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# DISCRETE DISTRIBUTIONS CONNECTED WITH THE BIVARIATE BINOMIAL

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**Abstract**

A new class of multivariate discrete distributions with binomial and multinomial marginals is studied. This class of distributions is obtained in a natural manner using probabilistic properties of the sampling model considered. Some possible applications in game theory, life testing and exceedance models for order statistics are discussed.

**Keywords:** Discrete multivariate distributions, Bivariate binomial distribution, Multinomial distribution, Probability density function, Poisson approximation.

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## 1. Introduction

Bivariate and multivariate binomial distributions have aroused the interest of many authors as a natural extension of the univariate binomial distribution. Aitken and Gonin [1] derived bivariate binomial probability functions by considering sampling with replacement from a fourfold population, and expressed the bivariate probability function as products of the corresponding univariate functions, multiplied by a terminating series bilinear in the appropriate orthogonal polynomials. Krishnamoorthy [17] studied the multivariate binomial distribution and extended the series of Aitken and Gonin [1] for a bivariate binomial distribution to any number of variables. In the papers of Hamdan [10, 11], Hamdan and Al-Bayati [12], Hamdan and Jensen [13], Papageorgiou and David [19, 20], Doss and Graham [8], Shanbhag and Basawa [21], the conditional distributions associated with trivariate and multivariate binomial distributions were studied, and characterizations of multivariate binomial distribution by univariate marginals established. For some discussions on bivariate binomial distributions see Kocherlakota and Kocherlakota [16] and Johnson *et al.* [15].

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Biswasa and Hwang [4] provide a new formulation of the bivariate binomial distribution in the sense that marginally each of the two random variables has a binomial distribution, and they have some non-zero correlation in the joint distribution. Chandrasekar and Balakrishnan [6] considered a trivariate binomial distribution and obtained regression equations of this distribution. They provided a set of necessary and sufficient conditions for the regression to be linear, and also established a characterization of the trivariate binomial distribution based on the distribution of the sum of two trivariate random vectors.

In the present paper we consider new trivariate and quadrivariate distributions constructed on the basis of a bivariate binomial distribution. These distributions appear in several models in the contexts of lifetesting and exceedances, and can also be applied in strategic games. We also consider an extension of the bivariate binomial model to the case when each individual of a population is being classified as one of $A_1, A_2, \ldots, A_m$ and simultaneously as one of $B_1, B_2, \ldots, B_m$, with probabilities given by $P(A_iB_j) = p_{ij}$, $i, j = 1, 2, \ldots, m$, $\sum p_{ij} = 1$, $P(A_i) = \sum_{j=1}^m P(A_iB_j)$, $P(B_j) = \sum_{i=1}^m P(A_iB_j)$. Let the experiment be repeated $n$ times. Assume that $\chi_1$, $\chi_2$, $\chi_{11}$, $\chi_{12}$ and $\chi_{21}$ are the numbers of occurrences of the events $A_1, B_1, A_1B_1, A_1B_2$ and $A_2B_1$ in these $n$ repetitions, respectively. We study the joint distributions of the random variables and discuss their possible applications.

For a description of a simple bivariate binomial distribution consider the fourfold model:

| $A\backslash B$ | $B_1$ | $B_2$ |
|---|---|---|
| $A_1$ | $\pi_{11}$ | $\pi_{12}$ |
| $A_2$ | $\pi_{21}$ | $\pi_{22}$ |

wherein each individual of a population can be classified as being one of $A_1, A_2$ and at the same time as one of $B_1, B_2$ with probabilities $P(A_iB_j) = \pi_{ij}$, $i, j = 1, 2$; $\sum_{ij} \pi_{ij} = 1$. Under random sampling with replacement $n$ times, let $\xi_1$ and $\xi_2$ denote the number of occurrences of $A_1$ and $B_1$, respectively. It is well known that

$$\begin{aligned} \mathbf{p}_1(k,l) &= P\left\{\xi_1 = k, \xi_2 = l\right\} \\ &= \sum_{i=\max(0,k+l-n)}^{\min(k,l)} \frac{n!}{i!(k-i)!(l-i)!(n-k-l+i)} \pi_{11}^i \pi_{12}^{k-i} \pi_{21}^{l-i} \pi_{22}^{n-k-l+i}, \end{aligned} \tag{1}$$

(see Aitken and Gonin [1] and Johnson, Kotz and Balakrishnan [15]). The bivariate discrete distribution given in (1) is called a *bivariate binomial distribution*. The corresponding probability generating function (pgf) is

$$\Phi_1(t,s) = (\pi_{11}ts + \pi_{12}t + \pi_{21}s + \pi_{22})^n. \tag{2}$$

A connection between a bivariate binomial distribution and a multinomial distribution can be shown as follows. In the fourfold model described above, let $A_1B_1 = C_1$, $A_1B_2 = C_2$, $A_2B_1 = C_3$, $A_2B_2 = C_4$ and $P(C_1) = p_{11}$, $P(C_2) = p_{12}, P(C_3) = p_{21}$, $P(C_4) = p_{22}$. Let $\zeta_i$ be the number of cases in which $C_i$ occurs in $n$ repetitions, $i = 1, 2, 3, 4$. Clearly, $(\zeta_1, \zeta_2, \zeta_3, \zeta_4)$ is multinomial. Then $\xi_1 = \zeta_1 + \zeta_2$ and $\xi_2 = \zeta_1 + \zeta_3$.

