Hacettepe Journal of Mathematics and Statistics
Volume 39 (1) (2010), 55 – 65

MAPPINGS AND COVERING PROPERTIES IN L-TOPOLOGICAL SPACES

T. Baiju*† and Sunil Jacob John*

Received 09 : 06 : 2009 : Accepted 15 : 10 : 2009

Abstract

The behavior of various types of noncompact covering properties such as paracompactness, metacompactness, subparacompactness, submetacompactness, etc., are studied under various types of fuzzy mappings such as open map, closed map, perfect map, etc. Moreover the concept of para-Lindelof space is introduced in L-topological spaces and its properties and behavior under maps are obtained.

Keywords: Paracompactness, Metacompactness, Subparacompactness, Submetacompactness, Para-Lindelofness, Perfect maps.

2000 AMS Classification: 54 D 20, 54 A 40.

1. Introduction

Zadeh [24] introduced fuzzy sets as a generalization of ordinary sets by means of membership functions. The concept of fuzzy set offers us a new framework of set theory, and in this new framework we are generalizing many of the concepts of general topology which form the content of fuzzy topology. In fact fuzzy topology, which was introduced by Chang [7], comes as a generalization of general topology.

The concept of compactness is one of the most important concepts in general topology. Locally finite families, point finite families, discrete families and locally countable families etc. are used to define several covering properties, namely, paracompactness, metacompactness, subparacompactness, submetacompactness, and para–Lindelofness, respectively. The class of paracompact spaces was introduced by J. Dieudonne in 1944 [9] as a natural generalization of compactness. Metacompact spaces were introduced by Arens and Dugundji in 1950 [1]. The concept of subparacompact spaces was introduced by McAuley [17], and further studies were conducted by Burke [6] and Creede [8]. In 1965, Worrel and Wicke [22] introduced the concept of θ- refinability and submetacompactness. The para–Lindelof space was introduced by J. Greever [11] in 1968.

*Department of Mathematics, National Institute of Technology, Calicut, Calicut-673 601, Kerala, India. E-mail: (T. Baiju) baijutmaths@gmail.com (S. J. John) sunil@nitc.ac.in

†Corresponding Author.

Also the preservation of covering properties under different maps, such as closed map, open map, perfect map etc. is of great importance in general topology.

Compactness and its various versions have been generalized to L-topological spaces by many authors (see [7, 10, 18, 19, 14]). The concept of paracompactness in fuzzy topology was introduced by Luo [16]. The present authors have done some work on metacompactness in [14]. In [4] and [3] we have introduced the concept of subparacompactness and submetacompactness, respectively, in L-topological spaces. In this paper we study the behavior of various types of noncompact covering properties such as paracompactness, metacompactness, subparacompactness, and submetacompactness and also introduce the concept of para–Lindelof spaces. Besides these, preservation of covering properties under various maps, such as closed map, open map, perfect map etc. are also studied.

Let L be a complete lattice. Its universal bounds are denoted by $\perp$ and $\top$. We presume that L is consistent. i.e., $\perp$ is distinct from $\top$. Thus $\perp \le \alpha \le \top$ for all $\alpha \in L$. We note $\bigvee \phi = \perp$ and $\bigwedge \phi = \top$. The two point lattice $\{\perp, \top\}$ is denoted by $\mathbf{2}$. A unary operation $'$ on L is a *quasi-complementation* if it is an involution (ie., $\alpha'' = \alpha$ for all $\alpha \in L$) that inverts the ordering. (ie., $\alpha \le \beta$ implies $\beta' \le \alpha'$).

In $(L,')$ the DeMorgan laws hold:

$$\big(\bigvee A\big)' = \bigwedge\{\alpha' : \alpha \in A\} \text{ and } \big(\bigwedge A\big)' = \bigvee\{\alpha' : \alpha \in A\}$$

for every $A \subset L$. Moreover, in particular, $\perp' = \top$ and $\top' = \perp$.

A *molecule* or *co-prime element* in a lattice L is a join irreducible element in L, and the set of all non-zero co-prime elements of L is denoted by $M(L)$. A complete lattice L is completely distributive if it satisfies either of the logically equivalent conditions CD1 or CD2 below:

CD1: $\bigwedge_{i\in I}\Big(\bigvee_{j\in J_i}\ a_{i,j}\Big) = \bigvee_{\phi\in\prod_{i\in I} J_i}\Big(\bigwedge_{i\in I}\ a_{i,\phi(i)}\Big)$,

CD2: $\bigvee_{i\in I}\Big(\bigwedge_{j\in J_i}\ a_{i,j}\Big) = \bigwedge_{\phi\in\prod_{i\in I} J_i}\Big(\bigvee_{i\in I}\ a_{i,\phi(i)}\Big)$,

for all $\{\{a_{ij} : j \in J_i\} : i \in I\} \subset P(L) \setminus \{\emptyset\}$.

If L is a complete lattice, then for a set X, L^X is the complete lattice of all maps from X into L, called L-sets or L-subsets of X, under the point-wise ordering, $a \le b$ in L^X if and only if $a(x) \le b(x)$ in L for all $x \in X$. If $A \subset X$, $1_A \in 2^X \subset L^X$ is the characteristic function of A. The constant member of L^X with value α is denoted by α itself. Usually we will not distinguish between a crisp set and its characteristic function.

Wang [20] proved that a complete lattice is completely distributive if and only if for each $\alpha \in L$, there exists $B \subseteq L$ such that

(i) $a = \bigvee A$, and
(ii) if $A \subseteq L$ and $a \le \bigvee B$, then for each $b \in B$, there exists $c \in A$ such that $b \le c$.

B is called a *minimal set of* a, and $\beta(a)$ denotes the union of all minimal sets of a. Also, $\beta^*(a) = \beta(a) \cap M(L)$. Clearly, $\beta(a)$ and $\beta^*(a)$ are minimal sets of a.

For $\alpha \in L$ and $A \in L^X$, we use the following notations.

$$A_{[\alpha]} = \{x \in X : A(x) \ge \alpha\};$$
$$A^{[\alpha]} = \{x \in X : A(x) \le \alpha\};$$
$$A^{(\alpha)} = \{x \in X : A(x) \not\ge \alpha\};$$
$$A_{(\alpha)} = \{x \in X : A(x) \not\le \alpha\}.$$

Clearly L^X has a quasi complementation $'$ defined point-wise, $\alpha'(x) = \alpha(x)'$ for all $\alpha \in L$ and $x \in X$. Thus the DeMorgan laws are inherited by $(L^X,')$.

