteq \overline{Q}_{(u,(t_1,t_2))}$ and $r_3 \not\subseteq \overline{Q}_{(u,t_3)}$. Now we have $v_1, v_2 \in T$ with $\overline{P}_{(u,(v_1,v_2))} \not\subseteq \overline{Q}_{(u,(t_1,t_2))}$ for which there exists $w \in S$ satisfying $P_u \not\subseteq Q_w$, $r_1 \not\subseteq \overline{Q}_{(w,v_1)}$ and $r_2 \not\subseteq \overline{Q}_{(w,v_2)}$. Choose $u' \in S$ satisfying $P_s \not\subseteq Q_{u'}$ and $P_{u'} \not\subseteq Q_u$. Then:

(i) $r_1 \not\subseteq \overline{Q}_{(u',v_1)}$.

(ii) $r_2 \not\subseteq \overline{Q}_{(u,v_2)}$. Choose $v'_2 \in T$ satisfying $r_2 \not\subseteq \overline{Q}_{(u,v'_2)}$ and $\overline{P}_{(u,v'_2)} \not\subseteq \overline{Q}_{(u,v_2)}$.

(iii) $r_3 \not\subseteq \overline{Q}_{(u,t_3)}$. Choose $v_3 \in T$ satisfying $r_3 \not\subseteq \overline{Q}_{(u,v_3)}$ and $\overline{P}_{(u,v_3)} \not\subseteq \overline{Q}_{(u,t_3)}$ and then $v'_3 \in T$ with $r_3 \not\subseteq \overline{Q}_{(u,v'_3)}$ and $\overline{P}_{(u,v'_3)} \not\subseteq \overline{Q}_{(u,v_3)}$.

From (iii) and (ii) we see $\overline{P}_{(u',(v'_2,v'_3))} \subseteq r_2 \cdot r_3$, whence $r_2 \cdot r_3 \not\subseteq \overline{Q}_{(u',(v_2,v_3))}$. Together with (i) this gives $\overline{P}_{(s,(u_1,(u_2,u_3)))} \subseteq r_1 \cdot (r_2 \cdot r_3)$, and hence $r_1 \cdot (r_2 \cdot r_3) \not\subseteq \overline{Q}_{(s,(t_1,(t_2,t_3)))}$. On the other hand $\overline{P}_{((t_1,(t_2,t_3)),((t_1,t_2),t_3))} \subseteq a$. Hence $a \not\subseteq \overline{Q}_{((t_1,(t_2,t_3)),((t'_1,t'_2),t'_3))}$, which leads to the contradiction $\overline{P}_{(s,((t'_1,t'_2),t'_3))} \subseteq a \circ (r_1 \cdot (r_2 \cdot r_3))$. This proves the inclusion

$$(r_1 \cdot r_2) \cdot r_3 \subseteq a \circ (r_1 \cdot (r_2 \cdot r_3)).$$

In view of Lemma 2.3 (1) an exactly analogous argument shows that $r_1 \cdot (r_2 \cdot r_3) \subseteq A^{\leftarrow} \circ ((r_1 \cdot r_2) \cdot r_3)$. Composing each side with a now gives

$$a \circ (r_1 \cdot (r_2 \cdot r_3)) \subseteq (a \circ A^{\leftarrow}) \circ ((r_1 \cdot r_2) \cdot r_3) = i \circ ((r_1 \cdot r_2) \cdot r_3) = (r_1 \cdot (r_2 \cdot r_3)),$$

by Lemma 2.3 (3). Combined with the previous inclusion this gives $(r_1 \cdot r_2) \cdot r_3 = a \circ (r_1 \cdot (r_2 \cdot r_3))$, as required. □

4.4. Proposition. *Let (r_k, R_k) be direlations from $(S_k, \mathcal{S}_k)$ to $(T_k, \mathcal{T}_k)$ and (p_k, P_k) direlations from $(S, \mathcal{S})$ to $(S_k, \mathcal{S}_k)$, $k = 1,2$. Then*

$$[(r_1, R_1) \times (r_2, R_2)] \circ [(p_1, P_1) \cdot (p_2, P_2)] = [(r_1, R_1) \circ (p_1, P_1)] \cdot [(r_2, R_2) \circ (p_2, P_2)].$$

Proof. We establish $(r_1 \times r_2) \circ (p_1 \cdot p_2) = (r_1 \circ p_1) \cdot (r_2 \circ p_2)$, leaving the proof of the dual result $(R_1 \times R_2) \circ (P_1 \cdot P_2) = (R_1 \circ P_1) \cdot (R_2 \circ P_2)$ to the reader.

