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# The modified beta Weibull distribution

Muhammad Nauman Khan*†

**Abstract**

A new five-parameter model called the modified beta Weibull probability distribution is being introduced in this paper. This model turns out to be quite flexible for analyzing positive data and has bathtub and upside down bathtub hazard rate function.

Our main objectives are to obtain representations of certain statistical functions and to estimate the parameters of the proposed distribution. As an application, the probability density function is utilized to model two actual data sets. The new distribution is shown to provide a better fit than related distributions.

The proposed distribution may serve as a viable alternative to other distributions available in the literature for modeling positive data arising in various fields of scientific investigation such as reliability theory, hydrology, medicine, meteorology, survival analysis and engineering.

**Keywords:** Weibull distribution, Modified beta distribution, Goodness–of–fit statistics, Lifetime data.

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## 1. Introduction

The Weibull distribution is a very popular distribution named after Waloddi Weibull, a Swedish physicist. He used it in 1939 to analyze the breaking strength of materials. Ever since, it has been widely used for analyzing lifetime data. However, this distribution does not have a bathtub or upside–down bathtub shaped hazard rate function, that is why it cannot be utilized to model the life time of certain systems. To overcome this shortcoming, several generalizations of the classical Weibull distribution have been discussed by different authors in recent years. Many authors introduced flexible distributions for modeling complex data and obtaining a better fit. Extensions of Weibull distribution arise in different areas of research as discussed for instance in

*Department of Mathematics, Kohat University of Science & Technology, Kohat, Pakistan 26000, Email: zaybasdf@gmail.com
†Corresponding Author.

[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 19, 20, 21, 24] and [27]. Many extended Weibull models have an upside–down bath tub shaped hazard rate, which is the case of the extensions discussed by [4], [14], [18] and [25], among others.
Adding parameters to an existing distribution enables one to obtain classes of more flexible distributions. Nadarajah *et al.* [17] introduced an interesting method for adding three new parameters to an existing distribution. The new distribution provides more flexibility to model various types of data. The baseline distribution has the cdf $G(x)$, then the new distribution is

$$F(x) = \frac{1}{B(a\,,\,b)} \int_0^{\left\{\frac{cG(x)}{(c-1)G(x)+1}\right\}} x^{a-1}(1-x)^{b-1}\,\mathrm{d}x\,. \tag{1.1}$$

The Modified beta Weibull probability density function obtained from (1.1) can be expressed in the following form:

$$f(x) = \frac{c^a g(x)\{G(x)\}^{a-1}\{1-G(x)\}^{b-1}}{B(a,b)\{1-(1-c)G(x)\}^{a+b}}. \tag{1.2}$$

The cdf and pdf of Weibull distribution are defined as follows:

$$G(x) = 1 - \mathrm{e}^{-\left(\frac{x}{\lambda}\right)^k}, \quad \lambda > 0\,,\, k > 0\,,\, x > 0 \tag{1.3}$$

and

$$g(x) = \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1} \mathrm{e}^{-\left(\frac{x}{\lambda}\right)^k}. \tag{1.4}$$

We further generalize this model by applying the modified beta technique [17], which results in what we are referring to as the modified beta Weibull (MBW) distribution. The cdf, survival function, pdf and hazard rate function of the modified beta Weibull distribution, for which $G(x)$ is the baseline function, are respectively given by

$$F(x) = \frac{1}{B(a,b)}\, B\left(\frac{1}{1-\frac{1}{c}+\left(c-c\left(\mathrm{e}^{-\frac{x}{\lambda}}\right)^k\right)^{-1}}; a, b\right), \tag{1.5}$$

$$S(x) = 1 - \frac{1}{B(a,b)}\, B\left(\frac{1}{1-\frac{1}{c}+\left(c-c\left(\mathrm{e}^{-\frac{x}{\lambda}}\right)^k\right)^{-1}}; a, b\right), \tag{1.6}$$