Let $(L,')$ be a complete lattice equipped with an order reversing involution and X any non empty set. A subfamily $\tau \subset L^X$ which is closed under the formation of sups and finite infs (both formed in L^X) is called an *L-topology on X* and its members are called *open L-sets*.

The pair (X,τ) is called an *L-topological space* (L-ts). The category of all L-topological spaces, together with L-continuous mappings and the composition and identities of **Set** is denoted by L-**Top**. Quasi complements of open L-sets are called *closed L-sets*.

We know that the set of all non-zero co-prime elements in a completely distributive lattice is $\vee$-generating. Moreover, for a continuous lattice L and a topological space (X,T), $T = i_L\omega_L(T)$ is not true in general. By Kubiak [15, Proposition 3.5] we know that one sufficient condition for $T = i_L\omega_L(T)$ is that L be completely distributive.

In [21], Wang extended the Lowen functor ω for completely distributive lattices as follows: For a topological space (X,T), $(X,\omega(T))$ is called the induced space of (X,T), where

$$\omega(T) = \{A \in L^X : \forall\, \alpha \in M(L),\ A^{(\alpha')} \in T\}.$$

In 1992 Kubiak also extended the Lowen functor ω_L for a complete lattice L. In fact, when L is completely distributive, $\omega_L = \omega$.

An L-topological space (X,τ) is called a *weakly induced space* if $\forall\, \alpha \in M(L)$ and $\forall\, A \in \tau$, it is true that $A^{(\alpha')} \in [\tau]$, where $[\tau]$ is the set of all crisp open sets in τ.

Based on these facts, in this paper we use a complete, completely distributive lattice L in L^X. For all standardized basic fixed-basis terminology, we follow Hohle and Rodabaugh [12].

2. Preliminaries and basic definitions

2.1. Definition. [23] Let (X,τ), (Y,μ) be L-topological spaces, $f : X \to Y$ an ordinary mapping. Based on this we define the L-fuzzy mapping $f^{\to} : L^X \to L^Y$ and its L-fuzzy reverse mapping $f^{\leftarrow} : L^Y \to L^X$ by

$$f^{\to} : L^X \to L^Y, \quad f^{\to}(A)(y) = \bigvee\{A(x) : x \in X,\ f(x) = y\}\ \forall\, A \in L^X,\ \forall\, y \in Y.$$
$$f^{\leftarrow} : L^Y \to L^X, \quad f^{\leftarrow}(B)(x) = B(f(x)),\ \forall\, B \in L^Y,\ \forall\, x \in X.$$

2.2. Definition. [23] Let (X,τ), (Y,μ) be L-topological spaces, $f^{\to} : L^X \to L^Y$ an L-fuzzy mapping. We say $f^{\to}$ is an *L-fuzzy continuous mapping* from (X,τ) to (Y,μ) if its L-fuzzy reverse mapping $f^{\leftarrow} : L^Y \to L^X$ maps every open subset in (Y,μ) as an open one in (X,τ). i.e., $\forall\, V \in \mu$, $f^{\leftarrow}(V) \in \tau$.

2.3. Definition. [23] Let (X,τ), (Y,μ) be L-topological spaces, $f^{\to} : L^X \to L^Y$ an L-fuzzy mapping. We say $f^{\to}$ is *open* if it maps every open subset in (X,τ) to an open set in (Y,μ). i.e., $\forall\, U \in \tau$, $f^{\to}(U) \in \mu$.

2.4. Definition. [23] Let (X,τ), (Y,μ) be L-topological spaces, $f^{\to} : L^X \to L^Y$ an L-fuzzy mapping. We say $f^{\to}$ is *closed* if it maps every closed subset in (X,τ) to a closed set in (Y,μ). i.e., $\forall\, F \in \tau'$, $f^{\to}(F) \in \mu'$.

2.5. Definition. [23] Let (X,τ) be an L-ts. A fuzzy point x_α is *quasi coincident with* $D \in L^X$ (and we write $x_\alpha \prec D$) if $x_\alpha \nleq D'$.

Also, D *quasi coincides with* E *at* x ($D\ q\ E$ at x) if $D(x) \nleq E'(x)$. We say D *is quasi coincident with* E, and write $D\ q\ E$, if $D\ q\ E$ at x for some $x \in X$. Further $D \neg\, q\ E$ means D does not quasi coincide with E.

We say $U \in \tau$ is a *quasi coincident nbd. of* x_α (Q - nbd.) if $x_\alpha \prec U$. The family of all Q- nbds. of x_α is denoted by $Q_\tau(x_\alpha)$ or by $Q(x_\alpha)$.

If $\alpha \in M(L)$, then $\mathbf{C} \in \tau$ is an *α-Q-nbd.* of A if $\mathbf{C} \in Q(x_\alpha)$ for every $x_\alpha \leq A$.

2.6. Definition. [23] Let (X, τ) be an L-ts, $A \in L^X$. Then $\Phi \subset L^X$ is called a *Q-cover of* A if for every $x \in \mathrm{Supp}\,(A)$, there exists $U \in \Phi$ such that $x_{A(x)} \prec U$. In particular, Φ is a Q-cover of (X, τ) if Φ is a Q-cover of $\top$.

Φ is called an *α-Q-cover of* A, if for each $x_\alpha \leq A$, there exists $U \in \Phi$ such that $x_\alpha \prec U$.

Φ is called an *open α-Q-cover of* A if $\Phi \subset \tau$ and Φ is an α-Q-cover of A.

$\Phi_0 \subset L^X$ is called a *sub α-Q-cover of* A if $\Phi_0 \subset \Phi$ and Φ_0 is also an α-Q-cover of A.

Φ is called an α^--Q cover of A, if there exists $\gamma \in \beta^*(\alpha)$ such that Φ is γ-Q-cover of A.

2.7. Definition. [23] Let (X, τ) be an L-ts, $D \in L^X$. Then, D is called *N-compact* if for every $\alpha \in M(L)$, every open α-Q cover of D has a finite sub family which is an α^--Q cover of D. In particular, (X, τ) is called *N-compact* if $\top$ is N-compact.