Suppose $(r_1 \times r_2) \circ (p_1 \cdot p_2) \not\subseteq (r_1 \circ p_1) \cdot (r_2 \circ p_2)$. Then we have $s \in S$, $t_k \in T_k$, $k = 1,2$ with $\overline{P}_{(s,(t_1,t_2))} \not\subseteq (r_1 \circ p_1) \cdot (r_2 \circ p_2)$ and $s_k \in S_k$, $k = 1,2$ so that $p_1 \cdot p_2 \not\subseteq \overline{Q}_{(s,(s_1,s_2))}$ and $r_1 \times r_2 \not\subseteq \overline{Q}_{((s_1,s_2),(t_1,t_2))}$. Now we have $s'_k \in S_k$, $k = 1,2$ with $\overline{P}_{(s,(s'_1,s'_2))} \not\subseteq \overline{Q}_{(s,(s_1,s_2))}$ and $u \in S$ satisfying $P_s \not\subseteq Q_u$ so that $p_k \not\subseteq \overline{Q}_{(u,s'_k)}$, $k = 1,2$. Also we have $v_k \in T_k$, $k = 1,2$ so that $\overline{P}_{((s_1,s_2),(v_1,v_2))} \not\subseteq \overline{Q}_{((s_1,s_2),(t_1,t_2))}$ and $r_k \not\subseteq \overline{Q}_{(s_k,v_k)}$, $k = 1,2$. Since $P_{s'_k} \not\subseteq Q_{s_k}$ we have $r_k \not\subseteq \overline{Q}_{(s'_k,v_k)}$, so $\overline{P}_{(u,v_k)} \subseteq r_k \circ p_k$, $k = 1,2$. But now $\overline{P}_{(s,(t_1,t_2))} \subseteq (r_1 \circ p_1) \cdot (r_2 \circ p_2)$, which is a contradiction.

Now suppose $(r_1 \circ p_1) \cdot (r_2 \circ p_2) \not\subseteq (r_1 \times r_2) \circ (p_1 \cdot p_2)$. Then we have $t_k \in T_k$ with $\overline{P}_{(s,(t_1,t_2))} \not\subseteq (r_1 \times r_2) \circ (p_1 \cdot p_2)$, for which there exists $P_s \not\subseteq Q_u$ so that $r_k \circ p_k \not\subseteq \overline{Q}_{(u,t_k)}$, $k = 1,2$. Now we have $t'_k \in T_k$ with $\overline{P}_{(u,t'_k)} \not\subseteq \overline{Q}_{(u,t_k)}$ for which there exists $s_k \in S_k$ with $p_k \not\subseteq \overline{Q}_{(u,s_k)}$ and $r_k \not\subseteq \overline{Q}_{(s_k,t'_k)}$, $k = 1,2$. Choose $s'_k \in S_k$, $k = 1,2$ satisfying $p_k \not\subseteq \overline{Q}_{(u,s'_k)}$ and $\overline{P}_{(u,s'_k)} \not\subseteq \overline{Q}_{(u,s_k)}$. Now we have $\overline{P}_{(s,(s'_1,s'_2))} \subseteq p_1 \cdot p_2$ and so $p_1 \cdot p_2 \not\subseteq \overline{Q}_{(s,(s_1,s_2))}$. On the other hand $\overline{P}_{((s_1,s_2),(t'_1,t'_2))} \subseteq r_1 \times r_2$ and so $r_1 \times r_2 \not\subseteq \overline{Q}_{((s_1,s_2),(t_1,t_2))}$. But now $\overline{P}_{(s,(t_1,t_2))} \subseteq (r_1 \times r_2) \circ (p_1 \cdot p_2)$, which is a contradiction. □

4.5. Corollary. *Let* (r_k, R_k), $k = 1, 2, 3$ *be direlations from* $(S, \mathcal{S})$ *to* $(T, \mathcal{T})$*,* (i, I) *the identity and* (c, C) *the commutativity direlation on* $(T, \mathcal{T})$*. Then*

$$((i, I) \times (c, C)) \circ ((r_1, R_1) \cdot ((r_2, R_2) \cdot (r_3, R_3))) = (r_1, R_1) \cdot ((r_3, R_3) \cdot (r_2, R_2)),$$
$$((c, C) \times (i, I)) \circ (((r_1, R_1) \cdot (r_2, R_2)) \cdot (r_3, R_3)) = ((r_2, R_2) \cdot (r_1, R_1)) \cdot (r_3, R_3).$$

Proof. $(i \times c) \circ (r_1 \cdot (r_2 \cdot r_3)) = (i \circ r_1) \cdot (c \circ (r_2 \cdot r_3)) = r_1 \cdot (r_3 \cdot r_2)$ by Proposition 4.4, [5, Proposition 2.17 (1)] and Proposition 4.2. The remaining equalities are proved likewise. □

We will also find the following result useful when we come to discuss continuity.

4.6. Lemma. *Let* (r_k, R_k) *be direlations from* $(S, \mathcal{S})$ *to* $(T_k, \mathcal{T}_k)$*,* $k = 1, 2$*. Then, for* $A_1 \in \mathcal{T}_1$*,*

(1) *If* $r_2^{\leftarrow}(\emptyset) = \emptyset$ *then* $(r_1 \cdot r_2)^{\leftarrow}(A_1 \times T_2) = r_1^{\leftarrow}(A_1)$*,*
(2) *If* $R_2^{\leftarrow}(T_2) = S$ *then* $(R_1 \cdot R_2)^{\leftarrow}(A_1 \times T_2) = R_1^{\leftarrow}(A_1)$*.*

Proof. We establish (1), the proof of (2) being dual.