A simple trivariate distribution in the fourfold model described above is of interest. Under random sampling $n$ times, let $\xi_1, \xi_2$ and $\xi_{11}$ be the number of occurrences of $A_1, B_1$ and $A_1B_1$, respectively. The joint probability function of the random variables $\xi_1$, $\xi_2$ and

$\xi_{11}$ can be obtained easily from combinatorial considerations, and it is

$$\begin{aligned} p_n(k,l,r) &= P\{\xi_1 = k, \xi_2 = l, \xi_{11} = r\} \\ &= \frac{n!}{r!(k-r)!(l-r)!(n-k-l+r)!}\pi_{11}^r \pi_{12}^{k-r}\pi_{21}^{l-r}\pi_{22}^{n-k-l+r}, \\ &\qquad (k,l = 0,1,2,\ldots,n \text{ and } r = \max(0,k+l-n),\ldots,\min(k,l)). \end{aligned} \tag{3}$$

The corresponding probability generating function is

$$\Psi(t,s,z) = (\pi_{11}tsz + \pi_{12}t + \pi_{21}s + \pi_{22})^n .$$

It is clear that the univariate marginals of the discrete random vector $(\xi_1, \xi_2, \xi_{11})$ are binomial, with cell probabilities $(\pi_{11}+\pi_{12})$, $(\pi_{11}+\pi_{21})$ and $\pi_{11}$, respectively. The joint distribution of $(\xi_1, \xi_2)$ is obviously the bivariate binomial distribution with probability function (1) and pgf $\Psi(t,s,1) = (\pi_{11}ts + \pi_{12}t + \pi_{21}s + \pi_{22})^n$, as in (2).

The joint probability function of $(\xi_1, \xi_{11})$ is

$$\begin{aligned} &P\{\xi_1 = k, \xi_{11} = r\} \\ &\qquad = \sum_{l=0}^{n} \frac{n!}{r!(k-r)!(l-r)!(n-k-l+r)!}\pi_{11}^r \pi_{12}^{k-r}\pi_{21}^{l-r}\pi_{22}^{n-k-l+r}, \end{aligned}$$

and the pgf is

$$\Psi_{1,11}(t,z) = \Psi(t,1,z) = (\pi_{11}tz + \pi_{12}t + \pi_{21} + \pi_{22})^n .$$

Similarly, the joint probability function of $(\xi_2, \xi_{11})$ is

$$\begin{aligned} &P\{\xi_2 = l, \xi_{11} = r\} \\ &\qquad = \sum_{k=0}^{n} \frac{n!}{r!(k-r)!(l-r)!(n-k-l+r)!}\pi_{11}^r \pi_{12}^{k-r}\pi_{21}^{l-r}\pi_{22}^{n-k-l+r}, \end{aligned}$$

and the corresponding pgf is

$$\Psi_{2,11}(s,z) = \Psi(1,s,z) = (\pi_{11}sz + \pi_{21}s + \pi_{12} + \pi_{22})^n .$$

The Poisson procedure allows us to obtain the formula that approximates the joint probability function $p_n(k,l,r)$ when the number of trials is large, $(n \to \infty)$ and $n\pi_{11} \to \lambda_{11}$, $n\pi_{12} \to \lambda_{12}$, $n\pi_{21} \to \lambda_{21}$. We have

$$\begin{aligned} &\lim_{n\to\infty} P\{\xi_1 = k, \xi_2 = l, \xi_{11} = r\} \\ &\quad = \lim_{n\to\infty} \frac{(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{k+l-r-1}{n})}{r!(k-r)!(l-r)!}\lambda_{11}^r \lambda_{12}^{k-r}\lambda_{21}^{l-r} \\ &\qquad\qquad \times \left(1 - \frac{\lambda_{11}+\lambda_{12}+\lambda_{21}}{n}\right)^n \left(1 - \frac{\lambda_{11}+\lambda_{12}+\lambda_{21}}{n}\right)^{-(k+l-r)} \\ &\quad = \frac{\lambda_{11}^r \lambda_{12}^{k-r}\lambda_{21}^{l-r}}{r!(k-r)!(l-r)!} \exp\left(-(\lambda_{11}+\lambda_{12}+\lambda_{21})\right). \end{aligned}$$

Therefore $p_n(k,l,r) \to p(k,l,r)$, where

$$\begin{aligned} p(k,l,r) &= \frac{\lambda_{11}^r \lambda_{12}^{k-r}\lambda_{21}^{l-r}}{r!(k-r)!(l-r)!} \exp\left(-(\lambda_{11}+\lambda_{12}+\lambda_{21})\right); \\ &\qquad (k,l = 0,1,2,\ldots \text{ and } r = 0,1,2,\ldots,\min(k,l)). \end{aligned}$$

This distribution is a trivariate Poisson distribution.

## 2. Extensions of the bivariate binomial distribution

**2.1. Example.** Consider a strategic game of two players $A$ and $B$. Player $A$ uses one of the strategies $A_1, A_2, \ldots, A_n$ together with one of the strategies $B_1, B_2, \ldots, B_n$ of Player $B$. The probability of the event "$A$ uses strategy $A_i$ and $B$ uses strategy $B_j$" is $P(A_i B_j) = p_{ij},\ i, j = 1, 2, \ldots, n$. If $A$ uses strategy $A_i$ against strategy $B_j$ used by $B$, then $A$ wins $a_{ij}$ units and $B$ loses $a_{ij}$ units. If the game is repeated $n$ times, then we are interested in the joint distribution of the random variables $\chi_1$ and $\chi_2$, where $\chi_1$ is the number of cases in which the strategy $A_1$ was used, $\chi_2$ is the number of cases in which the strategy $B_1$ was used. Clearly, $(\chi_1, \chi_2)$ is bivariate binomial.