2.8. Definition. [23] Let (X, τ) be an L-ts, $\mathbf{A} = \{A_t : t \in T\} \subseteq L^X$, and $x_\lambda \in M(L^X)$. Then $\mathbf{A}$ is called *locally finite at* x_λ, if there exist $U \in Q(x_\lambda)$ and a finite subset T_0 of T such that $t \in T \setminus T_0 \implies A_t \neg q\, U$. Likewise, $\mathbf{A}$ is called *$*$-locally finite at* x_λ if there exist $U \in Q(x_\lambda)$ and a finite subset T_0 of T such that $t \in T_0 \implies \chi_{At(0)} \neg q\, U$. Then, $\mathbf{A}$ is called *locally finite* (*$*$-locally finite*) for short, if $\mathbf{A}$ is locally finite ($*$-locally finite) at every molecule $x_\lambda \in M(L^X)$.

2.9. Definition. [14] Let (X, τ) be an L-ts, $\mathbf{A} = \{A_t : t \in T\} \subset L^X$, and $x_\lambda \in M(L^X)$. Then $\mathbf{A}$ is called *point finite at* x_λ if $x_\lambda \prec A_t$ for at most finitely many $t \in T$. Likewise, $\mathbf{A}$ is *$*$-point finite at* x_λ if there exists at most finitely many $t \in T$ such that $x_\lambda \prec \chi A_{t(0)}$.

Then $\mathbf{A}$ is called *point finite* (resp. *$*$-point finite*) for short, if $\mathbf{A}$ is point finite (resp. $*$-point finite) at every molecule x_λ of L^X.

2.10. Definition. Let (X, τ) be an L-ts, $\mathbf{A} = \{A_t : t \in T\} \subseteq L^X$, and $x_\lambda \in M(L^X)$. Then $\mathbf{A}$ is called *locally countable at* x_λ, if there exist $U \in Q(x_\lambda)$ and a countable subset T_0 of T such that $t \in T \backslash T_0 \implies A_t \neg q\, U$. Likewise, $\mathbf{A}$ is called *$*$-locally countable at* x_λ if there exist $U \in Q(x_\lambda)$ and a countable subset T_0 of T such that $t \in T_0 \implies \chi_{At(0)} \neg q\, U$.

Then $\mathbf{A}$ is called *locally countable* (*$*$-locally countable*) for short, if $\mathbf{A}$ is locally countable ($*$-locally countable) at every molecule $x_\lambda \in M(L^X)$.

2.11. Definition. Let (X, τ) be an L-ts, $\mathbf{A} = \{A_t : t \in T\} \subseteq L^X$, and $B \in L^X$. Then $\mathbf{A}$ is called *σ-locally countable in* B if $\mathbf{A}$ is a countable union of sub-families which are locally countable in B.

In particular, $\mathbf{A}$ is called *σ-locally countable* for short, if $\mathbf{A}$ is σ-locally countable in $\top$.

2.12. Definition. [23] Let (X, τ) be an L-ts. Then by $[\tau]$ we denote the family of support sets of all crisp subsets in τ. Clearly, $(X, [\tau])$ is a topology and it is the called the *background space*. Then (X, τ) is called *weakly induced* if $U \in \tau$ is a lower semi continuous function from the background space $(X, [\tau])$ to L.

2.13. Definition. [23] A collection $\mathbf{A}$ *refines* a collection $\mathbf{B}$ (written, $\mathbf{A} < \mathbf{B}$) if for every $A \in \mathbf{A}$, there exists $B \in \mathbf{B}$ such that $A \leq B$.

2.14. Definition. [23] Let (X, τ) be an L-ts. Then $\mathbf{A} = \{A_t : t \in T\} \subseteq L^X$ is a *closure preserving collection* if for every subfamily $\mathbf{A}_0$ of $\mathbf{A}$, $\mathrm{cl}\,[\bigvee \mathbf{A}_0] = \bigvee\, [\mathrm{cl}\,\mathbf{A}_0]$.

2.15. Definition. Let (X, τ) be an L-ts. Then $\mathbf{A} = \{A_t : t \in T\} \subseteq L^X$ is a *interior preserving collection* if for every subfamily $\mathbf{A}_0$ of $\mathbf{A}$, $\mathrm{int}\,[\bigwedge \mathbf{A}_0] = \bigwedge\,[\mathrm{int}\,\mathbf{A}_0]$.

2.16. Definition. [14] A collection $\mathbf{U}$ of fuzzy subsets of an L-topological space (X, τ) is said to be *well monotone* if the subset relation '$<$' is a well order on $\mathbf{U}$.

2.17. Definition. [14] A collection $\mathbf{U}$ of fuzzy subsets of an L-topological space (X, τ) is said to be *directed* if $U, V \in \mathbf{U}$ implies there exists $W \in \mathbf{U}$ such that $U \vee V < W$.

2.18. Definition. A sequence $\{\mathbf{G}_n\}$ of α-Q covers of $\top$ is said to be a *θ-sequence* (*$*$-θ-sequence*) of α-Q covers if for each $x_\alpha \in M(L^X)$, there is some $k \in N$ such that the family $\mathbf{G}_k$ is point finite ($*$-point finite) at x_α.

3. Noncompact covering properties

3.1. Definition. [23] Let (X, τ) be an L-ts, $A \in L^X$, $\alpha \in M(L)$. Then A is called *α-paracompact* (*α^*-paracompact*) if for every open α-Q-cover Φ of A, there exist an open refinement Ψ of Φ which is locally finite ($*$-locally finite) in A and Ψ is also an α-Q-cover of A.

In particular, A is *paracompact* (*$*$-paracompact*) if A is α-paracompact (α^*-paracompact) for every $\alpha \in M(L)$. Also, (X, τ) is *paracompact* (*$*$-paracompact*) if $\top$ is paracompact ($*$-paracompact).

3.2. Proposition. [23] *Let (X, τ) be a weakly induced L-ts. Then the following conditions are equivalent:*

(i) *(X, τ) is $*$-paracompact;*
(ii) *There exist $\alpha \in M(L)$ such that (X, τ) is α^*-paracompact;*
(iii) *$(X, [\tau])$ is paracompact.*

3.3. Proposition. [22] *Let (X, τ) be an L-ts, $A \in L^X$, and $\alpha \in M(L)$. Then:*

(i) *A is $*$-paracompact $\implies$ A is α-paracompact;*
(ii) *A is $*$-paracompact $\implies$ A is paracompact.*

3.4. Definition. [13] Let (X, τ) be an L-ts, $A \in L^X$, and $\alpha \in M(L)$. Then A is called *α-metacompact* (resp. *$\alpha*$-metacompact*) if every open α-Q-cover of A has a point finite (resp.$*$-point finite) open refinement which is also an α-Q-cover of A. In particular, A is called *metacompact* (resp. *$*$-metacompact*) if A is α-metacompact (resp. $\alpha*$-metacompact) for every $\alpha \in M(L)$.