$$\begin{aligned} f(x) &= c^a\, k\,(x)^{-1+k}\left(\mathrm{e}^{-\lambda^{-k}x^k}\right)^b\left(1-\mathrm{e}^{-\lambda^{-k}x^k}\right)^{-1+a} \\ &\quad\times \frac{\left\{1-(1-c)\left(1-\mathrm{e}^{-\lambda^{-k}x^k}\right)\right\}^{-a-b}}{\lambda^k\, B(a,b)} \end{aligned} \tag{1.7}$$

and

$$\begin{aligned} h(x) &= k\, c^a\, x^{-1+k}\, \lambda^{-k}\left(\mathrm{e}^{-x^k\lambda^{-k}}\right)^b\left(1-\mathrm{e}^{-x^k\lambda^{-k}}\right)^{-1+a} \\ &\quad\times \frac{\left\{1-(1-c)\left(1-\mathrm{e}^{-x^k\lambda^{-k}}\right)\right\}^{-a-b}}{B(a,b) -\ B\left(\frac{1}{1-\frac{1}{c}+\left(c-c\left(\mathrm{e}^{-\frac{x}{\lambda}}\right)^k\right)^{-1}}; a, b\right)}\,,\ x > 0\,, \end{aligned} \tag{1.8}$$

where $B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt,\ \ Re(a) > 0,\ Re(b) > 0$ and $B(z;a,b) = \int_0^z t^{a-1}(1-t)^{b-1}dt$.

Also $\lambda > 0$, $k > 0$, $a > 0$, $b > 0$, $c > 0$. Equations (1.5) to (1.8) can be easily evaluated numerically using computational packages such as *Mathematica*, *Maple*, *MATLAB* and *R*. The following *Mathematica* code can be used for integration purposes: Integrate[f(x),{x, 0, Infinity}]. Further, Figure 1 shows the correctness of the defined cdf.

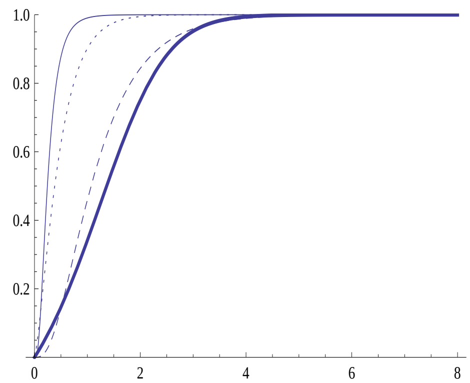


**Figure 1.** The MBW cdf. $\lambda = 0.8$, $k = 1.6$, $a = 1.4$, $b = 1.5$, $c = 0.8$, (dotted line), $\lambda = 4.8$, $k = 2.6$, $a = 3.4$, $b = 2.5$, $c = 1.8$, (dashed line), $\lambda = 10$, $k = 6$, $a = 4$, $b = 5$, $c = 5$, (solid line), $\lambda = 1$, $k = 1.2$, $a = 1$, $b = 2$, $c = 0.1$, (thick line).

Note that on making use of the identity

$$(1 - z)^{-\tau} = \sum_{n=0}^{\infty} \frac{\Gamma(\tau + n)}{\Gamma(\tau)\, n!}\, z^n\,, \quad |z| < 1\,, \tau > 0\,, \tag{1.9}$$

one has the following series representations of the pdf specified by (1.7)

$$\begin{aligned} f(x) = \frac{c^a\, x^{-1+k} k}{\lambda^k B(a,b)} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\Gamma(a+b+n)(1-c)^n}{\Gamma(a+b)n!} \frac{\Gamma(1-a-n+m)}{\Gamma(1-a-n)m!} \\ \times \left(\mathrm{e}^{-\lambda^{-k} x^k}\right)^{m+b}. \end{aligned} \tag{1.10}$$

Moreover, the first derivative of $h(x)$, which is used to study the shapes of hazard rate functions as explained in [13] is