Now assume that a third party is interested in this game, and has some profit in all cases when the strategy $A_1$ of the first player is used, or the strategy $B_1$ of the second player is used. Let $\chi_{11}$ be the number of cases in which $A_1$ and $B_1$ *were used simultaneously*. Then the number in which the third party is interested is $\chi_1 + \chi_2 - \chi_{11}$ – the number of cases when $A_1$ or $B_1$ were used in the $n$ times repeated game. It is clear that

$$P\left\{\chi_1 + \chi_2 - \chi_{11} = m\right\} = \sum_{k,l} P\left\{\chi_1 = k, \chi_2 = l, \chi_{11} = k + l - m\right\}.$$

Therefore, the joint probability function of $\chi_1, \chi_2$ and $\chi_{11}$ is required.

**2.2. Example.** Suppose $n$ independent units, each consisting of two components, are placed on a life-test with the corresponding failure times $(X_1, Y_1)$, $(X_2, Y_2)$, …, $(X_n, Y_n)$ being identically distributed with cumulative distribution function $F(x, y)$ and probability density function $f(x, y)$. For predefined numbers $a_1 < a_2$, if $X_i \le a_1$ we say that the $i^{\text{th}}$ unit fails test $A_1$. If $a_1 < X_i \le a_2$, then we say that the $i^{\text{th}}$ unit is successful in test $A_1$, but fails test $A_2$. Similarly, for $b_1 < b_2$, if $Y_i \le b_1$, then we say that the $i^{\text{th}}$ unit fails test $B_1$, $b_1 < Y_i \le b_2$ means that it passes test $B_1$ but fails test $B_2$. Under this setup we are interested in the following probabilities.

a) What is the probability that $q_1$ units fail test $A_1$, $q_2$ units fail test $B_1$, and at least $q_3$ pass both tests $A_1$ and $B_1$? If the number of units that fail test $A_1$ is $\chi_1$, the number of units that fail test $A_2$ is $\chi_2$, the number of units that fail both tests $A_1$ and $B_1$ is $\chi_{11}$, the number of units that fail both tests $A_2$ and $B_1$ is $\chi_{21}$, and the number of units that fail both tests $A_1$ and $B_2$ is $\chi_{12}$, then the required probability is $P\{\chi_1 = q_1, \chi_2 = q_2, \chi_{11} < q_3\}$.
b) What is the probability that $q_1$ units fail test $A_1$, $q_2$ units fail test $B_1$, $q_3$ units fail both tests $A_1$ and $B_2$, and $q_4$ units fail both tests $A_2$ and $B_1$? This probability is $P\{\chi_1 = q_1, \chi_2 = q_2, \chi_{12} = q_3, \chi_{21} = q_4\}$. Therefore the joint pmf of $(\chi_1, \chi_2, \chi_{12}, \chi_{21})$ is required.

**2.1. A model.** A general model, including the two examples, above can be described as follows. Suppose in an experiment that the results are observed as one of the events $A_1, A_2, \ldots, A_m$, and at the same time as one of the events $B_1, B_2, \ldots, B_m$ with probabilities $P(A_i B_j) = p_{ij},\ i, j = 1, 2, \ldots, m;\ \sum_{ij} p_{ij} = 1,\ m \ge 3$. This means that the outcomes in the experiment are pairs $A_i B_j,\ i, j = 1, 2, \ldots, m$. Assume that we repeat the experiment $n$ times and that the trials are independent.

**2.3. Definition.** Let $\chi_1, \chi_2, \chi_{11}, \chi_{12}, \chi_{21}$ denote the number of occurrences of $A_1$, $B_1$, $A_1 B_1$, $A_1 B_2$, $A_2 B_1$, respectively.

From (3) it is clear that the joint probability function of the random variables $\chi_1, \chi_2$ and $\chi_{11}$ is

$$P\{\chi_1 = k, \chi_2 = l, \chi_{11} = r\} = \frac{n!}{r!(k-r)!(l-r)!(n-k-l+r)!} \Pi_{11}^r \Pi_{12}^{k-r} \Pi_{21}^{l-r} \Pi_{22}^{n-k-l+r}, \tag{4}$$

where $\Pi_{11} = p_{11}$, $\Pi_{12} = \sum_{j=2}^m p_{1j}$, $\Pi_{21} = \sum_{i=2}^m p_{i1}$, $\Pi_{22} = 1 - p_{11} - \sum_{j=2}^m p_{1j} - \sum_{i=2}^m p_{i1}$.

**2.4. Theorem.** *Let* $P_{11} = p_{11}$, $P_{12} = p_{12}$, $P_{21} = p_{21}$, $P_{13} = \sum_{j=3}^m p_{1j}$ *and* $P_{31} = \sum_{i=3}^m p_{i1}$. *Then*

a) *The joint probability function of the random variables* $\chi_1$, $\chi_2$, $\chi_{12}$ *and* $\chi_{21}$ *is*

$$\begin{aligned} P_n\{k,l,r,h\} &= P\{\chi_1 = k, \chi_2 = l, \chi_{12} = r, \chi_{21} = h\} \\ &= \sum_{i=\max(0,k+l-n)}^{\min(k-r,l-h)} \frac{n!}{i!r!h!(k-i-r)!(l-i-h)!(n-k-l+i)!} \\ &\quad \times P_{11}^i P_{12}^r P_{13}^{k-i-r} P_{21}^h P_{31}^{l-i-h} (1 - P_{11} - P_{12} - P_{13} - P_{21} - P_{31})^{n-k-l+i} \\ &(k = r, \dots, n-h\ ;\ l = h, \dots, n-r;\ r = 0, \dots, n-h;\ h = 0, \dots, n). \end{aligned} \tag{5}$$

b) *The joint probability generating function is given by*

$$\mathbf{\Phi}(t,s,z,c) = (\alpha_1 ts + \alpha_2 tz + \alpha_3 sc + \alpha_4 t + \alpha_5 s + \alpha_6)^n, \tag{6}$$

*where* $\alpha_1 = P_{11}$, $\alpha_2 = P_{12}$, $\alpha_3 = P_{21}$, $\alpha_4 = P_{13}$, $\alpha_5 = P_{31}$ *and* $\alpha_6 = \sum_{i=2}^m \sum_{j=2}^m p_{ij}$.