Suppose first that $(r_1 \cdot r_2)^{\leftarrow}(A_1 \times T_2) \not\subseteq r_1^{\leftarrow}(A_1)$. Now we have $s \in S$ with $P_s \not\subseteq r_1^{\leftarrow}(A_1)$ for which $r_1 \cdot r_2 \not\subseteq \overline{Q}_{(s,(t_1,t_2))} \implies P_{(t_1,t_2)} \subseteq A_1 \times T_2 \implies P_{t_1} \subseteq A_1$. Let us take $u \in S$ with $P_s \not\subseteq Q_u$ and $P_u \not\subseteq r_1^{\leftarrow}(A_1)$, whence we have $t_1 \in T_1$ with $r_1 \not\subseteq \overline{Q}_{(u,t_1)}$ and $P_{t_1} \not\subseteq A_1$. Also, $P_u \not\subseteq \emptyset = r_2^{\leftarrow}(\emptyset)$, by hypothesis, so we have $t_2 \in T_2$ satisfying $r_2 \not\subseteq \overline{Q}_{(u,t_2)}$. We may now deduce $r_1 \cdot r_2 \not\subseteq \overline{Q}_{(s,(t_1,t_2))}$, and the above implications now lead to the contradiction $P_{t_1} \subseteq A_1$.

On the other hand, suppose $r_1^{\leftarrow}(A_1) \not\subseteq (r_1 \cdot r_2)^{\leftarrow}(A_1 \times T_2)$. Then we have $s \in S$ with $P_s \not\subseteq (r_1 \cdot r_2)^{\leftarrow}(A_1 \times T_2)$ for which $r_1 \not\subseteq \overline{Q}_{(s,t)} \implies P_t \subseteq A_1$. Now we have $t_k \in T_k$, $k = 1, 2$ with $r_1 \cdot r_2 \not\subseteq \overline{Q}_{(s,(t_1,t_2))}$ and $P_{(t_1,t_2)} \not\subseteq A_1 \times T_2$, i.e. $P_{t_1} \not\subseteq A_1$. Hence we have $t'_k \in T_k$, $k = 1, 2$, with $P_{(t'_1,t'_2)} \not\subseteq Q_{(t_1,t_2)}$ and $r_k \not\subseteq \overline{Q}_{(u,t'_k)}$, $k = 1, 2$. In particular we deduce $r_1 \not\subseteq \overline{Q}_{(s,t_1)}$ and hence the contradiction $P_{t_1} \subseteq A_1$ from the above implication. □

Naturally, the corresponding results for $A_2 \in \mathcal{T}_2$ also hold. If we note that the hypotheses of the above lemma are satisfied for difunctions [5, Proposition 2.28(1c)], while inverse images preserve meet and join, the following corollary is immediate:

4.7. Corollary. *Let* $(f, F), (g, G)$ *be difunctions from* $(S, \mathcal{S})$ *to* $(T_1, \mathcal{T}_1)$*,* $(T_2, \mathcal{T}_2)$*, respectively,* $A \in \mathcal{T}_1$ *and* $B \in \mathcal{T}_2$*. Then,*

(1) $(f \cdot g)^{\leftarrow}((A \times T_2) \cap (T_1 \times B)) = f^{\leftarrow}(A) \cap g^{\leftarrow}(B)$.
(2) $(F \cdot G)^{\leftarrow}((A \times T_2) \cup (T_1 \times B)) = F^{\leftarrow}(A) \cup G^{\leftarrow}(B)$.

Let us now make precise the notion of applying a di-operation to direlations.

4.8. Definition. Let (r_k, R_k), $k = 1, 2$ be direlations from $(S, \mathcal{S})$ to $(T, \mathcal{T})$ and $(\square, \Box)$ a di-operation on $(T, \mathcal{T})$. Then the result of applying $(\square, \Box)$ to (r_1, R_1) and (r_2, R_2) is the direlation $(r_1, R_1)(\square, \Box)(r_2, R_2) = (r_1 \square r_2, R_1 \Box R_2)$ from $(S, \mathcal{S})$ to $(T, \mathcal{T})$ defined by

$$(r_1, R_1)(\square, \Box)(r_2, R_2) = (\square, \Box) \circ ((r_1, R_1) \cdot (r_2, R_2)).$$

When (f, F) and (g, G) are difunctions from $(S, \mathcal{S})$ to $(T, \mathcal{T})$, $(f \square g, F \Box G)$ is also a difunction from $(S, \mathcal{S})$ to $(T, \mathcal{T})$. This follows from Lemma 4.1 (3) and the fact that difunctions are closed under composition ([6], Proposition 2.28(2)).

