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IFP IDEALS IN NEAR-RINGS

Akın Osman Atagün*

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Abstract

A near-ring N is called an IFP near-ring provided that for all $a, b, n \in N$, $ab = 0$ implies $anb = 0$. In this study, the IFP condition in a near-ring is extended to the ideals in near-rings. If N/P is an IFP near-ring, where P is an ideal of a near-ring N, then we call P as the IFP-ideal of N. The relations between prime ideals and IFP-ideals are investigated. It is proved that a right permutable or left permutable equiprime near-ring has no non-zero nilpotent elements and then it is established that if N is a right permutable or left permutable finite near-ring, then N is a near-field if and only if N is an equiprime near-ring. Also, attention is drawn to the fact that the concept of IFP-ideal occurs naturally in some near-rings, such as p-near-rings, Boolean near-rings, weakly (right and left) permutable near-rings, left (right) self distributive near-rings, left (right) strongly regular near-rings and left (w-) weakly regular near-rings.

Keywords: Near-ring, Prime ideal, IFP, Nilpotent element, Equiprime near-ring.

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1. Introduction

IFP near-rings have been studied by several authors since they were introduced in [2, 3, 12] and [14]. In this study, the IFP property in near-rings is broadened to the ideals of near-rings and these ideals, named IFP-ideals, of some certain classes of near-rings are considered. It is pointed out that a right permutable or left permutable equiprime near-ring has no non-zero nilpotent elements (Proposition 3.2). In [16], it was showed that if N is a finite near-ring, then N is a near-field if and only if N is an equiprime near-ring, and has no non-zero nilpotent elements. Using this result, we have that if N is a right permutable or left permutable finite near-ring, then N is a near-field if and only if N is an equiprime near-ring (Corollary 3.3). Besides, it is proved that every completely (semi) prime ideal of a zero-symmetric near-ring is an IFP-ideal and using this result some conclusions are obtained concerning under what conditions 0-prime and 3-(semi)

*Department of Mathematics, Bozok University, 66100 Yozgat, Turkey.
E-mail: `aosman.atagun@bozok.edu.tr`

prime ideals are IFP-ideals. Finally, some certain near-ring classes that bear the property of IFP-ideal on itself naturally are given.

2. Preliminary definitions and results

Throughout N will denote a right near-ring. It is assumed that the reader is familiar with the basic definitions of right near-ring, zero-symmetric near-ring, and ideal. (cf. [14]).

2.1. Definition. N is said to fulfil the *insertion-of-factors property* (IFP) provided that for all $a, b, n \in N$: $ab = 0$ implies $anb = 0$. Such near-rings are called IFP near-rings. If $P \lhd N$ and N/P is an IFP near-ring, then the ideal P is called an *IFP-ideal* of N.

The ideal P of N is called a *0-prime ideal* if for every $A, B \lhd N$, $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$. $P \lhd N$ is called a *3-prime* (*3-semiprime*) *ideal* if for $a, b \in N$, $aNb \subseteq P$ ($aNa \subseteq P$) implies $a \in P$ or $b \in P$ ($a \in P$) [10]. If for $a, b \in N$, $ab \in P$ ($a^2 \in P$) implies $a \in P$ or $b \in P$ ($a \in P$), then $P \lhd N$ is called a *completely prime* (*completely semi prime*) *ideal* [11]. $P \lhd N$ is called an *equiprime ideal* if $a \in N - P$ and $x, y \in N$ such that $anx - any \in P$ for all $n \in N$, then $x - y \in P$ [8, Proposition 2.2]. If the zero ideal of N is v-prime ($v = 0$, 3, completely, equi), then N is called a *v-prime near-ring*. It is easily seen that any completely prime near-ring has no non-zero nilpotent elements.

If for all $a, b, c, d \in N$, $abc = acb$ (resp. $abc = bac$, $abcd = acbd$), then N is called a *right permutable* (resp. *left permutable, medial*) *near-ring* [4]. If $abc = abac$ (resp.$abc = acbc$), then N is called a *left self distributive* (resp. *right self distributive*) *near-ring* [5].

For definitions of strongly regular near-ring, reduced near-ring, Boolean near-ring and p-near-ring the reader is referred to [14]. For left (w-) weakly regular near-ring we refer to [1] and [9]. For left (right) strongly regular near-rings we refer to [13].

