On coefficient estimates for a certain class of starlike functions

Ravinder Krishna Raina[*] and Janusz Sokół[†‡]

Abstract

The purpose of this paper is to consider coefficient estimates in a class of functions $S^*(q)$ consisting of analytic functions f normalized by $f(0) = f'(0) - 1 = 0$ in the open unit disk $\mathbb{U}$ which satisfies the subordination condition that

$$zf'(z)/f(z) \prec q(z), \quad z \in \mathbb{U},$$

where $q(z) = \sqrt{1 + z^2} + z$.

1. Introduction

Let $\mathcal{H}$ denote the class of analytic functions in the open unit disc $\mathbb{U} = \{z : |z| < 1\}$ on the complex plane $\mathbb{C}$. Also, let $\mathcal{A}$ denote the subclass of $\mathcal{H}$ comprising of functions f normalized by $f(0) = 0$, $f'(0) = 1$, and let $S \subset \mathcal{A}$ denote the class of functions which are univalent in $\mathbb{U}$. We say that an analytic function f is subordinate to an analytic function g, and write $f(z) \prec g(z)$, if and only if there exists a function ω, analytic in $\mathbb{U}$ such that $\omega(0) = 0$, $|\omega(z)| < 1$ for $|z| < 1$ and $f(z) = g(\omega(z))$. In particular, if g is univalent in $\mathbb{U}$, then we have the following equivalence:

$$f(z) \prec g(z) \iff f(0) = g(0) \text{ and } f(|z| < 1) \subset g(|z| < 1).$$

―――――――

[*]M.P. University of Agriculture and Technology, Udaipur, India, 10/11 Ganpati Vihar, Opposite Sector 5, Udaipur 313002, Rajasthan, India
Email : rkraina_7@hotmail.com

[†]Department of Mathematics, Rzeszów University of Technology, al. Powstańców Warszawy 12, 35-959 Rzeszów, Poland
Email : jsokol@prz.edu.pl

[‡]Corresponding Author.

Let a function f be analytic univalent in the unit disc $\mathbb{U} = \{z : |z| < 1\}$ on the complex plane $\mathbb{C}$ with the normalization $f(0) = 0$, then f maps $\mathbb{U}$ onto a starlike domain with respect to $w_0 = 0$ if and only if

$$(1.1) \qquad \mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \quad (z \in \mathbb{U}).$$

It is well known that if an analytic function f satisfies (1.1) and $f(0) = 0$, $f'(0) \neq 0$, then f is univalent and starlike in $\mathbb{U}$.

A set E is said to be convex if and only if it is starlike with respect to each of its points, that is if and only if the linear segment joining any two points of E lies entirely in E. Let f be analytic and univalent in $\mathbb{U}_r = \{z : |z| < r \leq 1\}$. Then f maps $\mathbb{U}_r$ onto a convex domain E if and only if

$$\mathfrak{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0 \quad (z \in \mathbb{U}_r).$$

If $r = 1$, then the function f is said to be convex in $\mathbb{U}$ (or briefly convex). The set of all functions $f \in \mathcal{A}$ that are starlike univalent in $\mathbb{U}$ will be denoted by $\mathcal{S}^*$ and the set of all functions $f \in \mathcal{A}$ that are convex univalent in $\mathbb{U}$ by $\mathcal{K}$.

1. Definition. [8] Let $\mathcal{S}^*(q)$ denote the class of analytic functions f in the unit disc $\mathbb{U}$ normalized by $f(0) = f'(0) - 1 = 0$ and satisfying the condition that

$$(1.2) \qquad \frac{zf'(z)}{f(z)} \prec \sqrt{1 + z^2} + z =: q(z), \quad z \in \mathbb{U},$$

where the branch of the square root is chosen to be $q(0) = 1$.