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{dx}}\, h(x) &= c^{1+a}\, k^2\, x^{-1+k}\, \lambda^{-1-k} \left(\mathrm{e}^{-\frac{x}{\lambda}}\right)^k \left(1-\mathrm{e}^{-x^k\lambda^{-k}}\right)^{-1+a} \left(\mathrm{e}^{-x^k\lambda^{-k}}\right)^b \\
&\times \frac{\left\{1-(1-c)\left(1-\mathrm{e}^{-x^k\lambda^{-k}}\right)\right\}^{-a-b}}{\left\{c-c\left(\mathrm{e}^{-\frac{x}{\lambda}}\right)^k\right\}^2 \left\{B(a,b)-B\left(\frac{1}{1-\frac{1}{c}+\left(c-c\left(\mathrm{e}^{-\frac{x}{\lambda}}\right)^k\right)^{-1}};a,b\right)\right\}^2} \\
&\times \left\{\frac{1}{1-\frac{1}{c}+\left(c-c\left(\mathrm{e}^{-\frac{x}{\lambda}}\right)^k\right)^{-1}}\right\}^{1+a} \\
&\times \left\{1-\frac{1}{1-\frac{1}{c}+\left(c-c\left(\mathrm{e}^{-\frac{x}{\lambda}}\right)^k\right)^{-1}}\right\}^{-1+b} \\
&+ k^2 x^{-2+2k}\lambda^{-2k}(a+b)(1-c)c^a \left(\mathrm{e}^{-x^k\lambda^{-k}}\right)^{1+b} \left(1-\mathrm{e}^{-x^k\lambda^{-k}}\right)^{-1+a} \\
&\times \frac{\left\{1-(1-c)\left(1-\mathrm{e}^{-x^k\lambda^{-k}}\right)\right\}^{-1-a-b}}{B(a,b)-\ B\left(\frac{1}{1-\frac{1}{c}+\left(c-c\left(\mathrm{e}^{-\frac{x}{\lambda}}\right)^k\right)^{-1}};a,b\right)} \\
&+ c^a k^2 \lambda^{-2k}(a-1)x^{-2+2k} \left(\mathrm{e}^{-x^k\lambda^{-k}}\right)^{1+b} \left(1-\mathrm{e}^{-x^k\lambda^{-k}}\right)^{-2+a} \\
&\times \frac{\left\{1-(1-c)\left(1-\mathrm{e}^{-x^k\lambda^{-k}}\right)\right\}^{-a-b}}{B(a,b)-\ B\left(\frac{1}{1-\frac{1}{c}+\left(c-c\left(\mathrm{e}^{-\frac{x}{\lambda}}\right)^k\right)^{-1}};a,b\right)} \\
&- b\, c^a \left(\mathrm{e}^{-x^k\lambda^{-k}}\right)^b \left(1-\mathrm{e}^{-x^k\lambda^{-k}}\right)^{-1+a} k^2 x^{-2+2k}\lambda^{-2k} \\
&\times \frac{\left\{1-(1-c)\left(1-\mathrm{e}^{-x^k\lambda^{-k}}\right)\right\}^{-a-b}}{B(a,b)-\ B\left(\frac{1}{1-\frac{1}{c}+\left(c-c\left(\mathrm{e}^{-\frac{x}{\lambda}}\right)^k\right)^{-1}};a,b\right)} \\
&+ c^a(k-1)k\,\lambda^{-k}x^{-2+k} \left(\mathrm{e}^{-x^k\lambda^{-k}}\right)^b \left(1-\mathrm{e}^{-x^k\lambda^{-k}}\right)^{-1+a} \\
&\times \frac{\left\{1-(1-c)\left(1-\mathrm{e}^{-x^k\lambda^{-k}}\right)\right\}^{-a-b}}{B(a,b)-\ B\left(\frac{1}{1-\frac{1}{c}+\left(c-c\left(\mathrm{e}^{-\frac{x}{\lambda}}\right)^k\right)^{-1}};a,b\right)}.
\end{aligned}$$