The following lemma gives formulae for directly calculating $r_1 \square r_2$ and $R_1 \Box R_2$.

4.9. Lemma. *With the notation as in* Definition 4.8*,*

$$r_1 \square r_2 = \bigvee\{\overline{P}_{(s,t)} \mid \exists u \in S, P_s \not\subseteq Q_u \text{ and } t_1, t_2 \in T \text{ with } r_k \not\subseteq \overline{Q}_{(u,t_k)},\ k = 1,2 \text{ and } \square \not\subseteq \overline{Q}_{((t_1,t_2),t)}\},$$

$$R_1 \Box R_2 = \bigcap\{\overline{Q}_{(s,t)} \mid \exists u \in S, P_u \not\subseteq Q_s \text{ and } t_1, t_2 \in T \text{ with } \overline{P}_{(u,t_k)} \not\subseteq R_k,\ k = 1,2 \text{ and } \overline{P}_{((t_1,t_2),t)} \not\subseteq \Box\}.$$

Proof. Immediate ☐

In case $(\square,\Box)$ is commutative or associative we have the following.

4.10. Theorem. *Let $(S,\mathcal{S})$, $(T,\mathcal{T})$ be textures and $(\square,\Box)$ a binary di-operation on $(T,\mathcal{T})$.*

(1) *If $(\square,\Box)$ is commutative then*

$$(r_1,R_1)(\square,\Box)(r_2,R_2) = (r_2,R_2)(\square,\Box)(r_1,R_1)$$

for all direlations (r_k,R_k), $k = 1,2$ from $(S,\mathcal{S})$ to $(T,\mathcal{T})$.

(2) *If $(\square,\Box)$ is associative then*

$$(r_1,R_1)(\square,\Box)((r_2,R_2)(\square,\Box)(r_3,R_3)) = ((r_1,R_1)(\square,\Box)(r_2,R_2))(\square,\Box)(r_3,R_3)$$

for all direlations (r_k,R_k), $k = 1,2,3$ from $(S,\mathcal{S})$ to $(T,\mathcal{T})$.

Proof. (1). By Definition 4.8 we have $r_2 \square r_1 = \square \circ (r_2 \cdot r_1) = \square \circ c \circ (r_1 \cdot r_2)$ by Proposition 4.2, where c is the commutativity relation on $(T,\mathcal{T})$. Since $\square$ is commutative we now have $r_2 \square r_1 = \square \circ (r_1 \cdot r_2) = r_1 \square r_2$, as required.

The proof of $R_2 \Box R_1 = R_1 \Box R_2$ is similar.

(2). Applying Definition 4.8, and letting i be the identity relation on $(T,\mathcal{T})$ we have $(r_1 \square r_2) \square r_3 = \square \circ ((\square \circ (r_1 \cdot r_2)) \cdot (i \circ r_3)) = \square \circ (\square \times i) \circ ((r_1 \cdot r_2) \cdot r_3)$ by Proposition 4.4. If a is the associativity relation on $(T,\mathcal{T})$, Proposition 4.3 now gives $(r_1 \square r_2) \square r_3 = \square \circ (\square \times i) \circ a \circ (r_1 \cdot (r_2 \cdot r_3)) = \square \circ (i \times \square) \circ (r_1 \cdot (r_2 \cdot r_3))$, since $\square$ is associative. Applying Proposition 4.4 again finally gives $(r_1 \square r_2) \square r_3 = \square \circ ((i \circ r_1) \cdot (\square \circ (r_2 \cdot r_3)) = r_1 \square (r_2 \square r_3)$, as required.

The proof of $(R_1 \Box R_2) \Box R_3 = R_1 \Box (R_2 \Box R_3)$ is similar.

☐

Now let (τ_1,κ_1) be a ditopology on $(S_1,\mathcal{S}_1)$ and (τ_2,κ_2) a ditopology on $(S_2,\mathcal{S}_2)$. We denote by (τ_2^2,κ_2^2) the product ditopology [6] on $(S_2 \times S_2, \mathcal{S}_2 \otimes \mathcal{S}_2)$. Hence, a base for τ_2^2 consists of elements of $\mathcal{S}_2 \otimes \mathcal{S}_2$ of the form $(G \times S_2) \cap (S_2 \times H) = G \times H$, $G, H \in \tau_2$, while a base for κ_2^2 consists of elements of the form $(F \times S_2) \cup (S_2 \times K)$, $F, K \in \kappa_2$. Let us first note the following:

4.11. Lemma. *With the notation above, let the difunctions $(f,F),(g,G)$ from $(S_1,\mathcal{S}_1)$ to $(S_2,\mathcal{S}_2)$ be (τ_1,κ_1)–(τ_2,κ_2) bicontinuous. Then $(f,F)\cdot(g,G)$ is (τ_1,κ_1)–(τ_2^2,κ_2^2) bicontinuous.*