We now mention some geometrical facts of curves defined in the open unit disk. For instance, the function $w(z) = \sqrt{1 + z}$ maps $\mathbb{U}$ onto a set bounded by a Bernoulli lemniscate, and a corresponding class of functions $f \in \mathcal{A}$ such that $zf'(z)/f(z) \prec \sqrt{1 + z}$ was considered in [10], while the class generated by the subordination that $zf'(z)/f(z) \prec \sqrt{1 + cz}$ was considered in [1]. This way the known class of k-starlike functions was seen to be connected with certain conic domains. For some recent results for k-starlike functions, we refer to [11]. In recent papers [2, 3, 4, 5, 6],certain function classes were considered and were defined by means of the subordination that $zf'(z)/f(z) \prec \widehat{q}(z)$, where $\widehat{q}(z)$ was not univalent and this obviously made the consideration of certain geometric properties for such classes of functions much more difficult. It may be noted from (1.2) of Definition 1 that the set $q(\mathbb{U})$ lies in the right half-plane and it is not a starlike domain with respect to the origin, see Fig. 1 (below).
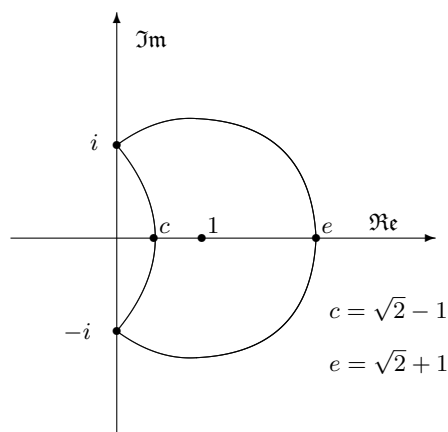
Fig. 1. The boundary of the set $q(\mathbb{U})$.

2. Main result

We first note the following result:

2.1. Lemma. [8] $\mathcal{S}^*(q) \subset \mathcal{S}^*$.

Therefore, if $f \in \mathcal{S}^*(q)$ and

$$(2.1) \qquad f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathbb{U}),$$

then $|a_n| \leq n$.

In this paper, we shall find estimations of first few coefficients of functions f of the form (2.1) belonging to $\mathcal{S}^*(q)$ and also consider the estimations of the familiar functionals like $|a_3 - \lambda a_2^2|$ and $|a_2 a_4 - a_3^2|$.

2.1. Theorem. Let the function f defined by (2.1) belong to the class $\mathcal{S}^*(q)$, then

$$(2.2) \qquad |a_2| \leq 1, \quad |a_3| \leq 3/4, \quad |a_4| \leq 1/2.$$

Proof. Since the function f defined by (2.1) belongs to the class $\mathcal{S}^*(q)$, therefore from (1.2), we have

$$(2.3) \qquad zf'(z) - \omega(z)f(z) = f(z)\sqrt{\omega^2(z) + 1},$$

where ω is such that $\omega(0) = 0$ and $|\omega(z)| < 1$ for $|z| < 1$.

Let us denote the function $\omega(z)$ by

$$(2.4) \qquad \omega(z) = \sum_{k=1}^{\infty} c_k z^k.$$

Then, (2.3) and (2.4) readily give

$$(2.5) \qquad \sqrt{\omega^2(z) + 1} = 1 + \frac{1}{2}c_1^2 z^2 + c_1 c_2 z^3 + \left(c_1 c_3 + \frac{1}{2}c_2^2 - \frac{1}{8}c_1^2\right) z^4 + \cdots$$

and

$$(2.6) \qquad f(z)\sqrt{\omega^2(z) + 1} = z + a_2 z^2 + \left(\frac{1}{2}c_1^2 + a_3\right) z^3 + \left(c_1 c_2 + \frac{1}{2}c_1^2 a_2 + a_4\right) z^4 + \cdots.$$

Moreover,

(2.7)
$$zf'(z)-\omega(z)f(z) = z+(2a_2-c_1)z^2+(3a_3-c_1a_2-c_2)z^3+(4a_4-c_1a_3-c_2a_2-c_3)z^4+\cdots .$$

Equating now the second, third and fourth coefficients in (2.6) and (2.7), we have

(i) $a_2 = 2a_2 - c_1$,
(ii) $\frac{1}{2}c_1^2 + a_3 = 3a_3 - c_1a_2 - c_2$,
(iii) $c_1c_2 + \frac{1}{2}c_1^2a_2 + a_4 = 4a_4 - c_1a_3 - c_2a_2 - c_3$.