Finally, (X, τ) is *metacompact* (resp. *$*$-metacompact*) if $\top$ is metacompact (resp.$*$-metacompact).

3.5. Theorem. [13] *Let (X, τ) be a weakly induced L-ts. Then the following are equivalent:*

(i) *(X, τ) is metacompact;*
(ii) *There exist $\alpha \in M(L)$ such that (X, τ) is α-metacompact;*
(iii) *$(X, [\tau])$ is metacompact;*
(iv) *For every $\alpha \in M(L)$, every well monotone open α-Q-cover of $\top$ has a point finite open refinement which is also an α-Q-cover of $\top$;*
(v) *There exists $\alpha \in M(L)$ such that every well monotone open α-Q-cover of $\top$ has a point finite open refinement which is also an α-Q-cover of $\top$;*
(vi) *For every $\alpha \in M(L)$, every directed open α-Q-cover of $\top$ has a closure preserving closed refinement which is also an α-Q-cover of $\top$;*
(vii) *There exist $\alpha \in M(L)$ such that every directed open α-Q-cover of $\top$ has a closure preserving closed refinement which is also an α-Q-cover of $\top$;*

(viii) *For every* $\alpha \in M(L)$ *and every open* α*-*Q*-cover* $\mathbf{U}$ *of* $\top$, $\mathbf{U}^F$ *has a closure preserving closed refinement which is also an* α*-*Q*-cover of* $\top$*;*

(ix) *There exist* $\alpha \in M(L)$ *such that for every open* α*-*Q*-cover* $\mathbf{U}$ *of* $\top$, $\mathbf{U}^F$ *has a closure preserving closed refinement which is also an* α*-*Q*-cover of* $\top$*.*

3.6. Definition. [4] Let (X, τ) be an L-ts, $A \in L^X$, and $\alpha \in M(L)$. Then A is called *α-subparacompact* (*$\alpha*$-subparacompact*) if for every open α-Q-cover Φ of A, there exist a closed refinement Ψ of Φ which is σ-discrete ($\sigma*$-discrete) in A and Ψ is also an α-Q-cover of A.

In particular, A is *subparacompact* (*$*$-subparacompact*) if A is α-subparacompact ($\alpha*$-subparacompact) for every $\alpha \in M(L)$. Finally, (X, τ) is *subparacompact* (*$*$-subparacompact*) if $\top$ is subparacompact ($*$-subparacompact).

3.7. Proposition. [4] *Let* (X, τ) *be an* L*-ts,* $A \in L^X$*, and* $\alpha \in M(L)$*. Then:*

(i) A *is* $\alpha*$*-subparacompact* $\Longrightarrow$ A *is* α*-subparacompact;*

(ii) A *is* $*$*-subparacompact* $\Longrightarrow$ A *is subparacompact.*

3.8. Theorem. [4] *Let* (X, τ) *be a weakly induced* L*-ts. Then the following conditions are equivalent.*

(i) (X, τ) *is* $*$*-subparacompact;*

(ii) *There exist* $\alpha \in M(L)$ *such that* (X, τ) *is* $\alpha*$*-subparacompact;*

(iii) $(X, [\tau])$ *is subparacompact.*

3.9. Definition. [4] Let (X, τ) be an L-ts, $A \in L^X$, and $\alpha \in M(L)$. Then A is called *α-submetacompact* (*$\alpha*$-submetacompact*) if for every open α-Q-cover of A there is a θ-sequence ($*$-θ-sequence) of open α-Q-cover refinements.

In particular, A is *submetacompact* (*$*$-submetacompact*) if A is α-submetacompact ($\alpha*$-submetacompact) for every $\alpha \in M(L)$. Finally, (X, τ) is submetacompact ($*$-submetacompact) if $\top$ is submetacompact ($*$-submetacompact).

3.10. Theorem. *Let* (X, τ) *be a weakly induced* L*-ts. Then the following conditions are equivalent:*

(i) (X, τ) *is submetacompact;*

(ii) *There exist* $\alpha \in M(L)$ *such that* (X, τ) *is* α*-submetacompact;*

(iii) $(X, [\tau])$ *is submetacompact.*

Proof. (i) $\Longrightarrow$ (ii). Clear.

(ii) $\Longrightarrow$ (iii). Let $\mathbf{U} \subset [\delta]$ be an open cover of X. So $\{\chi_U : U \in \mathbf{U}\}$ is an open α-Q-cover of $\top$. By (ii) it has a θ-sequence of open α-Q-cover refinements say $\mathbf{V} = \{\mathbf{V}_n\}$. For each $V_n \in \mathbf{V}_n$ take $V_{n(\alpha')} = \{x \in X : V_n(x) \not\leq \alpha'\}$, and consider the collection $\mathbf{W}_n = \{V_n(\alpha')) : V_n \in \mathbf{V}_n\}$. Now by the weakly induced property of (X, τ), $\mathbf{W}_n$ is an open cover of $(X, [\tau])$, and obviously $\mathbf{W}_n$ is a point finite open refinement of $\mathbf{U}$. Then $\mathbf{W} = \{\mathbf{W}_n\}$ is a θ-sequence of $\mathbf{U}$. Hence (ii) $\Longrightarrow$ (iii).

(iii) $\Longrightarrow$ (i). Let $\alpha \in M(L)$ and $\mathbf{U} \subset \tau$ be an α-Q cover of $\top$. Since (X, τ) is weakly induced, $\{U_{(\alpha')} : U \in \mathbf{U}\}$ is an open cover of $(X, [\tau])$. Therefore it has a θ-sequence of open refinements, say $\mathbf{V} = \{\mathbf{V}_n\}$.

For every $V_n \in \mathbf{V}_n$, take $U_{V_n} \in \mathbf{U}$ such that $V_n \subset U_{V_n(\alpha')}$. Let $\mathbf{W}_n = \{\chi_{v_n} \wedge U_{V_n} : V_n \in \mathbf{V}_n\}$. Clearly, $\mathbf{W}_n$ is an open α-Q-cover refinement of $\mathbf{U}$. Consider $\mathbf{W} = \{\mathbf{W}_n\}$. We will show that each $\mathbf{W}_n$ is point finite.

