Hacettepe Journal of Mathematics and Statistics
Volume 44 (6) (2015), 1435 – 1443

Applications of n-Gorenstein projective and injective modules

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Abstract

Over a commutative noetherian ring, we introduce a generalization of Gorenstein projective and injective modules, which we call, respectively, n-Gorenstein projective and injective modules. These last two classes of modules give us a new characterization of Gorenstein rings in terms of top local cohomology modules of flat modules. We also utilize the n-Gorenstein injective dimension to study an open question of Takahashi. Furthermore, we prove that a nonzero finite module with finite n-Gorenstein projective dimension satisfies the Auslander-Bridger formula.

Keywords: n-Gorenstein projective module; n-Gorenstein injective module; n-Gorenstein projective dimension; n-Gorenstein injective dimension.

2000 AMS Classification: 13C11; 13C15; 13D02; 13H10; 16D50; 16E10.

Received : 08.01.2014 *Accepted* : 15.10.2014 *Doi* : 10.15672/HJMS.2015449673

1. Introduction

Throughout this paper, R is a commutative noetherian ring with identity element, and all R-modules are unital. Also, for any R-module M, $Z(M)$ denotes the set of all zerodivisors of M.

When R is two-sided and noetherian, Auslander and Bridger [1] introduced the G-dimension for finitely generated modules. Several decades later, over a general ring R, Enochs and Jenda in [7] extended this homological dimension to Gorenstein projective dimension for arbitrary (non-finite) modules. Dually, they defined in [7] the Gorenstein injective dimension. The Gorenstein projective, injective dimension of a module is defined in terms of resolutions by Gorenstein projective, injective modules, respectively. Those modules are constructed from some special acyclic complexes. A complex of R-modules $\mathbf{A} = \cdots \to A_{i+1} \to A_i \to A_{i-1} \to A_{i-2} \to \cdots$ is *acyclic* if $\mathrm{H}(\mathbf{A})= 0$.

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1.1. Definition. (1) An R-module M is said to be *Gorenstein projective*, if there exists an acyclic complex of projective modules $\mathbf{P} = \cdots \to P_1 \to P_0 \to P_{-1} \to P_{-2} \to \cdots$ such that $M \cong \mathrm{Im}(P_0 \to P_{-1})$ and such that the complex $\mathrm{Hom}_R(\mathbf{P}, Q)$ is acyclic whenever Q is projective.

(2) An R-module M is said to be *Gorenstein injective*, if there exists an acyclic complex of injective modules $\mathbf{I} = \cdots \to I_1 \to I_0 \to I_{-1} \to I_{-2} \to \cdots$ such that $M \cong \mathrm{Im}(I_0 \to I_{-1})$ and such that the complex $\mathrm{Hom}_R(E, \mathbf{I})$ is acyclic whenever E is injective.

Inspired by the definitions of Gorenstein projective/injective modules and sesqui-acyclic complexes in [13], the main idea of this paper is to introduce and study two new classes of modules called n-Gorenstein projective/injective modules and related homological dimensions.

The structure of the paper is summarized below. Section 2 is devoted to introducing the concept of n-Gorenstein projective (resp., injective) modules. We will find some useful properties of these classes of modules. Section 3 discusses the n-Gorenstein projective (resp., injective) dimension. We prove that a module of finite n-Gorenstein projective dimension can be approximated by a module, for which the corresponding classical homological dimension is finite. Section 4 consists of three applications. Theorem 4.2 states that if a local ring R admits a nonzero finitely generated R-module M with $n\text{-Gid}_R M$ finite and $\dim_R M = \dim R$, then R is Cohen-Macaulay. This result in fact gives a partial answer to the following question of Takahashi: Is a local ring Cohen-Macaulay if it admits a nonzero finitely generated module of finite Gorenstein injective dimension? Theorem 4.3 can be regarded as the following expansion of a result of Yoshizawa (see [18, Theorem 2.6]). It shows that a complete Cohen-Macaulay local ring R of Krull dimension d is Gorenstein if and only if $\mathrm{H}^d_{\mathfrak{m}}(R)$ is n-Gorenstein injective for some positive integer n. The last Theorem 4.7, is in fact a generalized version of the Auslader-Bridge formula, which is proved by Lars in [4, Theorem 1.4.8]. However the method we use here is somewhat different from theirs.