Proof. For $G, H \in \tau_2$ we have $(f \cdot g)^{\leftarrow}(G \times H) = (f \cdot g)^{\leftarrow}((G \times S_2) \cap (S_2 \times H)) = f^{\leftarrow}(G) \cap g^{\leftarrow}(H) \in \tau_1$, by Corollary 4.7 (1). Since inverse images preserve join this is sufficient to show (τ_1,κ_1)-(τ_2^2,κ_2^2) continuity. Cocontinuity is proved likewise using Corollary 4.7 (2). ☐

4.12. Definition. The di-operation $(\square,\Box)$ is called *bicontinuous* on $(S_2,\mathcal{S}_2,\tau_2,\kappa_2)$ if it is bicontinuous as a difunction from $(S_2 \times S_2, \mathcal{S}_2 \otimes \mathcal{S}_2, \tau_2^2, \kappa_2^2)$ to $(S_2,\mathcal{S}_2,\tau_2,\kappa_2)$.

The following result in now a trivial consequence of Lemma 4.11 and the fact that the composition of two bicontinuous difunctions is bicontinuous ([7], Lemma 2.3 (2)).

4.13. Theorem. *With the notation as above, let* $(\sqcap, \square)$ *be a bicontinous di-operation on* $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ *and let* (f, F)*,* (g, G) *be bicontinuous difunctions from* $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ *to* $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$*. Then* $(f, F)(\sqcap, \square)(g, G)$ *is also a bicontinuous difunction from* $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ *to* $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$*.*

5. Real di-Operations and real difunctions

In this section we begin by considering certain natural di-operations on the real texture $(\mathbb{R}, \mathcal{R})$. This texture is clearly closed under arbitrary unions, and is therefore a plain texture [5]. It follows that the product texture $(\mathbb{R} \times \mathbb{R}, \mathcal{R} \otimes \mathcal{R})$ is also plain. We require the following result which characterizes difunctions on a plain texture in terms of ordinary (point) functions.

5.1. Theorem. *Let* $(S, \mathcal{S})$ *be a plain texture and* (f, F) *a difunction from* $(S, \mathcal{S})$ *to* $(T, \mathcal{T})$*. Then there exists a point function* φ *from* S *to* T *satisfying the conditions*

(1) $P_{s'} \subseteq P_s \implies P_{\varphi(s)} \not\subseteq Q_{\varphi(s')}$,
(2) $f = \bigvee\{\overline{P}_{(s,\varphi(s))} \mid s \in S\}$, $F = \bigcap\{\overline{Q}_{(s,\varphi(s))} \mid s \in S\}$, *and*
(3) $f^{\leftarrow}(B) = F^{\leftarrow}(B) = \varphi^{-1}(B)$ *for all* $B \in \mathcal{T}$*.*

Conversely, if φ *is any point function from* S *to* T *satisfying* (1)*, then setting* $f = \bigvee\{\overline{P}_{(s,\varphi(s))} \mid s \in S\}$, $F = \bigcap\{\overline{Q}_{(s,\varphi(s))} \mid s \in S\}$ *defines a difunction* (f, F) *satisfying* $f^{\leftarrow}(B) = F^{\leftarrow}(B) = \varphi^{-1}(B)$ *for all* $B \in \mathcal{T}$*.*

Proof. Clear from [5, Proposition 3.7] and [6, Lemma 3.8], since for a plain texture the conditions (b) and (c) mentioned there are automatically satisfied. □

We may apply this theorem to any di-operation $(\sqcap, \square)$ on $(\mathbb{R}, \mathcal{R})$ since this is just a difunction from the plain texture $(\mathbb{R} \times \mathbb{R}, \mathcal{R} \otimes \mathcal{R})$ to $(\mathbb{R}, \mathcal{R})$. Hence, bearing in mind that for the texture $(\mathbb{R} \times \mathbb{R}, \mathcal{R} \otimes \mathcal{R})$ we have $P_{(s_1,s_2)} = \{(r_1, r_2) \mid r_k \leq s_k,\ k = 1, 2\}$, while for $(\mathbb{R}, \mathcal{R})$, $P_s = \{r \mid r \leq s\}$ and $Q_s = \{r \mid r < s\}$, we see that $(\sqcap, \square)$ is equivalent, in the sense described in Theorem 5.1, to a point function $\varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfying the monotonicity property

MP : $s'_k \leq s_k,\ k = 1, 2 \implies \varphi(s'_1, s'_2) \leq \varphi(s_1, s_2)$.

The following relations hold between $(\sqcap, \square)$ and φ.