Fig. $2, 3$ and 4 plots some MBW curves for different choices of parameters for pdf and hazard rate function. Figures 2 and 3 indicate how the new parameters $a$, $b$ and $c$ affect the MBW density. These graphs illustrate the versatility of the MBW distribution. As can be seen from left panel of Figure 2 that $a$ is a scale parameter and from the right panel of Figure 2 and left panel of Figure 3 that $b$ and $c$ are shape parameters. Similarly

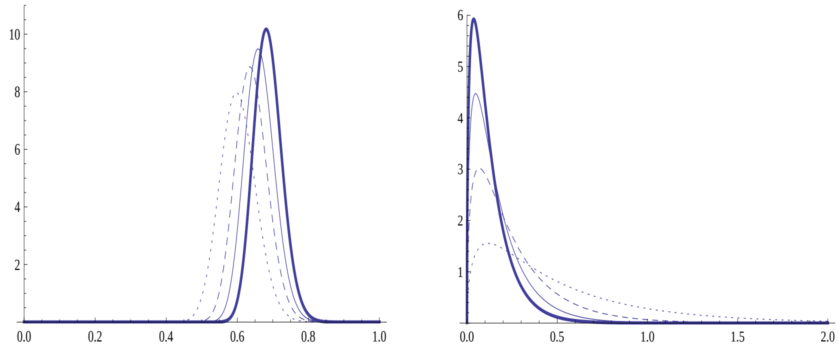


**Figure 2.** The MBW pdf. Left panel: $\lambda = 0.5$, $k = 3.5$, $b = 2.8$, $c = 2.1$ and $a = 30$ (dotted line) $a = 50$ (dashed line), $a = 70$ (solid line), $a = 100$ (thick line). Right panel: $\lambda = 0.5$, $k = 1$, $a = 1.5$, $c = 1.5$ and $b = 1$ (dotted line) $b = 2$ (dashed line), $b = 3$ (solid line), $b = 4$ (thick line).

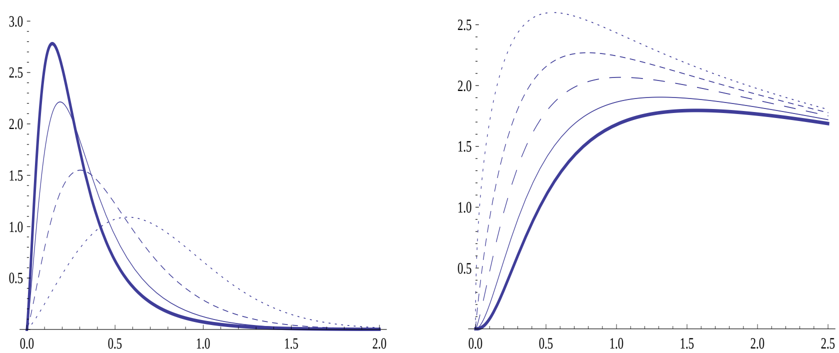


**Figure 3.** Left panel: The MBW pdf. $\lambda = 0.8$, $k = 1.6$, $a = 1.4$, $b = 1.5$ and $c = 0.8$ (dotted line), $c = 2$ (dashed line), $c = 4$ (solid line), $c = 6$ (thick line). Right panel: The MBW hazard rate function. $\lambda = 1.7$, $k = 1.2$, $b = 1.5$, $c = 3.5$ and $a = 1.2$ (dotted line) $a = 1.6$ (short dashes), $a = 2$ (long dashes), $a = 2.5$ (solid line), $a = 3$ (thick line).

right panel of Figure 3 and left and right panels of Figure 4 represent bathtub shaped and upside down bathtub shaped hazard rate function.

The rest of the paper is organized as follows. Representations of certain statistical functions are provided in Section 2. The parameter estimation technique described in Section 3 is utilized in connection with the modeling of two actual data sets originating from the engineering and biological sciences in Section 4, where the new model is compared with several related distributions.