From (i), we get

(2.8) $\quad a_2 = c_1$.

It is well known that the coefficients of the bounded function $\omega(z)$ satisfies the inequality that $|c_k| \leq 1$, so from (2.8), we have the first inequality that $|a_2| \leq 1$. Now, from (ii), we have

$$
\begin{aligned}
|2a_3| &= \left|\frac{1}{2}c_1^2 + c_1a_2 + c_2\right| \\
&= \left|\frac{1}{2}c_1^2 + c_1^2 + c_2\right| \\
&= \left|c_2 + \frac{3}{2}c_1^2\right|.
\end{aligned}
$$
(2.9)

Using the estimate (see [7]) that if $\omega(z)$ has the form (2.4), then

(2.10) $\quad |c_2 - \mu c_1^2| \leq \max\{1, |\mu|\}, \quad$ for all $\mu \in \mathbb{C}$,

and we obtain from (2.9) and (2.10) that

$$|2a_3| \leq \frac{3}{2},$$

which gives the second inequality that $|a_3| \leq 3/4$. Also, from $(i)-(iii)$, we find that

$$
\begin{aligned}
|3a_4| &= \left|c_1a_3 + c_2a_2 + c_3 + c_1c_2 + \frac{1}{2}c_1^2a_2\right| \\
&= \left|c_1\left(\frac{3}{4}c_1^2 + \frac{1}{2}c_2\right) + c_2c_1 + c_3 + c_1c_2 + \frac{1}{2}c_1^3\right| \\
&= \left|\frac{5}{4}c_1^3 + \frac{5}{2}c_1c_2 + c_3\right| \\
&= \left|\frac{5}{4}\left(c_1^3 + 2c_1c_2 + c_3\right) - \frac{1}{4}c_3\right| \\
&\leq \left|\frac{5}{4}\left(c_1^3 + 2c_1c_2 + c_3\right)\right| + \left|\frac{1}{4}c_3\right| \\
&\leq \left|\frac{5}{4}\left(c_1^3 + 2c_1c_2 + c_3\right)\right| + \frac{1}{4}.
\end{aligned}
$$
(2.11)

Next, we establish some properties of c_k involved in (2.4). It is known that the function $p(z)$ given by

(2.12) $\quad \dfrac{1+\omega(z)}{1-\omega(z)} = 1 + p_1z + p_2z^2 + \cdots =: p(z)$

defines a Caratheodory function with the property that $\mathfrak{Re}\{p(z)\} > 0$ in $\mathbb{U}$ and that $|p_k| \leq 2$ $(k = 1, 2, 3, \ldots)$. Equating of the coefficients in (2.12) yields that

$$p_2 = 2(c_1^2 + c_2)$$

and

$$p_3 = 2(c_1^3 + 2c_1c_2 + c_3).$$

Hence

(2.13) $|c_1^2 + c_2| \leq 1$

and

(2.14) $|c_1^3 + 2c_1c_2 + c_3| \leq 1.$

By applying (2.11) and (2.14), we find that

$$
\begin{aligned}
|3a_4| &\leq \left| \frac{5}{4} \left(c_1^3 + 2c_1c_2 + c_3 \right) \right| + \frac{1}{4} \\
&\leq \frac{5}{4} + \frac{1}{4} \\
&= \frac{3}{2},
\end{aligned}
$$

which gives the third inequality that $|a_4| \leq 1/2$. $\square$

2.2. Theorem. If the function defined by (2.1) belongs to the class $\mathcal{S}^*(q)$, then

(2.15) $|a_3 - \lambda a_2^2| \leq \max\{1/2, |\lambda - 3/4|\}$ $(\lambda \in \mathbb{C}).$

Furthermore, (2.15) is sharp.