*Proof.* It is clear that without loss of generality we can take $m = 3$. The model can be described symbolically as follows:

| $A\backslash B$ | $B_1$ | $B_2$ | $B_3$ |
|---|---|---|---|
| $A_1$ | $p_{11}$ $i$ | $p_{12}$ $r$ | $p_{13}$ $k-i-r$ |
| $A_2$ | $p_{21}$ $h$ | $p_{22}$ | $p_{23}$ |
| $A_3$ | $p_{31}$ $l-i-h$ | $p_{32}$ | $p_{33}$ |

It is clear that if we repeat the experiment $n$ times, then $r$ outcomes of the event $A_1$ can be observed together with $B_2$ in $\binom{n}{r}$ ways, together with $B_1$ in $\binom{n-r}{i}$ ways and together with $B_3$ in $\binom{n-r-i}{k-r-i}$ ways. Then, $h$ outcomes of the event $B_1$ can be realized together with $A_2$ in $\binom{n-r-i-(k-r-i)}{h} = \binom{n-k}{h}$ ways, and together with $A_3$ in $\binom{n-k-h}{l-i-h}$ ways. Therefore, in $n$ repeated independent trials, the number of possible cases when $A_1$ appears $i$ times, $B_1$ appears $l$ times, $A_1B_2$ appears $r$ times and $A_2B_1$ appears $h$ times is

$$\binom{n}{r}\binom{n-r}{i}\binom{n-r-i}{k-i-r}\binom{n-k}{h}\binom{n-k-h}{l-i-h} = \frac{n!}{i!r!h!(k-i-r)!(l-i-h)!(n-k-l+i)!},$$

and each case has the same probability of

$$P_{11}^i P_{12}^r P_{13}^{k-i-r} P_{21}^h P_{31}^{l-i-h} (1 - P_{11} - P_{12} - P_{13} - P_{21} - P_{31})^{n-k-l+i}.$$

It is then easy to see that $i$ changes from $\max(0, k+l-n)$ to $\min(k-r, l-h)$, and consequently we obtain (5).

To derive the joint probability generating function, let us write

$$\xi_1^i = \begin{cases} 1 & \text{if in the } i^{\text{th}} \text{ trial } A_1 \text{ appears,} \\ 0 & \text{otherwise,} \end{cases}$$

$$\xi_2^i = \begin{cases} 1 & \text{if in the } i^{\text{th}} \text{ trial } B_1 \text{ appears,} \\ 0 & \text{otherwise,} \end{cases}$$

$$\xi_{12}^i = \begin{cases} 1 & \text{if in the } i^{\text{th}} \text{ trial } A_1B_2 \text{ appears,} \\ 0 & \text{otherwise,} \end{cases}$$

$$\xi_{21}^i = \begin{cases} 1 & \text{if in the } i^{\text{th}} \text{ trial } A_2B_1 \text{ appears,} \\ 0 & \text{otherwise,} \end{cases}$$

for $i = 1, 2, \dots, n$. It is then clear that $\chi_1 = \sum\limits_{i=1}^{n} \xi_1^i$, $\chi_2 = \sum\limits_{i=1}^{n} \xi_2^i, \chi_{12} = \sum\limits_{i=1}^{n} \xi_{12}^i$ and $\chi_{21} = \sum\limits_{i=1}^{n} \xi_{21}^i$. Since the $n$ trials are independent, the pgf of the random vector $(\chi_1, \chi_2, \chi_{12}, \chi_{21})$ can then be written as

$$\text{(7)} \qquad \mathbf{\Phi}(t, s, z, c) = \Bigg( \sum_{x_1, x_2, x_3, x_4 = 0}^{1} t^{x_1} s^{x_2} z^{x_3} c^{x_4} q_{x_1, x_2, x_3, x_4} \Bigg)^n,$$

where

$$q_{x_1,x_2,x_3,x_4} = P\{\xi_1^i = x_1,\ \xi_2^i = x_2,\ \xi_{12}^i = x_3,\ \xi_{21}^i = x_4\};\ x_1,\ x_2,\ x_3,\ x_4 = 0, 1.$$