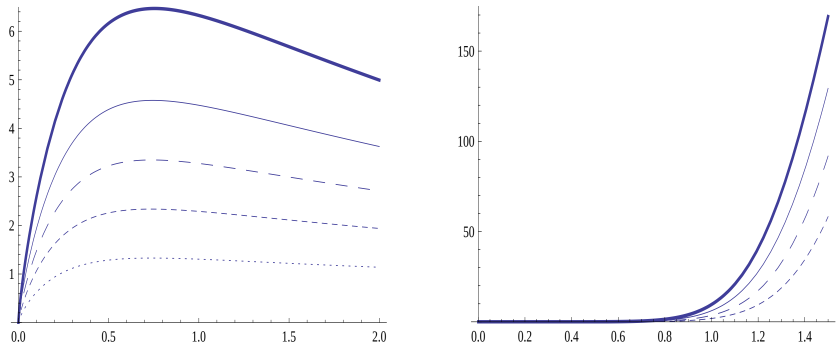


**Figure 4.** Left panel: The MBW hazard rate function. $\lambda = 1.7$, $k = 1.2$, $a = 1.5$, $c = 3.5$ and $b = 1$ (dotted line), $b = 1.5$ (dashed line), $b = 1.9$ (long dashes), $b = 2.3$ (solid line), $b = 2.8$ (thick line). Right panel: The MBW hazard rate function. $\lambda = 2$, $k = 4$, $a = 2$, $b = 1.5$ and $c = 1.6$ (dashed line), $c = 2$ (long dashes), $c = 2.4$ (solid line), $c = 2.8$ (thick line).

## 2. Statistical Functions of the MBW Distribution

Here, we derive computable representations of some statistical functions associated with the MBW distribution whose probability density function can be represented by (1.10). The resulting expressions can be evaluated exactly or numerically with symbolic computational packages such as *Mathematica*, *MATLAB* or *Maple*. In numerical applications, infinite sum can be truncated whenever convergence is observed.

**2.1. Moments.** We now derive closed form representations of the positive, negative and factorial moments of a MBW random variable. The $r^{\text{th}}$ raw moment of the MBW distribution is

$$E(X^r) = \frac{c^a k}{\lambda^k B(a,b)} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\Gamma(a+b+n)(1-c)^n}{\Gamma(a+b)n!} \frac{\Gamma(1-a-n+m)}{\Gamma(1-a-n)m!} \times \int_0^{\infty} x^r\, x^{-1+k} \left(\mathrm{e}^{-\lambda^{-k} x^k}\right)^{m+b} \mathrm{d}x \,. \tag{2.1}$$

Which gives

$$E(X^r) = \frac{c^a k}{\lambda^k B(a,b)} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\Gamma(a+b+n)(1-c)^n}{\Gamma(a+b)n!} \frac{\Gamma(1-a-n+m)}{\Gamma(1-a-n)m!} \times \frac{\left((b+m)\lambda^{-k}\right)^{-\frac{k+r}{k}} \Gamma\left(\frac{k+r}{k}\right)}{k} \,. \tag{2.2}$$

The $h^{th}$ order negative moment can readily be determined by replacing $r$ with $-h$ in (2.1):

$$E(X^{-h}) = \frac{c^a k}{\lambda^k B(a,b)} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\Gamma(a+b+n)(1-c)^n}{\Gamma(a+b)n!} \frac{\Gamma(1-a-n+m)}{\Gamma(1-a-n)m!} \times \int_0^{\infty} x^{-h}\, x^{-1+k} \left(\mathrm{e}^{-x^k/\lambda^k}\right)^{m+b} \mathrm{d}x$$

Which gives,

$$E(X^{-h}) = \frac{c^a k}{\lambda^k B(a,b)} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\Gamma(a+b+n)(1-c)^n}{\Gamma(a+b)n!} \frac{\Gamma(1-a-n+m)}{\Gamma(1-a-n)m!}$$

$$\times \frac{\left((b+m)\lambda^{-k}\right)^{-1+\frac{h}{k}} \Gamma\left(1-\frac{h}{k}\right)}{k}. \tag{2.3}$$