3. Prime ideals and IFP ideals

3.1. Lemma. [16, Corollary 4.5] *Let N be a zero-symmetric finite near-ring. Then N is a near-field if and only if N is equiprime and has no non-zero nilpotent elements.*

3.2. Proposition. (cf. [5]) *If N is a right (or left) permutable 3-prime near-ring, then N has no non-zero nilpotent elements.*

Proof. Let $ab = 0$ for $a, b \in N$. If N is right permutable (resp. left permutable), then $abN = aNb = 0$ (resp. $Nab = aNb = 0$). Since N is 3-prime, then $a = 0$ or $b = 0$, i.e. N is a completely prime near-ring. Hence N has no non-zero nilpotent elements. □

Since equiprimeness implies 3-primeness in near-rings, we have;

3.3. Corollary. *Let N be a finite near-ring.*

a) *If N is a zero symmetric right permutable near-ring, then N is a near-field if and only if N is an equiprime near-ring.*
b) *If N is a left permutable near-ring, then N is a near-field if and only if N is an equiprime near-ring.*

Proof. If N is left permutable, then for all $n \in N$ $n0 = n00 = 0n0 = 0$, i.e. N is zero-symmetric. Hence the result follows from Lemma 3.1 and Proposition 3.2. □

3.4. Proposition. *If P is an IFP-ideal and a 3-(semi) prime ideal of N, then P is a completely (semi) prime ideal.*

Proof. Let $ab \in P$ for $a, b \in N$. Since P is an IFP-ideal, then $aNb \subseteq P$. Hence $a \in P$ or $b \in P$, since P is a 3-prime ideal. Therefore P is a completely prime ideal. To prove the semiprime case, it is enough to take $a = b$. □

From now on, all near-rings will be zero-symmetric in this section.

3.5. Proposition. *Let P be a completely semiprime ideal of N. Then P is an IFP-ideal.*

Proof. Assume P is a completely semiprime ideal of N and $ab \in P$ for $a, b \in N$. It is easily seen that $NP \subseteq P$, since N is zero-symmetric. Then $(ba)^2 = baba \in NPN \subseteq P$ and then $ba \in P$ since P is completely semiprime. Hence $(anb)^2 = anbanb \in NPN \subseteq P$ for all $n \in N$, whence $anb \in P$ since P is completely semiprime. Therefore P is an IFP-ideal. □

3.6. Corollary. *Let P be a completely prime ideal of N. Then P is an IFP-ideal.*

Proof. If P is a completely prime ideal of N, then it is completely semiprime. Hence the result follows Proposition 3.5. □

3.7. Remark. The relation between completely (semi) prime ideals and IFP-ideals that is seen to hold naturally without the imposition of additional conditions, does not hold between 3-prime ideals and IFP-ideals because there are examples of near-rings which are IFP but not 3-prime [6, Example 1.3].

But we have the following:

3.8. Proposition. *Let N be a medial near-ring and P a 3-prime ideal of N. Then P is an IFP-ideal.*

Proof. If N is a medial near-ring and P is a 3-prime ideal of N, then P is completely prime by [4, Proposition 2.7]. Hence the result follows from Corollary 3.6. □

Since in near-rings 3-primeness implies 0-primeness, we see from Remark 3.7 that there is no relation between 0-prime ideals and IFP-ideals without imposing extra conditions. We have the following:

3.9. Proposition. *Let N be a reduced near-ring and let $P \lhd N$. Then*

a) *If P is a minimal 0-prime ideal, then P is an IFP-ideal.*
b) *If $N \in N_{pm} = \{N :$ every prime ideal of N is maximal$\}$ and P is a 0-prime ideal, then P is an IFP-ideal.*

Proof. The proof of a) is seen by [7, Corollary 2.2] and Corollary 3.6. The proof of b) follows from [7, Corollary 2.6] and Corollary 3.6. □

3.10. Proposition. *Let N be a strongly regular near-ring and $P \lhd N$. If P is a 0-prime ideal, then P is an IFP-ideal.*

Proof. The result follows from [1, Lemma 4.8] and Proposition 3.5. □

4. IFP ideals occurring naturally in some near-rings

The class consisting of zero-symmetric near-rings (resp., consisting of the near-rings with identity) will be denoted by R_o(resp., R_1).