Let $x_\alpha \in M(L^X)$. Since $\mathbf{V}_n$ is point finite, it follows clearly that $x \in V_1, V_2, \ldots, V_n$ for some $n \in N$ and $V_i \in \mathbf{V}_n$ for $i = 1, 2, \cdots, n$. Next we have to show that $x_\alpha \prec \chi_{V_i} \wedge U_{V_i}$ for at most finitely many i. For, if possible let $x_\alpha \prec \chi_{v_i} \wedge U_{V_i}$ for infinitely many

$V_i \in \mathbf{V}_n$. Then $x_\alpha \prec \chi_{V_i}$ or $x_\alpha \prec U_{V_i}$ for infinitely many $V_i \in \mathbf{V}_n$. In both cases $x \in V_i$ for infinitely many $V_i \in \mathbf{V}_n$. Therefore $\mathbf{W} = \{\mathbf{W}_n\}$ is a θ-sequence of $\mathbf{U}$, and thus (iii) $\Longrightarrow$ (i). This completes the proof. □

3.11. Definition. Let (X, τ) be an L-ts, $A \in L^X$, and $\alpha \in M(L)$. Then A is called *α-para-Lindelof* (*$\alpha*$-para-Lindelof*) if for every open α-Q-cover Φ of A, there exist an open refinement Ψ of Φ which is locally countable ($*$-locally countable) in A and Ψ is also an α-Q-cover of A.

In particular, A is *para-Lindelof* (*$*$-para-Lindelof*) if A is α-para-Lindelof ($\alpha*$-para-Lindelof) for every $\alpha \in M(L)$. Finally, (X, τ) is *para-Lindelof* (*$*$-para-Lindelof*) if $\top$ is para-Lindelof ($*$-para-Lindelof).

3.12. Theorem. *Let (X, τ) be a weakly induced L-ts. Then the following conditions are equivalent:*

(i) *(X, τ) is para-Lindelof;*
(ii) *There exist $\alpha \in M(L)$ such that (X, τ) is α-para-Lindelof;*
(iii) *$(X, [\tau])$ is para-Lindelof.*

Proof. (i) $\Longrightarrow$ (ii). Obvious.

(ii) $\Longrightarrow$ (iii). Let $\mathbf{U} \subset [\tau]$ be an open cover of X. Now $\mathbf{U}^* = \{\chi_U : U \in \mathbf{U}\}$ is an open α-Q-cover of $\top$, and it has a locally countable refinement $\mathbf{V}$ which is also an α-Q-cover of $\top$. Let

$$\mathbf{W} = \{V_{(\alpha')} : V \in \mathbf{V}\}.$$

Clearly $\mathbf{W}$ is both a refinement of $\mathbf{U}$ and a cover of X. Since (X, τ) is weakly induced, we have $\mathbf{W} \subset [\tau]$. Now we want to prove that $\mathbf{W}$ is locally countable. Let $x \in X$. Since (X, τ) is α-para-Lindelof, there exist $B \in Q(x_\alpha)$ such that B only quasi coincides with a countable number of members $V_0, V_1, V_2, \ldots$ of $\mathbf{V}$. Let $O = B_{(\perp)}$. By the weakly induced property of (X, τ), $O \in [\tau]$, for every $V \in \mathbf{V}$, if $O \cap V_{(\alpha')} \neq \phi$, then there exists an ordinary point $y \in O \cap V_{(\alpha')}$, and hence $B(y) \nleq \perp$, $V(y) \nleq \alpha'$.

Therefore $V(y)' < \alpha$, and it follows that $B(y) \nleq V(y)'$ and thus $B \; q \; V$. So $V \in \{V_0, V_1, V_2, \ldots\}$, O intersects only a countable number of members $V_{0(\alpha')}, V_{1(\alpha')}, V_{2(\alpha')}, \ldots$ of $\mathbf{W}$. Hence $(X, [\tau])$ is para-Lindelof.

(iii) $\Longrightarrow$ (i). Suppose that $\alpha \in M(L)$ and let $\mathbf{U} \subset \tau$ be an open α-Q-cover of $\top$. Since (X, τ) is weakly induced,

$$\mathbf{U}^* = \{U_{(\alpha')} : U \in \mathbf{U}\}$$

is an open cover of $(X, [\tau])$. Since $(X, [\tau])$ is para-Lindelof, there exist a refinement $\mathbf{V}$ of $\mathbf{U}^*$ which is also a locally countable cover of X.

For every $V \in \mathbf{V}$, let $U_V \in \mathbf{U}$ be such that $V \subset U_{V(\alpha')}$. Let

$$\mathbf{W} = \{\chi_V \wedge U_V : V \in \mathbf{V}\}.$$

Clearly $\mathbf{W}$ is both a refinement of $\mathbf{U}$ and an α-Q-cover of $\top$. Next we will prove that $\mathbf{W}$ is locally countable. Let $x_\alpha \in M(L^X)$. Then since $\mathbf{V}$ is locally countable, there exist a neighbourhood B of x such that B intersects with V_i for countably many $V_i \in \mathbf{V}$. Now we have $\chi_B \in Q(x_\alpha)$. We will show that $\chi_B \; q \; \chi_{V_i} \wedge U_{V_i}$ for at most countably many i. For if possible, let $\chi_B \; q \; \chi_V \wedge U_V$ for uncountably many $V \in \mathbf{V}$.

Then $\chi_B \; q \; \chi_V$ or $\chi_B \; q \; U_V$ for uncountably many $V \in \mathbf{V}$. In both cases B intersects with V for uncountably many $V \in \mathbf{V}$, which is a contradiction and hence W is locally countable. Therefore (X, τ) is α-para-Lindelof. This completes the proof. □

3.13. Definition. Let (X,τ) be an L-ts, $A \in L^X$, and $\mathbf{B} \subset L^X$. Then

$$\mathrm{st}(A,\mathbf{B}) = \bigvee\{B \in \mathbf{B} : B\ q\ A\}$$

is defined as the *star* of $\mathbf{B}$ about A. If $x_\lambda \in M(L^X)$, then $\mathrm{st}(\{x_\lambda\},\mathbf{B})$ is denoted by $\mathrm{st}(x_\lambda,\mathbf{B})$.