5.2. Theorem. *Let* $(\sqcap, \square)$ *and* φ *be related as above. Then*

(1) $(\sqcap, \square)$ *is commutative if and only if* φ *is commutative as a binary point operation on* $\mathbb{R}$*.*
(2) $(\sqcap, \square)$ *is associative if and only if* φ *is associative as a binary point operation on* $\mathbb{R}$*.*
(3) *Consider the usual ditopology*

$$\theta = \{(-\infty, s) \mid s \in \mathbb{R}\} \cup \{\mathbb{R}, \emptyset\},\ \phi = \{(-\infty, s] \mid s \in \mathbb{R}\} \cup \{\mathbb{R}, \emptyset\}$$

on $(\mathbb{R}, \mathcal{R})$ *and the product ditopology on* $(\mathbb{R} \times \mathbb{R}, \mathcal{R} \otimes \mathcal{R})$*. Then* $(\sqcap, \square)$ *is bicontinuous if and only if* φ *satisfies the following conditions.*

(a) *If* $s_1, s_2, s \in \mathbb{R}$ *satisfy* $\varphi(s_1, s_2) < s$ *then there exist* $r_1, r_2 \in \mathbb{R}$ *satisfying* $s_k < r_k$*,* $k = 1, 2$*, and* $\varphi(r_1, r_2) < s$*.*
(b) *If* $s_1, s_2, s \in \mathbb{R}$ *satisfy* $\varphi(s_1, s_2) > s$ *then there exist* $r_1, r_2 \in \mathbb{R}$ *satisfying* $s_k > r_k$*,* $k = 1, 2$*, and* $\varphi(r_1, r_2) > s$*.*

Proof. (1) Immediate from Theorem 3.5.

(2) Suppose that φ is associative. We first establish $(i) \implies (ii)$ for $\sqcap$ in Theorem 3.6 (1). Take $s_1, s_2, s_3, w \in \mathbb{R}$ and suppose we have $u \in \mathbb{R}$ with $\sqcap \not\subseteq \overline{Q}_{((s_1,s_2),u)}$ and

$\square \not\subseteq \overline{Q}_{((u,s_3),w)}$. Then $u \le \varphi(s_1,s_2)$ and $w \le \varphi(u,s_3)$. By MP we see $w \le \varphi(u,s_3) \le \varphi(\varphi(s_1,s_2),s_3) = \varphi(s_1,\varphi(s_2,s_3))$, since φ is associative. If we set $v = \varphi(s_2,s_3) \in \mathbb{R}$ we see that $\square \not\subseteq \overline{Q}_{((s_2,s_3),v)}$ and $\square \not\subseteq \overline{Q}_{((s_1,v),w)}$, which verifies (ii). The proof of $(ii) \implies (i)$ is similar, and likewise $(i) \iff (ii)$ for $\Box$ in Theorem 3.6 (2). Hence $(\square,\Box)$ is associative.

Suppose now that $\square$ is associative. If $\varphi(s_1,\varphi(s_2,s_3)) < \varphi(\varphi(s_1,s_2),s_3)$, set $u = \varphi(s_1,s_2) \in \mathbb{R}$ and take $w \in \mathbb{R}$ with $\varphi(s_1,\varphi(s_2,s_3)) < w < \varphi(\varphi(s_1,s_2),s_3)$. Then $\square \not\subseteq \overline{Q}_{((s_1,s_2),u)}$ and $\square \not\subseteq \overline{Q}_{((u,s_3),w)}$, so by Theorem 3.6 (1) there exists $v \in \mathbb{R}$ with $\square \not\subseteq \overline{Q}_{((s_2,s_3),v)}$ and $\square \not\subseteq \overline{Q}_{((s_1,v),w)}$. Now $v \le \varphi(s_2,s_3)$ and $w \le \varphi(s_1,v) \le \varphi(s_1,\varphi(s_2,s_3))$ by MP, which is a contradiction. In the same way $\varphi(\varphi(s_1,s_2),s_3) < \varphi(s_1,\varphi(s_2,s_3))$ also leads to a contradiction, and we have established that φ is associative. We may also establish the associativity of φ from that of $\Box$.

(3) By Theorem 5.1 we need only consider the inverse image with respect to φ. However, (a) is equivalent to

$$(s_1,s_2) \in (-\infty,r_1) \times (-\infty,r_2) \subseteq \varphi^{-1}((-\infty,s)),$$

and hence to the continuity of $(\square,\Box)$. Likewise, (b) is equivalent to

$$(s_1,s_2) \notin ((-\infty,r_1] \times \mathbb{R}) \cup (\mathbb{R} \times (-\infty,r_2]) \supseteq \varphi^{-1}((-\infty,s]),$$

and hence to the cocontinuity of $(\square,\Box)$. $\Box$

We now give the examples of di-operations on $(\mathbb{R},\mathcal{R})$ promised earlier.

5.3. Example. (1) Let $\varphi(s_1,s_2) = s_1+s_2$, $s_1,s_2 \in \mathbb{R}$. Clearly φ satisfies *MP* and is commutative and associative as a binary point operation on $\mathbb{R}$. Also, it is trivial to verify conditions (a) and (b) of Theorem 5.2 (3). Hence, by Theorem 5.1,

$$+ = \bigvee\{\overline{P}_{((s_1,s_2),s_1+s_2)} \mid s_1,s_2 \in \mathbb{R}\},$$
$$+ = \bigcap\{\overline{Q}_{((s_1,s_2),s_1+s_2)} \mid s_1,s_2 \in \mathbb{R}\},$$

define a bicontinous di-operation $(+, +)$ on $(\mathbb{R},\mathcal{R})$.