Let $\mathcal{X}$ be a class of R-modules and M an R-module. Following [10], a *left $\mathcal{X}$-resolution* of M is an exact sequence of R-modules in $\mathcal{X}$ of the form $\mathbf{X} = \cdots \to X_n \to X_{n-1} \to \cdots \to X_0 \to M \to 0$. Now let $\mathbf{X}$ be any left $\mathcal{X}$-resolution of M. We say that $\mathbf{X}$ is proper if the sequence $\mathrm{Hom}_R(Y, \mathbf{X})$ is exact for all $Y \in \mathbf{X}$. The *$\mathcal{X}$-projective dimension* of M is defined as $\mathcal{X}\text{-pd}_R(M) = \inf\{\sup\{n \geqslant 0 \mid X_n \neq 0\} \mid \mathbf{X}$ is an $\mathcal{X}$-resolution of $M\}$. The *right $\mathcal{X}$-resolution*, *co-proper right $\mathcal{X}$-resolution* and *$\mathcal{X}$-injective dimension* are defined dually. We write $\mathcal{P}(R)$, $\mathcal{I}(R)$ for the classes of projective, injective R-modules, respectively. Following established conventions, we use abbreviations (pd, id) for the homological dimensions ($\mathcal{P}(R)$-pd, $\mathcal{I}(R)$-id). For each positive integer n, we denote $\mathcal{X}^{\perp_n} := \{M \mid \mathrm{Ext}^i_R(X, M) = 0$ for any $X \in \mathcal{X}$ and $1 \leq i \leq n\}$. Dually, we have the class ${}^{\perp_n}\mathcal{X}$.

2. n-Gorenstein projective and injective modules

We begin with the following

2.1. Definition. (1) Suppose that n is a positive integer, an R-module M is said to be *n-Gorenstein projective*, if there exists an acyclic complex of projective modules $\mathbf{P} = \cdots \to P_1 \to P_0 \to P_{-1} \to P_{-2} \to \cdots$ such that $M \cong \mathrm{Im}(P_0 \to P_{-1})$ and such that for any projective module Q the complex $\mathrm{Hom}_R(\mathbf{P}, Q) = \cdots \to P_1^* \to P_0^* \to P_{-1}^* \to P_{-2}^* \to \cdots$ is exact at P_i^* for all $i \geq -n$, where $P_i^* = \mathrm{Hom}_R(P_{-i}, Q)$. The class of n-Gorenstein projective modules is denoted by $n\text{-}\mathcal{GP}(R)$.

(2) Suppose that n is a positive integer, an R-module M is said to be *n-Gorenstein injective*, if there exists an acyclic complex of injective modules $\mathbf{I} = \cdots \to I_1 \to I_0 \to I_{-1} \to I_{-2} \to \cdots$ such that $M \cong \mathrm{Im}(I_0 \to I_{-1})$ and such that for any injective module

E the complex $\mathrm{Hom}_R(E, \mathbf{I}) = \cdots \to I_1^* \to I_0^* \to I_{-1}^* \to I_{-2}^* \to \cdots$ is exact at I_i^* for all $i \geq -n$, where $I_i^* = \mathrm{Hom}_R(E, I_i)$. The class of n-Gorenstein injective modules is denoted by n-$\mathcal{GI}(R)$.

Almost by the definitions we have that Gorenstein projective (resp., injective) modules are n-Gorenstein projective (resp., injective) modules. However, there are n-Gorenstein projective (resp., injective) modules which are not Gorenstein projective (resp., injective)(see Example 2.4 below).

2.2. Proposition. *Suppose that M is an R-module, and m, n are positive integers such that $m < n$, then the following statements hold.*

(1) *M is n-Gorenstein projective if and only if M belongs to the class ${}^{\perp_n}\mathcal{P}(R)$, and admits a co-proper right $\mathcal{P}(R)$-resolution.*
(2) *n-Gorenstein projective modules are m-Gorenstein projective modules.*
(3) $\mathcal{GP}(R) = \bigcap_{n=1}^{\infty} n\text{-}\mathcal{GP}(R)$.
(4) *If M is n-Gorenstein projective, then there is an exact sequence $0 \to M \to P \to G \to 0$ such that P is projective and G is $(n+1)$-Gorenstein projective.*

Proof. (1). Since every module has a left $\mathcal{P}(R)$-resolution, the statement is clear from the definition of n-Gorenstein projective modules.

(2) follows immediately from (1).

(3) is true by [10, Proposition 2.3].

(4). Also by the definition of n-Gorenstein projective modules. □

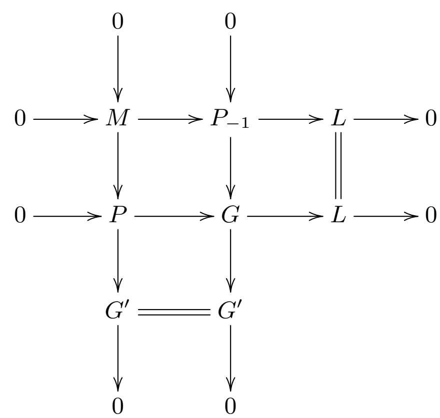
Next we discuss some connections between n-Gorenstein projective modules and n-Gorenstein injective modules.