3.14. Theorem. *Let (X,τ) be an L-ts. Then the following are equivalent:*

(i) *(X,τ) is para-Lindelof;*
(ii) *For every open α-Q-cover A of $\top$, there is a locally countable refinement $\mathbf{B}$ such that if $x_\alpha \in M(L^X)$ then $x_\alpha \in \mathrm{int}\,(\mathrm{st}(x_\alpha,\mathbf{B}))$.*

Proof. (i) $\Longrightarrow$ (ii). Obvious.

(ii) $\Longrightarrow$ (i). Suppose $\mathbf{A} = \{A_t : t \in T\}$ is an open α-Q-cover of $\top$. Let $\mathbf{B} = \{B_t : t \in T\}$ be a locally countable refinement, as given in (ii). Let $\mathbf{C}$ be an open α-Q-cover of $\top$ such that every element of $\mathbf{C}$ intersects at most countably many elements of $\mathbf{B}$. Then for every $x_\alpha \in M(L^X)$, there is a locally countable refinement $\mathbf{D}$ of $\mathbf{C}$ such that $x_\alpha \in \mathrm{int}\,(\mathrm{st}(x_\alpha,\mathbf{D}))$.

For each $B \in \mathbf{B}$, take $A_B \in \mathbf{A}$ such that $B \le A_B$ and let $G_B = \mathrm{int}\,(\mathrm{st}(B,\mathbf{D})) \wedge A_B$. Then clearly $\mathbf{G} = \{G_B : B \in \mathbf{B}\}$ is an α-Q-cover of $\top$, and hence is an open refinement of $\mathbf{A}$. To show $\mathbf{G}$ is locally countable, let $x_\alpha \in M(L^X)$ and take $W \in Q(x_\alpha)$ such that W intersects only countably many elements of $\mathbf{D}$. Since each $D \in \mathbf{D}$ intersects only countably many elements of $\mathbf{B}$, it follows that W intersects only countably many elements of $\{st(B,\mathbf{D}) : B \in \mathbf{B}\}$. Hence $\mathbf{G}$ is locally countable, and the theorem is proved. □

3.15. Definition. Let (X,τ) be an L-ts and $\alpha \in M(L)$. Then (X,τ) is called *σ-para-Lindelof* if for every open α-Q-cover Φ of $\top$, there exist an open refinement Ψ of Φ which is σ-locally countable in $\top$ and is also an α-Q-cover of $\top$.

The methodology used to prove Theorem 3.14 can be applied to the following Theorem:

3.16. Theorem. *Let (X,τ) be an L-ts. Then the following are equivalent:*

(i) *(X,τ) is σ-para-Lindelof;*
(ii) *For any open α-Q-cover $\mathbf{A}$ of (X,τ), there is a σ-locally countable refinement $\mathbf{B} = \bigcup \mathbf{B}_i$ such that if $x_\alpha \in M(L^X)$ then $x_\alpha \in \mathrm{int}\,(\mathrm{st}(x_\alpha,\mathbf{B}_k))$ for some $k \in \mathbf{N}$.* □

4. Covering properties under mappings

4.1. Definition. [2] Let (X,τ), (Y,μ) be L-ts's, and $f^{\rightarrow} : L^X \to L^Y$ an L-fuzzy mapping. Then $f^{\rightarrow}$ is *perfect* if it is continuous, closed and $f^{\leftarrow}(y)$ is N-compact for every $y \in Y$.

4.2. Results. [23] *If (X,τ), (Y,μ) are two weakly induced L-topological spaces, then*

(i) *If the map $f^{\rightarrow} : L^X \to L^Y$ is L-fuzzy continuous, then $f : (X,[\tau]) \to (Y,[\mu])$ is continuous;*
(ii) *If the map $f^{\rightarrow} : L^X \to L^Y$ is L-fuzzy closed, then $f : (X,[\tau]) \to (Y,[\mu])$ is closed;*
(iii) *If the map $f^{\rightarrow} : L^X \to L^Y$ is L-fuzzy open, then $f : (X,[\tau]) \to (Y,[\mu])$ is open.*

4.3. Theorem. *Let (X,τ), (Y,μ) be two weakly induced L-topological spaces. Then if $f^{\rightarrow} : L^X \to L^Y$ is perfect, so is $f : (X,[\tau]) \to (Y,[\mu])$.*

Proof. Let $y_\alpha \in M(L^Y)$. Since $f^\rightarrow : L^X \to L^Y$ is perfect, $f^\leftarrow(y_\alpha)$ is N-compact. Now to prove $f : (X, [\tau]) \to (Y, [\mu])$ is perfect, it is enough to prove that $f^\leftarrow(y_\alpha)$ is compact for every $y \in Y$.

Now let $\mathbf{U} \in [\tau]$ be an open cover of $f^{-1}(y)$. Consider $\mathbf{U}* = \{\chi_U : U \in \mathbf{U}\}$. This is an open α-Q-cover of $f^\leftarrow(y_\alpha)$. For, let $x_\alpha \leq f^\leftarrow(y_\alpha)$. i.e., $f^\leftarrow(y_\alpha)(x) = y_\alpha(f(x)) \geq \alpha$. Now let $U \in \mathbf{U}$ be such that $x \in U$. This is possible since U is a cover of $f^{-1}(y)$. Then it follows that $\chi_U(x) \geq y_\alpha \geq \alpha$. i.e., $\chi_U(x) \geq \alpha$ or $x_\alpha \leq \chi_U$. Hence clearly $x_\alpha \prec \chi_U$. Therefore $\{\chi_U : U \in \mathbf{U}\}$ is an open α-Q-cover of $f^\leftarrow(y_\alpha)$.

Again, $f^\leftarrow(y_\alpha)$ being N-compact, there exists a finite sub collection $\mathbf{U}_s^*$ of $\mathbf{U}^*$ which is also an α^--Q cover of $f^\leftarrow(y_\alpha)$. Let

$$\mathbf{U}_s^* = \{\chi_{U1}, \chi_{U2}, \ldots, \chi_{Uk}\}.$$

Then clearly $\{U_1, U_2, \ldots, U_k\}$ will be a finite sub cover of $f^{-1}(y)$. This completes the proof. □

4.4. Theorem. *Let (X, τ), (Y, μ) be two weakly induced L-ts's and $f^\rightarrow : L^X \to L^Y$ a closed continuous map. Then:*

(i) *If (X, τ) is paracompact, then so is (Y, μ);*
(ii) *If (X, τ) is metacompact, then so is (Y, μ);*
(iii) *If (X, τ) is subparacompact, then so is (Y, μ);*
(iv) *If (X, τ) is submetacompact, then so is (Y, μ).*

Proof. We prove only (ii). The proofs of (i), (iii) and (iv) can be arrived at in a similar manner.