4.1. Proposition. *Let $N \in R_o$ be a Boolean near-ring and $P \lhd N$. Then P is an IFP-ideal.*

Proof. Assume $ab \in P$ for $a, b \in N$. Since $ba = (ba)^2 = baba \in NPN \subseteq P$, then $anb = (anb)^2 = anbanb \in NPN \subseteq P$ for all $n \in N$. Therefore P is an IFP-ideal of N. □

4.2. Proposition. *Let $N \in R_o$ be a p-near-ring and $P \lhd N$. Then P is an IFP-ideal.*

Proof. If $p = 2$, the result follows Proposition 4.1. Assume $p > 2$ and that $ab \in P$ for $a, b \in N$. Since $ba = (ba)^p \in NPN \subseteq P$, then $anb = (anb)^p \in NPN \subseteq P$ for all $n \in N$. Therefore P is an IFP-ideal of N. □

4.3. Remark. A near-ring N has the strong IFP if every homomorphic image of N has the IFP [14, p.288]. Plasser [15] obtained that N has the strong IFP iff for all $I \lhd N$ and for all $a, b, n \in N$, $ab \in I$ implies $anb \in I$. Then, every ideal of an IFP near-ring is an IFP-ideal iff N has the strong IFP.

We have the following:

4.4. Proposition. *Let N be an IFP near-ring. Then for all $x \in N$, $(0 : x)$ is an IFP-ideal of N.*

Proof. By [14, Proposition 9.3], $(0 : x) \lhd N$ for all $x \in N$ when N is an IFP near-ring. Let $ab \in (0 : x)$ for $a, b \in N$. Then $abx = 0$. Since N is an IFP near-ring, then $an(bx) = 0$ for all $n \in N$. Hence $anb \in (0 : x)$ for all $n \in N$, i.e. $(0 : x)$ is an IFP-ideal of N. □

4.5. Proposition. *If P is an IFP-ideal of a near-ring N, then $(P : P)$ is an ideal of N. Furthermore $(P : P)$ is also an IFP-ideal.*

Proof. To prove $(P : P) \lhd N$, it is enough to show that $(P : P)N \subseteq (P : P)$. Let $y \in (P : P)N$. Then there exist an $a \in (P : P)$ and an $n \in N$ such that $y = an$. Since $a \in (P : P)$, then $ap \in P$ for all $p \in P$. Since P is an IFP-ideal, then $anp \in P$ for all $n \in N$. Then $yp \in P$ for all $p \in P$. Hence $y \in (P : P)$. Now, we show that $(P : P)$ is an IFP-ideal. Assume $xy \in (P : P)$ for $x, y \in N$. Then $xyp \in P$ for all $p \in P$. Since P is an IFP-ideal, $xnyp \in P$ for all $n \in N$ and for all $p \in P$. Therefore, $xny \in (P : P)$, which completes the proof. □

4.6. Proposition. *Let $N \in R_o \cap R_1$ be a reduced left (w-) weakly regular near-ring and let $P \lhd N$. Then P is an IFP-ideal of N.*

Proof. The result follows [9, Lemma 3 and Corollary 1] and Proposition 3.5. □

4.7. Proposition. *Let $P \lhd N$. Then,*

a) *If N is right permutable, then P is an IFP-ideal.*
b) *If N is left permutable, then P is an IFP-ideal.*
c) *If N is right self distributive, then P is an IFP-ideal.*
d) *If $N \in R_o$ is left self distributive, then P is an IFP-ideal.*

Proof. For $a, b \in N$, assume $ab \in P$. Then for all $n \in N$;

a) $anb = abn \in PN \subseteq P$.

b) $anb = nab \in NP \subseteq P$, since $N \in R_o$ by the proof of Corollary 3.3.

c) $anb = abnb \in PN \subseteq P$.

d) $anb = anab \in NP \subseteq P$, since $N \in R_o$. □

4.8. Proposition. *Let N be a medial near-ring and $P \lhd N$. Then,*

a) *If N is regular, then P is an IFP-ideal.*
b) *If N is right strongly regular, then P is an IFP-ideal.*
c) *If $N \in R_o$ is left strongly regular, then P is an IFP-ideal.*

Proof. For $x, y \in N$, assume $xy \in P$.

a) Since N is regular, then there exist $a, b \in N$ such that $x = xax$ and $y = yby$. Then for all $n \in N$, $xny = xaxnyby = x(axn)y(by)$. Since N is medial, then $xny = xy(axn)by \in PN \subseteq P$.

b) Since N is right strongly regular, then there exist $a, b \in N$ such that $x = x^2a$ and $y = y^2b$. Then for all $n \in N$, $xny = xxanyyb = x(xan)y(yb)$. Since N is medial, then $xny = xy(xan)(yb) \in PN \subseteq P$.

c) Since $N \in R_o$ is left strongly regular, then there exist $a, b \in N$ such that $x = ax^2$ and $y = by^2$. Then for all $n \in N$, $xny = axxnbyy = (ax)(xnb)y(y)$. Since N is medial, then $xny = (ax)y(xnb)y = a(xy)xnby \in NPN \subseteq P$. □

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