2.3. Proposition. *Let M and E be R-modules. Then the following statements hold.*

(1) *If M is a finitely generated n-Gorenstein projective module and E is injective, then $\mathrm{Hom}_R(M, E)$ is n-Gorenstein injective.*
(2) *If R is a complete local ring with the residue field k, M is an artinian n-Gorenstein injective module, then $\mathrm{Hom}_R(M, E(k))$ is n-Gorenstein projective.*
(3) *If M is finitely generated, then M is n-Gorenstein projective module if and only if M is reflexive, $\mathrm{Ext}_R^i(M, R) = 0$ for $1 \leq i \leq n$, and $\mathrm{Ext}_R^i(M^*, R) = 0$ for $i \geq 1$, where $M^* = \mathrm{Hom}_R(M, R)$.*

Proof. (1). Since M is n-Gorenstein projective and finitely generated, there exists an exact sequence $0 \to M \to P_{-1} \to L \to 0$ such that P_{-1} is a finitely generated projective module and such that the exact sequence remains exact after applying $\mathrm{Hom}_R(-, \mathcal{P}(R))$ to it. Hence we obtain that $\mathrm{Ext}_R^1(L, Q) = 0$ for all projective modules Q. Since M is n-Gorenstein projective, there also exists an exact sequence $0 \to M \to P \to G' \to 0$ with

$P \in \mathcal{P}(R)$, $G' \in n$-$\mathcal{GP}(R)$. Consider the following pushout diagram.



So L is n-Gorenstein projective by Proposition 2.6 and Corollary 2.7 below. Continuing in this way gives a co-proper right $\mathcal{P}(R)$-resolution $\mathbf{P} = 0 \to M \to P_{-1} \to P_{-2} \to \cdots$ with each term finitely generated. Because R is commutative, for each injective R-module I, $\mathrm{Hom}_R(I, \mathrm{Hom}_R(\mathbf{P},\ E)) \cong \mathrm{Hom}_R(I \otimes_R \mathbf{P},\ E) \cong \mathrm{Hom}_R(\mathbf{P},\ \mathrm{Hom}_R(I, E))$. Note that $\mathrm{Hom}_R(I, E)$ is flat by [8, Theorem 3.2.16] and every flat module is a direct limit of finitely generated projective modules. Hence $\mathrm{Hom}_R(\mathbf{P}, E)$ becomes a proper left $\mathcal{I}(R)$-resolution of $\mathrm{Hom}_R(M, E)$ by [9, Lemma 3.1.6]. For each $i \geq 1$, we have $\mathrm{Ext}^i_R(I, \mathrm{Hom}_R(M, E)) \cong \mathrm{Ext}^i_R(\mathrm{Tor}^R_i(I, M), E) \cong \mathrm{Ext}^i_R(M, \mathrm{Hom}_R(I, E))$ by [8, Theorem 3.2.1]. Therefore $\mathrm{Hom}_R(M, E)$ is n-Gorenstein injective by a dual statement of Proposition 2.2(1) and [9, Lemma 3.1.6].

(2) is proved in a fashion similar to [6, Theorem 4.8].

(3) is easy by the definition of n-Gorenstein projective modules.

□

2.4. Example. *Let R be the local artinian ring in* [14]*, there exists a family $\{M_s\}_{s \geq 1}$ of reflexive modules such that*

(1) $\mathrm{Ext}^i_R(M_s, R) = 0$ if and only if $1 \leq i \leq s - 1$.

(2) $\mathrm{Ext}^i_R(M_s^*, R) = 0$ for all $i > 0$.

Then the following statements hold for any $s > 1$.

(a) M_s is $(s-1)$-Gorenstein projective but not s-Gorenstein projective.

(b) $\mathrm{Hom}_R(M_s, E(k))$ is $(s-1)$-Gorenstein injective but not s-Gorenstein injective.

Proof. (a) is easy since the reflexive module M_s satisfies conditions (1) and (2).

(b). Since M_s is (s-1)-Gorenstein projective by (a), it is deduced from Proposition 2.3(1) that $\mathrm{Hom}_R(M_s, E(k))$ is (s-1)-Gorenstein injective. Now suppose that $\mathrm{Hom}_R(M_s, E(k))$ is s-Gorenstein injective. Indeed, since artinian local rings are complete and $M_s \cong \mathrm{Hom}_R(\mathrm{Hom}_R(M_s, E(k)), E(k))$, the Matlis duality between noetherian modules and artinian modules implies that M_s must be s-Gorenstein projective. It is impossible. So $\mathrm{Hom}_R(M_s, E(k))$ is not s-Gorenstein injective for $s > 1$. □

2.5. Remark. From the construction of M_s (see [14]), we know that there is an exact sequence $0 \to M_s \to R^2 \to M_{s+1} \to 0$ such that M_{s+1} is s-Gorenstein projective, M_s is (s-1)-Gorenstein projective but not s-Gorenstein projective. This implies that n-$\mathcal{GP}(R)$ is not closed under kernels of epimorphisms. Hence n-$\mathcal{GP}(R)$ is not a projectively resolving class (see [10, 1.1]).