Suppose (X, τ) is metacompact. Let $\mathbf{A}$ be an open α-Q-cover of (Y, μ). By Theorem 3.5 it is enough to show that $\mathbf{A}^F$ has a closure preserving closed refinement which is also an α-Q-cover of (Y, μ). Let

$$\mathbf{W} = \{f^\leftarrow(A) : A \in \mathbf{A}\}.$$

Clearly $\mathbf{W}$ is an open α-Q-cover of (X, τ), and since (X, τ) is metacompact, $\mathbf{W}^F$ has a closure preserving closed refinement $\mathbf{B}$ which is also an α-Q-cover of (X, τ). Now $\{f^\rightarrow(B) : B \in \mathbf{B}\}$ is the desired closure preserving closed refinement of $\mathbf{A}^F$. Hence the proof is complete. □

4.5. Theorem. *Let (X, τ), (Y, μ) be two weakly induced L-ts's and $f^\rightarrow : L^X \to L^Y$ a perfect map. Then:*

(i) *(X, τ) is paracompact if and only if (Y, μ) is paracompact;*
(ii) *(X, τ) is metacompact if and only if (Y, μ) is metacompact;*
(iii) *(X, τ) is subparacompact if and only if (Y, μ) is subparacompact;*
(iv) *(X, τ) is submetacompact if and only if (Y, μ) is submetacompact;*
(v) *(X, τ) is para-Lindelof if and only if (Y, μ) is para-Lindelof.*

Proof. (i). By Theorem 4.3 and Propositions 3.2 and 3.3.

(ii). By Theorem 4.3 and Theorem 3.5.

(iii). By Theorem 4.3, Proposition 3.7 and Theorem 3.8.

(iv). By Theorem 4.3 and Theorem 3.10.

(v). By Theorem 4.3 and Theorem 3.12. □

4.6. Definition. Let (X, τ) and (Y, μ) be L-topological spaces, $f^\rightarrow : L^X \to L^Y$ a closed or open L-fuzzy mapping. We say $A \subseteq L^X$ is *saturated* with respect to $f^\rightarrow$; whenever it is the complete inverse image of some set in L^Y. That is A is saturated if and only if $A = f^\leftarrow(f^\rightarrow(A))$.

4.7. Theorem. *Let (X,τ) and (Y,μ) be two weakly induced L-ts's. If (X,τ) is para-Lindelof and $f^{\rightarrow}: L^X \rightarrow L^Y$ a closed map with $f^{\leftarrow}(y_\alpha)$ Lindelof for each $y_\alpha \in M(L^Y)$, then (Y,μ) is para-Lindelof.*

Proof. Let $\mathbf{U}$ be an open α-Q-cover of (Y,μ). Then $\{f^{\leftarrow}(U) : U \in \mathbf{U}\}$ is an α-Q-cover of (X,τ), and we let $\mathbf{W} = \{W_t : t \in T\}$ be a locally countable open α-Q-cover refinement of $\{f^{\leftarrow}(U) : U \in \mathbf{U}\}$.

Now for any $y_\alpha \in M(L^Y)$, $f^{\leftarrow}(y_\alpha)$ is Lindelof so there is an open set $G_{y\alpha}$ in L^X such that $f^{\leftarrow}(y_\alpha) \leq G_{y\alpha}$ and $G_{y\alpha} \leq W_t$ for countably many $t \in T$. Take $V_{y\alpha}$ as the saturated part of $G_{y\alpha}$. Then $f^{\rightarrow}(V_{y\alpha})$ is an open set about y_α. Consider

$$\mathbf{H} = \{f^{\rightarrow}(W_t) : W_t \in \mathbf{W}\}.$$

Now $f^{\rightarrow}(V_{y\alpha})$ meets only countably many elements of $\mathbf{H}$. Hence $\mathbf{H}$ is locally countable, and it is clear that $y_\alpha \in \operatorname{int}(\operatorname{st}(y_\alpha, \mathbf{H}))$ for every $y_\alpha \in L^Y$. Since $\mathbf{H}$ is a refinement of $\mathbf{U}$, it follows from Theorem 3.14 that (Y,μ) is para-Lindelof. □

4.8. Theorem. *Let (X,τ), (Y,μ) be two weakly induced L-ts's. If (X,τ) is σ-para-Lindelof and $f^{\rightarrow}: L^X \rightarrow L^Y$ a perfect map then (Y,μ) is σ-para-Lindelof.*

Proof. Let $\mathbf{U}$ be an open α-Q-cover of Y and $\mathbf{W} = \bigcup \mathbf{W}_i$ an open refinement of $\{f^{\leftarrow}(U) : U \in \mathbf{U}\}$, where each $\mathbf{W}_i$ is locally countable. Since each $f^{\leftarrow}(y_\alpha)$ is N-compact, without loss of generality we may assume that $f^{\leftarrow}(y_\alpha) \subset \bigcup \mathbf{W}_k$ for some k. Consider

$$\mathbf{H}_i = \{f^{\rightarrow}(W) : W \in \mathbf{W}_i\}.$$

Then $\mathbf{H} = \bigcup \mathbf{H}_i$ is a refinement of $\mathbf{U}$, where each $\mathbf{H}_i$ is locally countable, and if $y_\alpha \in L^Y$, then $y_\alpha \in \operatorname{int}(\operatorname{st}(y_\alpha, \mathbf{H}_i))$ for some i. The result now follows from Theorem 3.16. □

4.9. Theorem. *Let (X,τ) be an L-ts and $\alpha \in M(L)$. Then (X,τ) is metacompact if and only if for every open α-Q-cover A of $\top$, there is a point finite open refinement $\mathbf{B}$ such that $x_\alpha \in \operatorname{int}(\operatorname{st}(x_\alpha, \mathbf{B}))$ for every $x_\alpha \in M(L^X)$.*

Proof. (i) $\Longrightarrow$ (ii). Obvious.