Proof. Applying the notations used in the proof of Theorem 2.1, we obtain from (2.8) and (2.9) that

(2.16) $|a_3 - \lambda a_2^2| = \left| \frac{1}{2}c_2 + \frac{3}{4}c_1^2 - \lambda c_1^2 \right| = \left| \frac{1}{2}c_2 - \left(\lambda - \frac{3}{4} \right) c_1^2 \right|.$

In view of (2.10), we have then

$$
\begin{aligned}
|a_3 - \lambda a_2^2| &= \left| \frac{1}{2}c_2 - \left(\lambda - \frac{3}{4} \right) c_1^2 \right| \\
&= \frac{1}{2} \left| c_2 - \left(\frac{4\lambda - 3}{2} \right) c_1^2 \right| \\
&\leq \frac{1}{2} \max \left\{ 1, \left| \frac{4\lambda - 3}{2} \right| \right\} \\
&= \max\{1/2, |\lambda - 3/4|\}.
\end{aligned}
$$

If

(2.17) $\dfrac{z f_1'(z)}{f_1(z)} = q(z) = \sqrt{1 + z^2} + z, \quad f_1(z) = z + \displaystyle\sum_{n=2}^{\infty} b_n z^n,$

then $f_1 \in \mathcal{S}^*(q)$. Moreover, we have from (2.17) that

(2.18) $z f_1'(z) - z f_1(z) = f_1(z)\sqrt{1 + z^2}.$

Hence,

$$z + (2b_2 - 1)z^2 + (3b_3 - b_2)z^3 + \cdots = (z + b_2 z^2 + b_3 z^3 + \cdots)(1 + z^2/2 - z^4/8 + z^6/16 + \cdots).$$

Equating the coefficients of like powers of z, we obtain the following first few coefficients of the series involved in (2.18):

$$b_2 = 1, \ b_3 = 3/4, \ b_4 = 5/12.$$

Upon integrating (2.17), we can express the function $f_1(z)$ by

$$
\begin{aligned}
f_1(z) &= z \exp \int_0^z \frac{\sqrt{1+t^2}+t-1}{t} \mathrm{d}t \\
&= \frac{2\sqrt{1+z^2}-2}{z} \exp\left\{z-1+\sqrt{1+z^2}\right\} \\
&= z + z^2 + \frac{3}{4}z^3 + \frac{5}{12}z^4 + \frac{2}{9}z^5 + \cdots \quad (z \in \mathbb{U}).
\end{aligned}
$$

(2.19)

For the above function f_1, we have

$$\left|b_3 - \lambda b_2^2\right| = |\lambda - 3/4|.$$

Next, if a function f_2 is such that

(2.20) $\quad \dfrac{zf_2'(z)}{f_2(z)} = q(z^2) = \sqrt{1+z^4}+z^2,$

then $f_2 \in \mathcal{S}^*(q)$ and

$$
\begin{aligned}
f_2(z) &= z \exp \int_0^z \frac{\sqrt{1+t^4}+t^2-1}{t} \mathrm{d}t \\
&= z + \frac{1}{2}z^3 + \cdots \\
&= z + \sum_{n=2}^{\infty} d_n z^n \quad (z \in \mathbb{U}).
\end{aligned}
$$

(2.21)

Evidently, then for f_2 given by (2.21), we have

$$\left|d_3 - \lambda d_2^2\right| = |1/2|.$$

$\square$

Conjecture. *Let* $f \in \mathcal{S}^*(q)$ *and* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. *Then*

(2.22) $\quad |a_n| \le |b_n|, \ n = 2, 3, 4, \ldots,$

where the coefficients b_n are those given in (2.17).

From (2.19), we have

$$b_2 = 1, \ b_3 = 3/4, \ b_4 = 5/12, \ b_5 = 2/9, \ldots,$$

and from (2.22), we get $|a_3| \le 3/4$, as is in Theorem 2.1. Also, for $n = 4$, the inequality (2.22) gives $|a_4| \le 5/12 = 0.416\ldots$, while Theorem 2.1 gives $|a_4| \le 5/10$.

The second Hankel determinant for the function $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ is given by $a_2 a_4 - a_3^2$. For more details and applications of this determinant, we refer to the recent paper [9]. We next find the second Hankel determinant estimation in the class $\mathcal{S}^*(q)$.