The next proposition provides ways to create n-Gorenstein projective modules.

2.6. Proposition. *n-$\mathcal{GP}(R)$ is closed under direct sums and extensions.*

Proof. By [10, Proposition 1.6], [15, Proposition 7.21] and Proposition 2.2(1), n-$\mathcal{GP}(R)$ is closed under direct sums. It follows from [10, 1.7] that n-$\mathcal{GP}(R)$ is closed under extensions. □

Although n-$\mathcal{GP}(R)$ is not projectively resolving, we may show that the class of n-Gorenstein projective modules is closed under direct summands without using Eilenberg's technique (see [10, Proposition 1.4]). This technique is applied by Holm in [10, Theorem 2.5] to show that the class of Gorenstein projective modules is closed under direct summands.

2.7. Corollary. *n-$\mathcal{GP}(R)$ is closed under direct summands.*

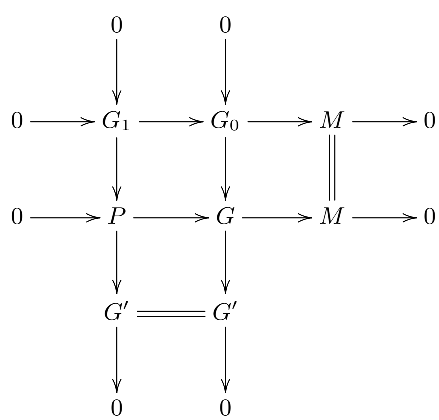
Proof. Let G be an n-Gorenstein projective module and H be a direct summand of G. Since $G \in {}^{\perp_n}\mathcal{P}(R)$, $H \in {}^{\perp_n}\mathcal{P}(R)$. By [12, Theorem 4.6(1)], one sees that H has a co-proper right $\mathcal{P}(R)$-resolution. Now Proposition 2.2(1) gives the result. □

3. n-Gorenstein projective and injective dimensions

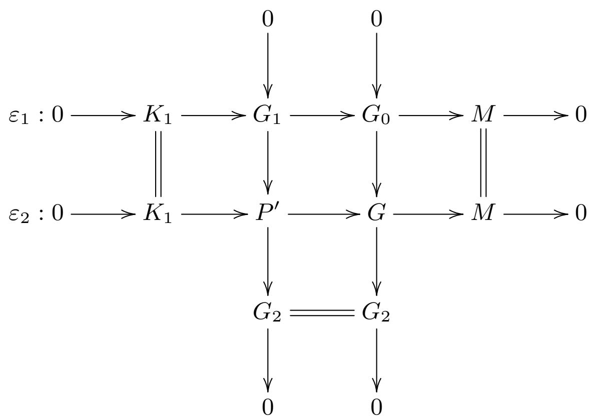
In this section, we turn to studying n-Gorenstein projective and injective dimensions. For an R-module M, we use n-$\mathrm{Gpd}_R M$ (resp., n-$\mathrm{Gid}_R M$) to denote the n-Gorenstein projective (resp., injective) dimension of M. Holm in [10, Theorem 2.10] showed that an R-module M with Gorenstein projective dimension n admits such an exact sequence $0 \to K \to G \to M \to 0$ with G Gorenstein projective and $\mathrm{pd}_R K = n-1$. The proof of this result depends on the fact that for an R-module M with Gorenstein projective dimension n every projective resolution of M has its nth syzygy Gorenstein projective. However we don't know weather the same fact is true for modules with finite n-Gorenstein projective dimension. But by showing in a different way we still have the following result which is similar to that of Holm.

3.1. Proposition. *Let M be an R-module with finite n-Gorenstein projective dimension m. Then there exists an exact sequence $0 \to K \to G \to M \to 0$, where G is n-Gorenstein projective and $pd_R K = m-1$.*

Proof. If $m = 0$, our claim clearly holds. If $m = 1$, we have an exact sequence $0 \to G_1 \to G_0 \to M \to 0$ with G_0, $G_1 \in n$-$\mathcal{GP}(R)$. Since G_1 is n-Gorenstein projective, there also exists an exact sequence $0 \to G_1 \to P \to G' \to 0$ with $P \in \mathcal{P}(R)$, $G' \in n$-$\mathcal{GP}(R)$. Consider the following pushout diagram.