We have

$$\begin{aligned}
q_{1,1,1,1} &= P(A_1B_1(A_1B_2)(A_2B_1)) = 0 \\
q_{0,1,1,1} &= P({A_1}^cB_1(A_1B_2)(A_2B_1)) = 0 \\
q_{1,1,1,0} &= P(A_1B_1(A_1B_2)(A_2B_1)^c) = 0 \\
q_{0,1,1,0} &= P({A_1}^cB_1(A_1B_2)(A_2B_1)^c) = 0 \\
q_{1,1,0,1} &= P(A_1B_1(A_1B_2)^c(A_2B_1)) = 0 \\
q_{0,1,0,1} &= P({A_1}^cB_1(A_1B_2)^c(A_2B_1)) = P_{21} \\
q_{1,1,0,0} &= P(A_1B_1(A_1B_2)^c(A_2B_1)^c) = P_{11} \\
q_{0,1,0,0} &= P({A_1}^cB_1(A_1B_2)^c(A_2B_1)^c) = P_{31} \\
q_{1,0,1,1} &= P(A_1{B_1}^c(A_1B_2)(A_2B_1)) = 0 \\
q_{0,0,1,1} &= P({A_1}^c{B_1}^c(A_1B_2)(A_2B_1)) = 0 \\
q_{1,0,1,0} &= P(A_1{B_1}^c(A_1B_2)(A_2B_1)^c) = P_{12} \\
q_{0,0,1,0} &= P({A_1}^c{B_1}^c(A_1B_2)(A_2B_1)^c) = 0 \\
q_{1,0,0,1} &= P(A_1{B_1}^c(A_1B_2)^c(A_2B_1)) = 0 \\
q_{0,0,0,1} &= P({A_1}^c{B_1}^c(A_1B_2)^c(A_2B_1)) = 0 \\
q_{1,0,0,0} &= P(A_1{B_1}^c(A_1B_2)^c(A_2B_1)^c) = P_{13}, \\
q_{0,0,0,0} &= P(A_1^cB_1^c(A_1B_2)^c(A_2B_1)^c) = \sum\nolimits_{i=2}^{3} \sum\nolimits_{j=2}^{3} p_{ij}.
\end{aligned}$$

Substituting for these values in (7) and simplifying, we obtain (6). Observe that $k = r, r+1, \dots, n-h$; $l = h, h+1, \dots, n-r$; $r = 0, 1, \dots, n-h$; $h = 0, 1, \dots, n$. □

**2.1.1.** *Marginal distributions.* The univariate marginals of the discrete random vector $(\chi_1, \chi_2, \chi_{12}, \chi_{21})$ are binomial with cell probabilities $(P_{11}+P_{12}+P_{13})$, $(P_{11}+P_{21}+P_{31})$, $P_{12}$ and $P_{21}$, respectively.

The joint distribution of $(\chi_1, \chi_2)$ is obviously the bivariate binomial distribution with probability function (1) and pgf

$$\Psi_{1,2}(t,s) = \mathbf{\Phi}(t,s,1,1) = (b_1 ts + b_2 t + b_3 s + b_4)^n \,,$$

where $b_1 = \alpha_1$, $b_2 = \alpha_2 + \alpha_4$, $b_3 = \alpha_3 + \alpha_5$ and $b_4 = \alpha_6$, as in (2).

The joint pgf of $(\chi_1, \chi_{12})$

$$\Psi_{1,12}(t,z) = \mathbf{\Phi}(t,1,z,1) = (c_1 tz + c_2 t + c_3)^n,$$

where $c_1 = \alpha_2$, $c_2 = \alpha_1 + \alpha_4$, $c_3 = \alpha_3 + \alpha_5 + \alpha_6$.

The joint pgf of $(\chi_1, \chi_{21})$ is

$$\Psi_{1,21}(t,c) = \mathbf{\Phi}(t,1,1,c) = (d_1 t + d_2 c + d_3)^n \,,$$

where $d_1 = \alpha_1 + \alpha_2 + \alpha_4$, $d_2 = \alpha_3$, $d_3 = \alpha_5 + \alpha_6$, which is the pgf of a trinomial distribution.

The joint pgf of $(\chi_2, \chi_{12})$ is

$$\Psi_{2,12}(s,z) = \mathbf{\Phi}(1,s,z,1) = (e_1 s + e_2 z + e_3)^n \,,$$

where $e_1 = \alpha_1 + \alpha_3 + \alpha_5$, $e_2 = \alpha_2$, $e_3 = \alpha_4 + \alpha_6$, which is the pgf of a trinomial distribution.

The joint pgf of $(\chi_2, \chi_{21})$ is

$$\Psi_{2,21}(s,c) = \mathbf{\Phi}(1,s,1,c) = (f_1 cs + f_2 s + f_3)^n \,,$$

where $f_1 = \alpha_3$, $f_2 = \alpha_1 + \alpha_5$, $f_3 = \alpha_2 + \alpha_4 + \alpha_6$.

The joint pgf $(\chi_{12}, \chi_{21})$ is

$$\Psi_{12,21}(z,c) = \mathbf{\Phi}(1,1,z,c) = (g_1 c + g_2 z + g_3)^n \,,$$

where $g_1 = \alpha_3$, $g_2 = \alpha_2$, $g_3 = \alpha_1 + \alpha_4 + \alpha_5 + \alpha_6$, which is the pgf of a trinomial distribution.

The trivariate marginals of the discrete random vector $(\chi_1, \chi_2, \chi_{12}, \chi_{21})$ are as follows.

The joint pgf of $(\chi_1, \chi_2, \chi_{12})$ is

$$\Psi_{1,2,12}(t,s,z) = \mathbf{\Phi}(t,s,z,1) = (h_1 ts + h_2 tz + h_3 t + h_4 s + h_5)^n \,,$$

where $h_1 = \alpha_1$, $h_2 = \alpha_2$, $h_3 = \alpha_4$, $h_4 = \alpha_3 + \alpha_5$, $h_5 = \alpha_6$.

The joint pgf of $(\chi_1, \chi_2, \chi_{21})$ is

$$\Psi_{1,2,21}(t,s,c) = \mathbf{\Phi}(t,s,1,c) = (j_1 ts + j_2 cs + j_3 t + j_4 s + j_5)^n \,,$$

where $j_1 = \alpha_1$, $j_2 = \alpha_3$, $j_3 = \alpha_2 + \alpha_4$, $j_4 = \alpha_5$, $j_5 = \alpha_6$.