2.3. Theorem. If the function defined by (2.1) belongs to the class $\mathcal{S}^*(q)$, then

$$|a_2 a_4 - a_3^2| \le 39/48.$$

Proof. Applying the notations used in the proof of Theorem 2.1, we obtain the following relations from (2.8), (2.9) and (2.11):

$$a_2 = c_1, \quad a_3 = \frac{1}{2}\left(c_2 + \frac{3}{2}c_1^2\right)$$

and

$$a_4 = \frac{1}{3}\left(\frac{5}{4}c_1^3 + \frac{5}{2}c_1 c_2 + c_3\right),$$

where c_k are coefficients of a Schwarz function. Therefore, we have

$$
\begin{aligned}
a_2 a_4 - a_3^2 &= \frac{c_1}{3} \left(\frac{5}{4} c_1^3 + \frac{5}{2} c_1 c_2 + c_3 \right) - \frac{1}{4} \left(c_2 + \frac{3}{2} c_1^2 \right)^2 \\
&= \frac{c_1}{3} \left(c_1^2 + 2 c_1 c_2 + c_3 \right) - \frac{1}{4} \left(c_2 + c_1^2 \right)^2 - \frac{1}{12} c_1^2 \left(c_2 + c_1^2 \right) - \frac{7}{48} c_1^4.
\end{aligned}
$$

From (2.13) and (2.14), we obtain that

$$
\begin{aligned}
&\left| a_2 a_4 - a_3^2 \right| \\
&= \left| \frac{c_1}{3} \left(c_1^3 + 2 c_1 c_2 + c_3 \right) - \frac{1}{4} \left(c_2 + c_1^2 \right)^2 - \frac{1}{12} c_1^2 \left(c_2 + c_1^2 \right) - \frac{7}{48} c_1^4 \right| \\
&\leq \left| \frac{c_1}{3} \left(c_1^3 + 2 c_1 c_2 + c_3 \right) \right| + \left| \frac{1}{4} \left(c_2 + c_1^2 \right)^2 \right| + \left| \frac{1}{12} c_1^2 \left(c_2 + c_1^2 \right) \right| + \left| \frac{7}{48} c_1^4 \right| \\
&\leq \frac{1}{3} + \frac{1}{4} + \frac{1}{12} + \frac{7}{48} \\
&= 39/48.
\end{aligned}
$$

$\square$

Remark. In view of (2.22), it may be observed that the value of the Hankel determinant $\left| b_2 b_4 - b_3^2 \right|$ is $7/48$ for the function f_1 (defined above by (2.17)).

References

[1] M. K. Aouf, J. Dziok, J. Sokół, On a subclass of strongly starlike functions, Appl. Math. Letters 24(2011) 27-32.

[2] J. Dziok, R. K. Raina, J. Sokół, On alpha-convex functions related to shell-like functions connected with Fibonacci numbers, Appl. Math. Comput. 218(2011) 966-1002.

[3] J. Dziok, R. K. Raina, J. Sokół, Certain results for a class of convex functions related to a shell-like curve connected with Fibonacci numbers, Comput. Math. Appl. 61(9)(2011) 2605-2613.

[4] J. Dziok, R. K. Raina, J. Sokół, On a class of starlike functions related to a shell-like curve connected with Fibonacci numbers, Math. Comput. Modelling 57(2013) 1203-1211.

[5] J. Dziok, R. K. Raina, J. Sokół, Differential subordinations and alpha-convex functions, Acta Math. Scientia 33B(2013) 609-620.

[6] R. Jurasińska, J. Sokół, Some problems for certain family of starlike functions, Math. Comput. Modelling 55(2012) 2134-2140.

[7] F. R. Keogh and E. P. Merkes, A coefficient inequality for certain classes of analytic functions, Proc. Amer. Math. Soc. 20(1969) 8–12.

[8] R. K. Raina, J. Sokół, On some class of starlike functions, submitted.

[9] See K. Lee, V. Ravichandran, S. Supramaniam, Bounds for the second Hankel determinant of certain univalent functions, J. Ineq. Appl. 2013, 2013:281.

[10] J. Sokół, Coeffcient Estimates in a Class of Strongly Starlike Functions, Kyungpook Math. J. 49(2009) 349–353.

[11] J. Sokół, A. Wiśniowska-Wajnryb, On certain problem in the clases of k-starlike functions, Comput. Math. Appl. 62(2011) 4733-4741.