According to Proposition 2.6, G is n-Gorenstein projective. Therefore $0 \to P \to G \to M \to 0$ is the desired sequence. Now suppose $m > 1$. We will use induction to show this statement. Since $n\text{-Gpd}_R M = m$, we have such an exact sequence $0 \to K \to G_0 \to M \to 0$ with $G_0 \in n\text{-}\mathcal{GP}(R)$ and $n\text{-Gpd}_F K = m-1$. Hence there exists an exact sequence $0 \to K_1 \to G_1 \to K \to 0$ with $\text{pd}_F K_1 = m-2$ and $G_1 \in n\text{-}\mathcal{GP}(R)$ by induction. Observe that G_1 is n-Gorenstein projective, there is an exact sequence $0 \to G_1 \to P' \to G_2 \to 0$ with $P' \in \mathcal{P}(R)$, $G_2 \in n\text{-}\mathcal{GP}(R)$. Similarly, consider the following pushout diagram.



Since G_0, $G_2 \in n\text{-}\mathcal{GP}(R)$, $G \in n\text{-}\mathcal{GP}(R)$. Let $K = \text{Ker}(G \to M)$, $0 \to K \to G \to M \to 0$ is the desired sequence. □

A complement to Proposition 2.6 is given below.

3.2. Corollary. *Let $0 \to G' \to G \to M \to 0$ be an exact sequence, where G and G' are n-Gorenstein projective and $\text{Ext}^1_R(M, Q) = 0$ for all projective modules Q. Then M is n-Gorenstein projective.*

If M is n-Gorenstein projective, $\text{Ext}^i_R(M, L) = 0$ for all $i \geq 1$ and all modules L with finite injective dimension. If N is n-Gorenstein injective, $\text{Ext}^i_R(H,\text{N}) = 0$ for all $i \geq 1$ and all modules H with finite projective dimension. Using this fact, we may provide the following results. Their proofs are similar to those in [11, Theorem 2.1, 2.2].

3.3. Corollary. *Let M be an R-module, then:*
(1) *If $pd_R M < \infty$, $n\text{-}Gid_R M = id_R M$.*
(2) *If $id_R M < \infty$, $n\text{-}Gpd_R M = pd_R M$.*

Using Propositions 2.2 and 3.1, we get the following result which is analogous to [5, Lemma 2.17].

3.4. Proposition. *Let M be an R-module with $n\text{-}Gpd_R M < \infty$. There is an exact sequence $0 \to M \to H \to A \to 0$, where A is n-Gorenstein projective and $pd_R H = n\text{-}Gpd_R M$.*

4. Applications

For an R-module M, the *ith local cohomology module* of M with respect to an ideal I is defined as $\text{H}^i_I(M) = \varinjlim_n \text{Ext}^i_R(R/I^n, M)$. Let $(R, \mathfrak{m}, k)$ be a local ring. We say that R is *Cohen-Macaulay* if $\text{depth}R = \dim R$. R is *Gorenstein* if it has finite self-injective dimension.

A careful reading of the proof in [17, Lemma 1.1], combined with some basic facts about the n-Gorenstein injective dimension, gives the following vanishing result of local cohomology modules.

4.1. Lemma. *Let M be an R-module with $n\text{-Gid}_R M$ finite, and let I be a nonzero ideal of R. Then $\mathrm{H}_I^i(M) = 0$ for all $i > n\text{-Gid}_R M$.*

With the aid of Lemma 4.1, We are now in a position to give one of the main results in this paper, which partially answers a question of Takahashi in [16]. A similar result was proved by Yassemi([17, Theorem 1.3]).

4.2. Theorem. *Let $(R, \mathfrak{m}, k)$ be a local ring. If R admits a nonzero finitely generated R-module M with $n\text{-Gid}_R M$ finite and $\dim_R M = \dim R$, then R is Cohen-Macaulay.*