The joint pgf of $(\chi_1, \chi_{12}, \chi_{21})$ is

$$\Psi_{1,12,21}(t,z,c) = \mathbf{\Phi}(t,1,z,c) = (k_1 tz + k_2 t + k_3 c + k_4)^n \,,$$

where $k_1 = \alpha_2$, $k_2 = \alpha_1 + \alpha_4$, $k_3 = \alpha_3$, $k_4 = \alpha_5 + \alpha_6$.

The joint pgf of $(\chi_2, \chi_{12}, \chi_{21})$ is

$$\Psi_{2,12,21}(s,z,c) = \Psi(1,s,z,c) = (n_1 cs + n_2 s + n_3 z + n_4)^n \,,$$

where $n_1 = \alpha_3$, $n_2 = \alpha_1 + \alpha_5$, $n_3 = \alpha_2$, $n_4 = \alpha_4 + \alpha_6$.

**Example 2.1 (continued)** a) It is clear that

$$P\{\chi_1 = q_1, \chi_2 = q_2, \chi_{11} < q_3\} = \sum_{r=1}^{q_3-1} P\{\chi_1 = q_1, \chi_2 = q_2, \chi_{11} = r\}.$$

It can be calculated from (4), for $m = 3$ and the probabilities

$$\begin{aligned}
&p_{11} = P(A_1B_1) = P\{X \le a_1, Y \le b_1\},\\
&p_{12} = P(A_1B_2) = P\{X \le a_1, b_1 < Y \le b_2\},\\
&p_{21} = P(A_2B_1) = P\{a_1 < X \le a_2, Y \le b_1\},\\
&p_{22} = P(A_2B_2) = P\{a_1 < X \le a_2, b_1 < Y \le b_2\},\\
&p_{13} = P(A_1B_3) = P\{X \le a_1, Y > b_2\},\\
&p_{23} = P(A_2B_3) = P\{a_1 < X \le a_2, Y > b_2\}\\
&p_{31} = P(A_3B_1) = P\{X > a_2, Y \le b_1\},\\
&p_{32} = P(A_3B_2) = P\{X > a_2, b_1 < Y \le b_2\},\\
&p_{33} = P(A_3B_3) = P\{X > a_2, Y > b_2\}.
\end{aligned} \tag{8}$$

b) The probability is

$$P\{\chi_1 = q_1, \chi_2 = q_2, \chi_{12} = q_3, \chi_{21} = q_4\},$$

which can be calculated from (5) for $m = 3$ by using the probabilities (8).

Below, in Table 1, we provide some numerical values of

$$f(n, k, l, r, h) = P\{\chi_1 = k, \chi_2 = l, \chi_{12} = r, \chi_{21} = h\}$$

for $n = 2, m = 3$ and $p_{ij} = \frac{1}{9}$, $i, j = 1, 2, 3$.

**Table 1. Numerical values of** $f(n, k, l, r, h) = P\{\chi_1 = k, \chi_2 = l, \chi_{12} = r, \chi_{21} = h\}$ **for** $n = 2, m = 3$.

| $n$ | $k$ | $l$ | $r$ | $h$ | $f(n,k,l,r,h)$ | $n$ | $k$ | $l$ | $r$ | $h$ | $f(n,k,l,r,h)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 0 | 0 | 0 | 0 | 0.198 | 2 | 1 | 1 | 1 | 0 | 0.025 |
| 2 | 0 | 1 | 0 | 0 | 0.099 | 2 | 2 | 0 | 1 | 0 | 0.025 |
| 2 | 0 | 2 | 0 | 0 | 0.012 | 2 | 2 | 1 | 1 | 0 | 0.025 |
| 2 | 1 | 0 | 0 | 0 | 0.099 | 2 | 2 | 0 | 2 | 0 | 0.012 |
| 2 | 1 | 1 | 0 | 0 | 0.123 | 2 | 0 | 1 | 0 | 1 | 0.099 |
| 2 | 1 | 2 | 0 | 0 | 0.025 | 2 | 0 | 2 | 0 | 1 | 0.025 |
| 2 | 2 | 0 | 0 | 0 | 0.012 | 2 | 1 | 1 | 0 | 1 | 0.025 |
| 2 | 2 | 1 | 0 | 0 | 0.025 | 2 | 1 | 2 | 0 | 1 | 0.025 |
| 2 | 2 | 2 | 0 | 0 | 0.012 | 2 | 1 | 1 | 1 | 1 | 0.025 |
| 2 | 1 | 0 | 1 | 0 | 0.099 | 2 | 0 | 2 | 0 | 2 | 0.012 |

**2.2. The Poisson approximation.** The Poisson procedure allows us to obtain the formula that approximates the probability mass function $p_n(k, l, r, h)$ when the number of trials is large ($n \to \infty$), and $P_{11}, P_{12}, P_{21}, P_{13}, P_{31} \to 0$, $nP_{11} \to \lambda_{11}$, $nP_{12} \to \lambda_{12}$, $nP_{21} \to \lambda_{21}, nP_1 \to \lambda_1, nP_2 \to \lambda_2$, where $P_1 = P_{11} + P_{12} + P_{13}$ and $P_2 = P_{11} + P_{21} + P_{31}$.