(ii) $\Longrightarrow$ (i). Suppose that $\alpha \in M(L)$ and that $\mathbf{U} \subset \tau$ is an open α-Q-cover of $\top$. By (ii), there exists a point finite refinement $\mathbf{B}$ such that $x_\alpha \in \operatorname{int}(\operatorname{st}(x_\alpha, \mathbf{B}))$. That is, $x_\alpha \in \operatorname{int}(\bigvee\{B \in \mathbf{B} : x_\alpha \prec B\})$. Then $\mathbf{B}$ is an α-Q-cover of $\top$ and hence (X,τ) is metacompact. □

4.10. Definition. Let (X,τ), (Y,μ) be L-ts's and $f^{\rightarrow}: L^X \rightarrow L^Y$ an L-fuzzy mapping. Then $f^{\rightarrow}$ is said to be an *open compact map* if it is open and $f^{\leftarrow}(y)$ is N-compact for every $y \in Y$.

4.11. Definition. Let (X,τ), (Y,μ) be L-ts's and $f^{\rightarrow}: L^X \rightarrow L^Y$ an L-fuzzy mapping. We say $f^{\rightarrow}$ is *pseudo-open* if whenever $f^{\leftarrow}(y_\alpha) \leq U$, where $y_\alpha \in L^Y$ and $U \in L^X$, then $y_\alpha \in \operatorname{int}(f^{\rightarrow}(U))$.

4.12. Theorem. *Let (X,τ) and (Y,μ) be L-ts's. If (X,τ) is paracompact and $f^{\rightarrow}: L^X \rightarrow L^Y$ a pseudo-open compact mapping, then (X,τ) is metacompact.*

Proof. Suppose $\mathbf{U}$ is an open α-Q-cover of (Y,μ). Then $\mathbf{V} = \{f^{\leftarrow}(U) : U \in \mathbf{U}\}$ is an open α-Q-cover of (X,τ), and $\mathbf{W}$ is a locally finite refinement of $\mathbf{V}$. Consider

$$\mathbf{H} = \{f^{\rightarrow}(W) : W \in \mathbf{W}\}.$$

Since $\mathbf{W}$ is locally finite, every $f^{\leftarrow}(y_\alpha)$ intersects at most finitely many elements of $\mathbf{W}$. Now it follows that $\mathbf{H}$ is a point finite refinement, and using the pseudo open condition it is clear that $y_\alpha \in \operatorname{int}(\operatorname{st}(y_\alpha, \mathbf{H}))$ for every $y_\alpha \in L^Y$. Now by applying Theorem 4.9, the proof is complete. □

Acknowledgments The authors are thankful to the referees who suggested some changes to the original version of the paper.

References

[1] Arens, R. and Dugundji, J. *Remark on the concept of compactness*, Portugal. Math. **9**, 141–143, 1950.

[2] Baiju, T. and Jacob John, S. *Finitistic spaces in L-topological spaces*, Proyecciones Journal of Mathematics **28** (1), 47–56, 2009.

[3] Baiju, T. and Jacob John, S. *Fuzzy submetacompact spaces*, (communicated).

[4] Baiju, T. and Jacob John, S. *Subparacompactness in L-topological spaces*, (communicated).

[5] Burke, D. K. *Covering properties*, in K. Kunen, J. E Vaughan (Eds.), *Hand Book of Set Theoretic Topology* (Elsevier Science Publishers, 1984), 349–422.

[6] Burke, D. K. *On subparacompact spaces*, Proc. Amer. Math. Soc. **23**, 655–663, 1969.

[7] Chang, C. L. *Fuzzy topological spaces*, J. Math. Anal. Appl. **24**, 182–190, 1968.

[8] Creede, G. D. *Concerning semi-stratifiable spaces*, Pacific Journal of Mathematics **32** (1), 47–54, 1970.

[9] Dieudonne, J. *Une generalization des espaces compact*, J. Math. Pures. Appl. **23**, 65–76, 1944.

[10] Gantner, T. E., Steinlage, R. C. and Warren, R. H. *Compactness in fuzzy topological spaces*, J. Math. Anal. Appl. **62**, 547–562, 1978.

[11] Greever, J. *On some generalized compactness properties*, Publ. Res. Inst. Math. Soc. Ser. A **4** (1), 39–49, 1968.

[12] Hohle, U. and Rodabaugh, S. E. *Mathematics of Fuzzy Sets : Logic, Topology and Measure Theory* (The Hand Book of Fuzzy Set Series 3, Kluwer Academic Pub., 1999).

[13] Jacob John, S. *Metacompactness in the L-fuzzy context*, The Journal of Fuzzy Mathematics **8** (3), 661–668, 2000.

[14] Jacob John, S. and Baiju, T. *Metacompactness in L-topological spaces*, Iranian Journal of Fuzzy Systems **5** (3), 71–79, 2008.

[15] Kubiak, T. *The topological modification of the L-fuzzy unit interval*, in: S. E. Rodabaugh, E. P. Klement, U. Hohle (Eds.), *Applications of Category Theory to Fuzzy Subsets*, (Kluwer Academic Publishers, Dordrecht, 1992), 275–305.

[16] Mao-Kang, L. *Paracompactness in fuzzy topological spaces*, J. Math. Anal. Appl. **130** (1), 55–77, 1988.

[17] McAuley, L. F. *A note on complete collectionwise normality and paracompactness*, Proc. Amer. Math. Soc. **9**, 796–799, 1958.

[18] Shi, F.-G. *A new notion of fuzzy compactness in L-topological spaces*, Information Sciences **173**, 35–48, 2005.

[19] Shi, F.-G. and Zheng, C.-Y. *Paracompactness in L-topological spaces*, Fuzzy Sets and Systems **129**, 29–37, 2002.

[20] Wang, G. J. *On the structure of fuzzy lattices*, Acta math. Sinica **29**, 539–543, 1986.

[21] Wang, G. J. *Theory of L-fuzzy topological spaces* (Shaanxi Normal University Pub., Xian, 1988).

[22] Worrel, J. M. and Wicke, H. H. *Characterizations of developable spaces*, Canad. J. Math. **17**, 820–830, 1965.

[23] Ying-Ming, L. and Mao-Kang, L. *Fuzzy Topology* (Advances in Fuzzy Systems Applications and Theory **9**, World Scientific, 1997).

[24] Zadeh, L. A. *Fuzzy sets*, Information and Control **8**, 338–353, 1965.