Proof. Since $n\text{-Gid}_R M$ is finite, by Lemma 4.1, $\mathrm{H}_{\mathfrak{m}}^i(M) = 0$ for all $i > n\text{-Gid}_R M$. On the other hand, we have $\mathrm{H}_{\mathfrak{m}}^{dim_R M}(M) \neq 0$ by [2, Theorem 7.3.2]. Hence $\dim_R M \leqslant n\text{-Gid}_R M$ follows. Also by a dual argument of Proposition 3.4, the finiteness of n-Gorenstein injective dimension of M means that there is an R-module L such that $\mathrm{id}_R L = n\text{-Gid}_R M$. But $\mathrm{id}_R L = \sup\{\mathrm{depth} R_{\mathfrak{p}} - \mathrm{width}_{R_{\mathfrak{p}}} L_{\mathfrak{p}} \mid \mathfrak{p} \in \mathrm{Supp}(L)\}$ by [3, 3.1]. Thus $\dim R = \dim_R M \leq n\text{-Gid}_R M = \mathrm{id}_R L \leq \mathrm{depth} R_{\mathfrak{p}} \leq \dim R_{\mathfrak{p}}$ for some prime ideal $\mathfrak{p}$. Hence $\mathfrak{p}$ must be the maximal ideal $\mathfrak{m}$. So we get that $\mathrm{depth} R = \dim R$, and R is Cohen-Macaulay. □

Enochs and Jenda in [8, Corollary 9.5.13] proved that a local ring R is Gorenstein if and only if R is Cohen-Macaulay and the top local cohomology module of R, $\mathrm{H}_{\mathfrak{m}}^{dim R}(R)$, is isomorphic to $E(k)$. Recently, Yoshizawa in [18] generalized this result. It says that a complete Cohen-Macaulay local ring R of krull dimension d is Gorenstein if and only if the top local cohomology module $\mathrm{H}_{\mathfrak{m}}^{dim R}(R)$ is Gorenstein injective. Motivated by these established facts, an extended version of this result is presented as follows.

4.3. Theorem. *Let R be a complete Cohen-Macaulay local ring of krull dimension d, then the following statements are equivalent.*

(1) *R is Gorenstein.*
(2) *For any positive integer n, $H_{\mathfrak{m}}^d(F)$ is n-Gorenstein injective for every flat R-module F.*
(3) *For some positive integer n, $H_{\mathfrak{m}}^d(F)$ is n-Gorenstein injective for every flat R-module F.*
(4) *For some positive integer n, $H_{\mathfrak{m}}^d(R)$ is n-Gorenstein injective.*

Proof. (1) ⇒ (2). Since flat modules are direct limits of finitely generated free modules and the local cohomology functors commute with direct limits, the result follows from [18, Theorem 2.6].

(2) ⇒ (3) and (3) ⇒ (4) are trivial.

(4) ⇒ (1). By assumption, $\mathrm{H}_{\mathfrak{m}}^d(R)$ is n-Gorenstein injective for some positive integer n. Since $\mathrm{H}_{\mathfrak{m}}^d(R)$ is artinian, by Proposition 2.3, $n\text{-Gpd}_R(\mathrm{Hom}_R(\mathrm{H}_{\mathfrak{m}}^d(R), E(k))) = 0$. By [8, Theorem 9.5.16], we know that $\mathrm{Hom}_R(\mathrm{H}_{\mathfrak{m}}^d(R), E(k))$ is a dualizing module. Now Corollary 3.3 says that $\mathrm{Hom}_R(\mathrm{H}_{\mathfrak{m}}^d(R), E(k))$ is projective. Note that projective modules over local rings are free. Hence $\mathrm{Hom}_R(\mathrm{H}_{\mathfrak{m}}^d(R), E(k)) \cong R^n$. Thus R is Gorenstein. □

Now it is natural to ask what can we say about R when the top local cohomology module has finite n-Gorenstein injective dimension?

4.4. Corollary. *The following statements are equivalent for a commutative artinian local ring R.*

(1) *R is quasi-Frobenius.*
(2) *For any positive integer n, $H_{\mathfrak{m}}^0(R)$ is n-Gorenstein injective.*

(3) *For some positive integer n, all modules are n-Gorenstein injective.*
(4) *For some positive integer n, all modules are n-Gorenstein projective.*

Proof. (1) $\Leftrightarrow$ (2) is easy directly by Theorem 4.3.

(1) $\Rightarrow$ (4). Because all n-Gorenstein projective modules are Gorenstein projective over Gorenstein rings, it is a consequence of [8, Exercise 10.3.5].

(4) $\Rightarrow$ (1). Since all modules are n-Gorenstein projective, all injective modules are also n-Gorenstein projective. Thus every injective module can be embedded into a projective module. So every injective module is projective. It means that R is quasi-Frobenius.

(1) $\Leftrightarrow$ (3) can be shown dually. □

Finally we aim at investigating some applications of n-Gorenstein projective dimensions among the category of all finitely generated R-modules. For convenience, all modules appeared below are finitely generated.