The limiting form of $P\{k,l,r,h\}$ is given by

$$\lim_{n\to\infty} P\{\chi_1=k,\chi_2=l,\chi_{12}=r,\chi_{21}=h\}$$

$$= \lim_{n\to\infty} \sum_{i=\max(0,k+l-n)}^{\min(k-r,l-h)} \frac{n(n-1)\cdots(n-k)\cdots(n-k-l+i+1)(n-k-l+i)!}{i!r!h!(k-i-r)!(l-i-h)!(n-k-l+i)!}$$

$$\times \left(\frac{\lambda_{11}}{n}\right)^i \left(\frac{\lambda_{12}}{n}\right)^r \left(\frac{\lambda_1-\lambda_{11}-\lambda_{12}}{n}\right)^{k-i-r} \left(\frac{\lambda_{21}}{n}\right)^h$$

$$\times \left(\frac{\lambda_2-\lambda_{11}-\lambda_{21}}{n}\right)^{l-i-h} \left(1-\frac{\lambda_1+\lambda_2-\lambda_{11}}{n}\right)^{n-k-l+i}$$

$$= \lim_{n\to\infty} \sum_{i=\max(0,k+l-n)}^{\min(k-r,l-h)} \frac{1(1-\frac{1}{n})\cdots(1-\frac{k}{n})\cdots(1-\frac{k+l-i-1}{n})}{i!r!h!(k-i-r)!(l-i-h)!} \lambda_{11}^i\lambda_{12}^r$$

$$\times (\lambda_1-\lambda_{11}-\lambda_{12})^{k-i-r}\lambda_{21}^h(\lambda_2-\lambda_{11}-\lambda_{21})^{l-i-h}$$

$$\times \left(1-\frac{\lambda_1+\lambda_2-\lambda_{11}}{n}\right)^{n-k-l+i}$$

$$= \mathrm{e}^{-(\lambda_1+\lambda_2-\lambda_{11})} \sum_{i=0}^{\min(k-r,l-h)} \frac{\lambda_{11}^i}{i!}\frac{\lambda_{12}^r}{r!}\frac{\lambda_{21}^h}{h!}\frac{(\lambda_1-\lambda_{11}-\lambda_{12})^{k-i-r}}{(k-i-r)!}$$

$$\times \frac{(\lambda_2-\lambda_{11}-\lambda_{21})^{l-i-h}}{(l-i-h)!},$$

$$(k=r,r+1,\ldots;\ l=h,h+1,\ldots;\ r=0,1,2,\ldots;\ h=0,1,2,\ldots).$$

Therefore $P_n(k,l,r,h)\to p(k,l,r,h)$, where

$$p(k,l,r,h) = \mathrm{e}^{-(\lambda_1+\lambda_2-\lambda_{11})} \sum_{i=0}^{\min(k-r,l-h)} \frac{\lambda_{11}^i}{i!}\frac{\lambda_{12}^r}{r!}\frac{\lambda_{21}^h}{h!}\frac{(\lambda_1-\lambda_{11}-\lambda_{12})^{k-i-r}}{(k-i-r)!}$$

$$\times \frac{(\lambda_2-\lambda_{11}-\lambda_{21})^{l-i-h}}{(l-i-h)!};$$

$$(k=r,r+1,\ldots;\ l=h,h+1,\ldots;\ r=0,1,2,\ldots;\ h=0,1,2,\ldots).$$

This distribution is a version of the quadrivariate Poisson distribution.

**2.5. Remark.** It should be noted that Theorem 2.4 enables one to calculate the joint probability function of any random variables $X_i$, $X_j$, $X_{ij}$ counting, respectively, the number of occurrences of $A_i, B_j$ and $A_iB_j$ in $n$ repetitions of the experiment.

**2.6. Remark.** In this paper we do not deal with statistical inferences for the proposed family of distributions. The estimating and testing techniques for similar multivariate distributions are discussed in the statistical literature, see e.g. Voinov and Nikulin [22], Voinov *et al.* [23].

**2.7. Remark.** The problem addressed in this paper is that associated with determining the joint distribution of overlapping sums of the coordinates of a multinomial density in certain specific cases. The general problem would begin with

$$X \sim \text{ Multinomial}(n; p_1, p_2, \ldots, p_m),$$

where $p_i \ge 0$ and $\sum_{i=1}^{m} p_i = 1$, and for each $i$, $X_i$ denotes the number of outcomes of type $i$, $i=1,2,\ldots,m$. In such a setting, we can consider the joint density of the random vector $Z=AX$, where $A$ is a $k\times m$ matrix of zeroes and ones. It is random vectors of this type that are considered in the paper. In particular, we focus on a cross-classified version in which there are $m^2$ possible outcomes of the experiment which can be indexed by $\{(i;j)\colon i=1,2,\ldots,m;\ j=1,2,\ldots,m\}$ with corresponding probabilities $p_{ij}$.

## 3. Some possible applications

**3.1. Empirical distribution function and dependence measures.** Let $(X_1,Y_1)$, $(X_2,Y_2),\ldots,(X_n,Y_n)$ be a sample from a bivariate distribution with distribution function (df) $F(x,y)$ and marginal df's $F_X(x)$ and $F_Y(y)$. Write $\xi_1 = \#\{i : X_i \le x\}$, $\xi_2 = \#\{j : Y_j \le y\}$, $\xi_{11} = \#\{i : X_i \le x, Y_i \le y\}$, $\xi_{12} = \#\{i : X_i \le x, Y_i > y\}$, $A_1 = \{X \le x\}$, $A_2 = \{X > x\}$ and $B_1 = \{Y \le y\}$, $B_2 = \{Y > y\}$. It is easy to observe that $\xi_1$ represents the number of elements of the sample $X_1, X_2, \ldots, X_n$ falling below the threshold $x$, $\xi_2$ represents the number of elements of the sample $Y_1, Y_2, \ldots, Y_n$ falling below the threshold $y$, and $\xi_{11}$ represents the number of elements of the sample $(X_1,Y_1)$, $(X_2,Y_2),\ldots,(X_n,Y_n)$ belonging to the set $\{(u,v) : u \le x, v \le y\}$. It can also be observed that $\xi_1 \equiv nF_X^*(x)$ is the empirical df of the sample $X_1, X_2, \ldots, X_n$; $\xi_2 \equiv nF_Y^*(y)$ is the empirical df of the sample $Y_1, Y_2, \ldots, Y_n$, and $\xi_{11} = nF_{X,Y}^*(x,y)$ is the empirical df of the bivariate sample $(X_1,Y_1),(X_2,Y_2),\ldots,(X_n,Y_n)$. In this case we have