(2) Let $\varphi(s_1,s_2) = \max(s_1,s_2)$, $s_1,s_2 \in \mathbb{R}$. Clearly φ satisfies *MP* and is commutative and associative as a binary point operation on $\mathbb{R}$. Also, it is trivial to verify conditions (a) and (b) of Theorem 5.2 (3). Hence, by Theorem 5.1,

$$\vee = \bigvee\{\overline{P}_{((s_1,s_2),s_1\vee s_2)} \mid s_1,s_2 \in \mathbb{R}\},$$
$$\vee = \bigcap\{\overline{Q}_{((s_1,s_2),s_1\vee s_2)} \mid s_1,s_2 \in \mathbb{R}\},$$

define a bicontinous di-operation $(\vee, \vee)$ on $(\mathbb{R},\mathcal{R})$.

(3) Let $\varphi(s_1,s_2) = \min(s_1,s_2)$, $s_1,s_2 \in \mathbb{R}$. Clearly φ satisfies *MP* and is commutative and associative as a binary point operation on $\mathbb{R}$. Also, it is trivial to verify conditions (a) and (b) of Theorem 5.2 (3). Hence, by Theorem 5.1,

$$\wedge = \bigvee\{\overline{P}_{((s_1,s_2),s_1\wedge s_2)} \mid s_1,s_2 \in \mathbb{R}\},$$
$$\wedge = \bigcap\{\overline{Q}_{((s_1,s_2),s_1\wedge s_2)} \mid s_1,s_2 \in \mathbb{R}\},$$

define a bicontinous di-operation $(\wedge, \wedge)$ on $(\mathbb{R},\mathcal{R})$.

(4) The point function $\varphi(s_1,s_2) = s_1s_2$ does not define a di-operation on $(\mathbb{R},\mathcal{R})$ in the above sense since φ does not satisfy *MP*.

Now let $(S,\mathcal{S})$ be a texture with ditopology (τ,κ). We denote by $\mathrm{DF}(S,\mathcal{S})$ the set

$$\mathrm{DF}(S,\mathcal{S}) = \{(f,F) \mid (f,F) : (S,\mathcal{S}) \to (\mathbb{R},\mathcal{R}), \text{ is a difunction}\}$$

of *real difunctions* on $(S,\mathcal{S})$, and by $\mathrm{BDF}(S,\mathcal{S},\tau,\kappa)$ the set

$$\mathrm{BDF}(S,\mathcal{S},\tau,\kappa) = \{(f,F) \in \mathrm{DF}(S,\mathcal{S}) \mid (f,F), (\tau,\kappa) - (\theta,\phi) \text{ bicontinuous}\}$$

of *bicontinuous real difunctions* on $(S, \mathcal{S}, \tau, \kappa)$.

If $(\square,\Box)$ is a binary di-operation on $(\mathbb{R}, \mathcal{R}, \theta, \phi)$ then we may apply $(\square,\Box)$ to $(f, F), (g, G)$ in $\mathrm{DF}(S, \mathcal{S})$ to give the element $(f, F)(\square,\Box)(g, G)$ of $\mathrm{DF}(S, \mathcal{S})$. That is, $(\square,\Box)$ induces a binary operation on the set $\mathrm{DF}(S, \mathcal{S})$, which is commutative and associative if and only if $(\square,\Box)$ is. Likewise it induces a binary operation on the set $\mathrm{BDF}(S, \mathcal{S}, \tau, \kappa)$. Moreover, if φ is the point function corresponding to $(\square,\Box)$ as described above, then from Lemma 4.9, Theorem 5.1 and the fact that $(\mathbb{R}\times\mathbb{R}, \mathcal{R}\otimes\mathcal{R})$ is plain, we may easily deduce the following formulae for $f\square g$ and $F\Box G$.

(a) $f\square g = \bigvee\{\overline{P}_{(s,\varphi(r_1,r_2))} \mid \exists P_s \not\subseteq Q_u \text{ with } f \not\subseteq \overline{Q}_{(u,r_1)} \text{ and } g \not\subseteq \overline{Q}_{(u,r_2)}\}$.
(b) $F\Box G = \bigcap\{\overline{Q}_{(s,\varphi(r_1,r_2))} \mid \exists P_u \not\subseteq Q_s \text{ with } \overline{P}_{(u,r_1)} \not\subseteq F \text{ and } \overline{P}_{(u,r_2)} \not\subseteq G\}$.

If we consider the di-operations $(\vee, \bigvee)$, $(\wedge, \bigwedge)$ and $(+, +)$ on the sets $\mathrm{DF}(S, \mathcal{S})$ and $\mathrm{BDF}(S, \mathcal{S}, \tau, \kappa)$ we obtain the following.