4.5. Lemma. *Let M be an R-module, and let x be an M- and R-regular element. If M is n-Gorenstein projective, then M/xM is n-Gorenstein projective $R/(x)$-module.*

Proof. Set $\bar{M} = M/xM$ and $\bar{R} = R/(x)$. Since M is n-Gorenstein projective, there is an exact complex of free modules $\mathbf{F} = \cdots \to F_1 \stackrel{d_1}{\to} F_0 \stackrel{d_0}{\to} F_{-1} \stackrel{d_{-1}}{\to} F_{-2} \to \cdots$ such that $M \cong \mathrm{Im}(F_0 \to F_{-1})$ and such that the complex $\mathbf{F}^* = \cdots \to F_1^* \to F_0^* \to F_{-1}^* \to F_{-2}^* \to \cdots$ is exact at F_i^* for all $i \geq -n$, where $F_i^* = \mathrm{Hom}_R(F_{-i}, R)$. Let $M_i = \mathrm{Im} d_i$ for each i. Since x is M_i-regular for each i, by [4, Lemma 1.3.4(a)], applying $- \otimes_R \bar{R}$ to the exact complex $\mathbf{F}$ gives an exact complex $\bar{\mathbf{F}} = \cdots \to \bar{F}_1 \to \bar{F}_0 \to \bar{F}_{-1} \to \bar{F}_{-2} \to \cdots$. Again by [4, Lemma 1.3.4(a)], there is a complex $\cdots \to \overline{F_1^*} \to \overline{F_0^*} \to \overline{F_{-1}^*} \to \overline{F_{-2}^*} \to \cdots$, which is exact at $\overline{F_i^*}$ for all $i \geq -n$. It is deduced from [4, Lemma 1.3.4(b)] that $\mathrm{Hom}_{\bar{R}}(\bar{F}_{-i}, \bar{R}) \cong \overline{F_i^*}$. Therefore $\bar{M}$ is n-Gorenstein projective $\bar{R}$-module. □

4.6. Proposition. *If R is a local ring, and suppose that there exists an exact sequence $0 \to M \to N \to H \to 0$ with n-$Gpd_R M \leq pd_R N < \infty$ and H n-Gorenstein projective. Then $depth_R M = depth_R N$.*

Proof. Case 1: $\mathrm{depth} R = 0$. Since $\mathrm{pd}_R N < \infty$, we infer by the Auslander-Buchsbaum formula that N is projective and $\mathrm{depth}_R N = 0$, and hence M is n-Gorenstein projective. Now assume that $\mathrm{depth}_R M \geq 1$, so there is a regular element x of M. Note that M is n-Gorenstein projective. Applying $\mathrm{Hom}_R(-, R)$ to the exact sequence $0 \to M \stackrel{x}{\to} M \to \bar{M} \to 0$ yields an exact sequence $0 \to \bar{M}^* \to M^* \stackrel{x}{\to} M^* \to \mathrm{Ext}_R^1(\bar{M}, R) \to 0$. Applying $\mathrm{Hom}_R(-, R)$ to this exact sequence gives the following exact sequence $0 \to \mathrm{Ext}_R^1(\bar{M}, R)^* \to M^{**} \stackrel{x}{\to} M^{**}$. As M is reflexive by Proposition 2.3(3), $\mathrm{Ext}_R^1(\bar{M}, R)^* = 0$. We get from the formula $\mathrm{Ass}(\mathrm{Ext}_R^1(\bar{M}, R)^*) = \mathrm{Supp}(\mathrm{Ext}_R^1(\bar{M}, R)) \cap \mathrm{Ass} R$ that $\mathrm{Ext}_R^1(\bar{M}, R) = 0$. Therefore $M^* = xM^*$. It follows from the Nakayama's lemma that $M^* = 0$. Hence $M = 0$, which is a contradiction.

Case 2: $\mathrm{depth} R \geq 1$. Because M is a submodule of N, $\mathrm{depth}_R M = 0$ implies that $\mathrm{depth}_R N = 0$. So we may assume that $\mathrm{depth}_R M \geq 1$. Since $\mathrm{depth} R \geq 1$, there must be an element $x \in R \setminus Z(M) \cup Z(R)$. Moreover x is H-regular element as H can be embedded into a free module. By Lemma 4.5 and [4, Lemma 1.3.4(a)], tensoring the exact sequence $0 \to M \to N \to H \to 0$ gives an exact sequence $0 \to \bar{M} \to \bar{N} \to \bar{H} \to 0$ such that $\bar{H}$ is n-Gorenstein projective and n-$\mathrm{Gpd}_{\bar{R}} \bar{M} \leq n$-$\mathrm{Gpd}_R M$. Note that $\mathrm{pd}_{\bar{R}}(\bar{N}) = \mathrm{pd}_R N$. We have that n-$\mathrm{Gpd}_{\bar{R}} \bar{M} \leq \mathrm{pd}_{\bar{R}}(\bar{N})$. Hence, as $\mathrm{depth} \bar{R} = \mathrm{depth} R - 1$, we obtain, by induction on $\mathrm{depth} R$, $\mathrm{depth}_{\bar{R}} \bar{M} = \mathrm{depth}_{\bar{R}} \bar{N}$. Therefore $\mathrm{depth}_R M = \mathrm{depth}_R N$. □