$$P\{nF_X^*(x)=k, nF_Y^*(y)=l, nF_{X,Y}^*(x,y)=r\} = P\left\{\xi_1=k, \xi_2=l, \xi_{11}=r\right\}, \tag{9}$$

where the joint probability function of the exceedances $(\xi_1,\xi_2,\xi_{11})$ is as given in (4) with the probabilities

$$\begin{aligned}
\pi_{11} &= P(A_1B_1) = P\left\{X \le x, Y \le y\right\} = F(x,y)\\
\pi_{12} &= P(A_1B_2) = P\left\{X \le x, Y > y\right\} = F_X(x) - F(x,y)\\
\pi_{21} &= P(A_2B_1) = P\left\{X > x, Y \le y\right\} = F_Y(y) - F(x,y)\\
\pi_{22} &= P(A_2B_2) = P\left\{X > x, Y > y\right\} = 1 - F_X(x) - F_Y(y) + F(x,y)
\end{aligned}$$

The probabilities (9) can be used to construct a criteria for testing the independence of random variables $X$ and $Y$, based on the empirical distribution functions $F_X^*(x)$, $F_Y^*(y)$ and $F_{X,Y}^*(x,y)$. In recent years, several statistical papers have appeared, discussing local dependence measures that can characterize the dependence structure of two random variables localized at the fixed point. For more details on local dependence functions, see Bjerve and Doksum [5], Jones [14], Bairamov and Kotz [2], Bairamov *et al.* [3], Kotz and Nadarajah [18]. Assume that $J(u,v,w)$ is any function on the unit cube and that $J(F_X^*(x), F_Y^*(y), F_{X,Y}^*(x,y))$ leads to a test statistic for testing independence between $X$ and $Y$ at the point $(x,y)$. Then the distribution of this statistic is $P\{J(F_X^*(x), F_Y^*(y), F_{X,Y}^*(x,y)) \le t\}$ given by

$$P\{J(F_X^*(x), F_Y^*(y), F_{X,Y}^*(x,y)) \le t\} = \sum_{\{(k,l,r): J(\frac{k}{n},\frac{l}{n},\frac{r}{n}) \le t\}} P\left\{\xi_1=k, \xi_2=l, \xi_{11}=r\right\}.$$

**3.2. Exceedances.** Let $(X_1,Y_1)$, $(X_2,Y_2),\ldots,(X_n,Y_n)$ be a sample as in Example 2.2. Denote by $X_{r:n}$ the $r^{\text{th}}$ order statistic constructed from the sample $X_1, X_2, \ldots, X_n$ and let $Y_{[r:n]}$ be the corresponding concomitant of $X_{r:n}$. The joint probability density function of the $r^{\text{th}}$ order statistic and its concomitant $Y_{[r:n]}$ is

$$f_{X_{r:n},Y_{[r:n]}}(x,y) = f(y \mid x) f_{r:n}(x).$$

The concomitants of order statistics arise in different selection procedures (see David [7]). Assume that $(X_{n+1},Y_{n+1}),(X_{n+2},Y_{n+2}),\ldots,(X_{n+m},Y_{n+m})$ are the next $m$ observations obtained from the same population with df $F(x,y)$ that are independent of the first sample. In this case let $r < s$ and $\eta_1 = \#\left\{i : X_i \le X_{r:n}\right\}$, $\eta_2 = \#\left\{j : Y_j \le Y_{[r:n]}\right\}$, $\eta_{11} = \#\left\{i : X_i \le X_{r:n}, Y_i \le Y_{[r:n]}\right\}$, $\eta_{12} = \#\left\{i : X_i \le X_{r:n}, Y_i \le Y_{[s:n]}\right\}$ and $\eta_{21} =$

$\#\{i : X_i \le X_{s:n}, Y_i \le Y_{[r:n]}\}$. The random variable $\eta_1 + 1$ shows the rank of $X_{r:n}$ among the $X_{n+1}, X_{n+2}, \dots, X_{n+m}$ and $\eta_2 + 1$ shows the rank of $Y_{[r:n]}$ among the $Y_{n+1}, Y_{n+2}, \dots, Y_{n+m}$. The joint distribution of $\eta_1 + 1$ and $\eta_2 + 1$ can be obtained from (1) as follows:

$$
\begin{aligned}
&P\{\xi_1 = k-1, \xi_2 = l-1\} \\
&\qquad = \sum_{i=\max(0,k+l-n-2)}^{\min(k-1,l-1)} \frac{n!}{i!(k-1-i)!(l-1-i)!(n-k-l+i+2)!} \\
&\qquad\qquad \times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \pi_{11}^{i}(x,y)\pi_{12}^{k-i-1}(x,y)\pi_{21}^{l-i-1}(x,y) \\
&\qquad\qquad\qquad \times \pi_{22}^{n-k-l+i+2}(x,y) f(y \mid x) f_{r:n}(x)\, dx\, dy.
\end{aligned} \tag{10}
$$

Formula (10) is obtained in Eryilmaz and Bairamov [9] by conditioning on $X_{r:n}$ and $Y_{[r:n]}$. The joint distribution of the exceedance statistics $\eta_1$, $\eta_2$, $\eta_{11}$ and $\eta_1$, $\eta_2$, $\eta_{12}$, $\eta_{21}$ can be obtained in a similar way to (4) and (5).

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