5.4. Theorem. *Let $(S, \mathcal{S})$ be a texture and (τ, κ) a ditopology on $(S, \mathcal{S})$. Then*

(1) *The spaces* $(\mathrm{DF}(S, \mathcal{S}), (\wedge, \bigwedge), (\vee, \bigvee))$ *and* $(\mathrm{BDF}(S, \mathcal{S}, \tau, \kappa), (\wedge, \bigwedge), (\vee, \bigvee))$ *are distributive lattices.*
(2) *For all (f, F), (g, G), (h, H) in* $\mathrm{DF}(S, \mathcal{S})$ *or* $\mathrm{BDF}(S, \mathcal{S}, \tau, \kappa)$ *we have*
 (i) $(f + (g \wedge h), F + (G \wedge H)) = ((f + g) \wedge (f + h), (F + G) \wedge (F + H))$, *and*
 (ii) $(f + (g \vee h), F + (G \vee H)) = ((f + g) \vee (f + h), (F + G) \vee (F + H))$.

Proof. (1). Bearing in mind that the di-operations $(\vee, \bigvee)$ and $(\wedge, \bigwedge)$ are commutative and associative, it will be sufficient to verify the following equalities and define the required partial order $\le$ on $\mathrm{DF}(S, \mathcal{S})$ or $\mathrm{BDF}(S, \mathcal{S}, \tau, \kappa)$ by one of the equivalent conditions $(f, F) \le (g, G) \iff (f \wedge g, F \wedge G) = (f, F)$ or $(f, F) \le (g, G) \iff (f \vee g, F \vee G) = (g, G)$:

(i) $(f \wedge f, F \wedge F) = (f, F)$ and $(f \vee f, F \vee F) = (f, F)$.
(ii) $(f \wedge (f \vee g), F \wedge (F \vee G)) = (f, F)$.
(iii) $(f \vee (g \wedge h), F \vee (G \wedge H)) = ((f \vee g) \wedge (f \vee h), (F \vee G) \wedge (F \vee H))$.

Here, (f, F), (g, G) and (h, H) are arbitrary elements of the space concerned. We will verify (ii), leaving the remaining equalities to the interested reader.

Firstly, $f \wedge (f \vee g) \subseteq f$ is trivial, so suppose $f \not\subseteq f \wedge (f \vee g)$. Then we have $s \in S$ and $r_1 \in \mathbb{R}$ satisfying $f \not\subseteq \overline{Q}_{(s,r_1)}$ and $\overline{P}_{(s,r_1)} \not\subseteq f \wedge (f \vee g)$. By *R2* we have $u \in S$ with $P_s \not\subseteq Q_u$ and $f \not\subseteq \overline{Q}_{(u,r_1)}$. Take $u' \in S$ with $P_s \not\subseteq Q_{u'}$ and $P_{u'} \not\subseteq Q_u$. Since $g^{\leftarrow}(\emptyset) = \emptyset$ we have $r_2 \in \mathbb{R}$ with $g \not\subseteq \overline{Q}_{(u,r_2)}$ and $P_{r_1} \ne \emptyset$ so by formula (a) above for $\square = \vee$ we have $\overline{P}_{(u',r_1\vee r_2)} \subseteq f \vee g$, whence $f \vee g \not\subseteq \overline{Q}_{(u',r_1\vee r_2)}$ since $(\mathbb{R}, \mathcal{R})$ is plain. On the other hand $f \not\subseteq \overline{Q}_{(u',r_1)}$ by *R1*, so by formula (a) above for $\square = \wedge$ we have $\overline{P}_{(s,r_1\wedge(r_1\vee r_2))} \subseteq f \wedge (f \vee g)$. Since $r_1 \wedge (r_1 \vee r_2) = r_1$ we obtain the contradiction $\overline{P}_{(s,r_1)} \subseteq f \wedge (f \vee g)$.

This verifies $f = f \wedge (f \vee g)$, and the proof of $F = F \wedge (F \vee G)$ is dual to this. This establishes (ii) as required.

(2). Much as in the proof of (ii) above, this reduces to the equalities $r + (r_1 \wedge r_2) = (r + r_1) \wedge (r + r_2)$ and $r + (r_1 \vee r_2) = (r + r_1) \vee (r + r_2)$, which hold trivially in $\mathbb{R}$. $\Box$

The lattice $(\mathrm{BDF}(S, \mathcal{S}, \tau, \kappa), (\wedge, \bigwedge), (\vee, \bigvee))$ has already found applications in the work of F. Yıldız on real dicompactness of ditopological texture spaces [9], see also, for example [11]. When $(S, \mathcal{S})$ is plain, Theorem 5.1 may be used to represent the elements of $\mathrm{DF}(S, \mathcal{S})$ and of $\mathrm{BDF}(S, \mathcal{S}, \tau, \kappa)$ as real-valued point functions on S. The reader is referred to [10] for a discussion of the general case.

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