4.7. Theorem. *If R is a local ring, M is a nonzero R-module with n-$Gpd_R M$ finite. Then n-$Gpd_R M + depth_R M = depth R$.*

Proof. There exists an exact sequence $0 \to M \to H \to A \to 0$ with n-$\mathrm{Gpd}_R M = \mathrm{pd}_R H$ and A n-Gorenstein projective by Proposition 3.4. Then by Proposition 4.6 we have that $\mathrm{depth}_R M = \mathrm{depth}_R H$. But we know from the Auslander-Buchsbaum formula that $\mathrm{pd}_R H + \mathrm{depth}_R H = \mathrm{depth} R$. Hence we get the result. □

We end this section with the following corollary.

4.8. Corollary. *If R is a Gorenstein local ring and M is a nonzero R-module. Then $Gpd_R M = n$-$Gpd_R M$.*

Proof. Since R is Gorenstein, $\mathrm{Gpd}_R M < \infty$ by [8, Theorem 12.3.1]. Furthermore we have $\mathrm{Gpd}_R M = \mathrm{depth} R - \mathrm{depth}_R M$ by [4, Theorem 1.4.8]. But n-$\mathrm{Gpd}_R M < \mathrm{Gpd}_R M$, it now follows from Theorem 4.7 that n-$\mathrm{Gpd}_R M = \mathrm{depth} R - \mathrm{depth}_R M$. Hence the result is true. □

ACKNOWLEDGEMENTS

This research was partially supported by NSFC(Grant Nos. 11501144, 11401128) and NSF of Guangxi Province of China (Grant No. 2013GXNSFBA019005). The author is grateful to the referee for the helpful suggestions which have improved this paper.

References

[1] Auslander, M. and Bridger, M. *Stable module theory*, Mem. Amer. Math. Soc., 1969.
[2] Brodmann M. and Sharp, R. Y. *Local cohomology: An algebraic introduction with geometric applications,* Cambridge Studies in Advanced Mathematics, Cambridge University Press, 1998.
[3] Chouinard II, L. G. *On finite weak and injective dimension*, Proc. Amer. Math. Soc. **60**, 57-60, 1976.
[4] Christensen, L.W. *Gorenstein dimensions*, Lecture Notes in Math. **1747**, Springer-Verlag, 2000.
[5] Christensen, L. W., Frankild, A. and Holm, H. *On Gorenstein projective, injective and flat dimensions–A functorial description with applications*, J. Algebra **302**, 231-279, 2006.
[6] Enochs, E. E. and Jenda, O. M. G. *On Gorenstein injective modules*, Comm. Algebra **21**, 3489-3501, 1993.
[7] Enochs, E. E. and Jenda, O. M. G. *Gorenstein injective and projective modules*, Math. Z. **220**, 611-633, 1995.
[8] Enochs, E. E. and Jenda, O. M. G. *Relative Homological Algebra*, Berlin, New York: De Gruyter, 2000.
[9] Göbel, R. and Trlifaj, J. *Approximations and Endomorphism Algebras of Modules*, Walter de Gruyter, Berlin-New York, 2006.
[10] Holm, H. *Gorenstein homological dimensions*, J. Pure Appl. Algebra **189,** 167-193, 2004.
[11] Holm, H. *Rings with finite Gorenstein injective dimension*, Proc. Amer. Math. Soc. **132,** 1279-1283, 2004.
[12] Huang, Z. Y., *Proper resolutions and Gorenstein categories*, J. Algebra **393,** 142-169, 2013.
[13] Hughes, M. T., Jorgensen, D. A. and Şega, L. M. *Acyclic complexes of finitely generated free modules over local rings*, Math. Scand. **105,** 85-98, 2009.
[14] Jorgensen, D. A. and Şega, L. M. *Independence of the total reflexivity conditions for modules*, Algebr. Represent. Theory **9,** 217-226, 2006.
[15] Rotman, J. J. *An introduction to homological algebra*, Academic Press, New York, 1979.
[16] Takahashi, R. *The existence of finitely generated modules of finite Gorenstein injective dimension*, Proc. Amer. Math. Soc. **134,** 3115-3121, 2006.
[17] Yassemi, S. *A generalization of Bass theorem*, Comm. Algebra **35,** 249-251, 2007.
[18] Yoshizawa, T. *On Gorenstein injectivity of top local cohomology modules*, Proc. Amer. Math. Soc. **140,** 1897-1907, 2012.