The factorial moments of $X$ are

$$E(X(X-1)(X-2)\cdots(X-\gamma+1)) \equiv \sum_{m=0}^{\gamma-1} \phi_m (-1)^j E(X^{\gamma-m}), \tag{2.4}$$

where $E(X^{\gamma-m})$ can be evaluated by replacing $r$ by $\gamma - m$ in (2.1).

**2.2. Moment Generating Function.** The moment generating function of the MBW distribution whose density function is specified by (1.10) will be derived here. First, we consider a result developed in [23]:

$$\int_0^\infty x^{\eta-1} \mathrm{e}^{-\theta\, x^k} \mathrm{e}^{s\, x} \mathrm{d}x = \frac{(2\pi)^{1-(q+p)/2} q^{1/2} p^{\eta-1/2}}{(-s)^\eta}$$

$$\times G^{q,p}_{p,q}\left( \left(-\frac{p}{s}\right)^p \left(\frac{\theta}{q}\right)^q \,\middle|\, \begin{array}{ll} 1-\frac{i+\eta}{p}, & i=0,1,....,p-1 \\ j/q, & j=0,1,....,q-1 \end{array} \right), \tag{2.5}$$

where $\Re(\eta), \Re(\theta), \Re(s) < 0$ and $k$ is rational number such that $k = p/q$, where $p$ and $q \neq 0$ are integers.

The moment generating function of the MBW distribution whose density function is specified by (1.10) is

$$M(t) = \frac{c^a}{B(a,b)} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\Gamma(a+b\,+n)\,\Gamma(1-a-n+m)(1-c)^n}{\Gamma(a+b)\,\Gamma(1-a-n)\,n!\,m!}$$

$$\times \int_0^\infty x^{k-1} \mathrm{e}^{-(\lambda^{-k}\,(m+b))x^k}\, \mathrm{e}^{t\,x}\, \mathrm{d}x$$

On replacing $\eta$ with $k$, $\theta$ with $\lambda^{-k}\,(m+b))$ and $s$ with $t$. In the integrand of integral and making use of (2.5), we have the following representation of the moment generating function when $k = p/q$:

$$M(t) = \frac{c^a k}{\lambda^k B(a,b)} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\Gamma(a+b+n)(1-c)^n}{\Gamma(a+b)n!} \frac{\Gamma(1-a-n+m)}{\Gamma(1-a-n)m!}$$

$$\times \frac{(2\pi)^{1-(q+p)/2}\, q^{1/2} p^{\,k-1/2}}{(-t)^k}$$

$$\times G^{q,p}_{p,q}\left( \left(\frac{-p}{t}\right)^p \left(\frac{\lambda^{-k}\,(m+b)}{q}\right)^q \,\middle|\, \begin{array}{ll} 1-\frac{i+k}{p}\,, & i=0,1,\ldots,p-1 \\ j/q\,, & j=0,1,\ldots,q-1 \end{array} \right) \tag{2.6}$$

**2.3. Entropy.** Entropy is a concept encountered in Physics and Engineering. An extension of Shannon's entropy for the continuous case can be defined as follows:

$$H(f) = -\int_0^\infty f(x)\log(f(x))\,\mathrm{d}x\,. \tag{2.7}$$

Combining (1.10) with (2.7), one has the following representation:

$$\begin{aligned}
H(f) = &-\frac{c^a k}{\lambda^k B(a,b)} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\Gamma(a+b+n)(1-c)^n}{\Gamma(a+b)n!}\frac{\Gamma(1-a-n+m)}{\Gamma(1-a-n)m!} \\
&\times \log\left(\frac{c^a k}{\lambda^k B(a,b)} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\Gamma(a+b+n)(1-c)^n}{\Gamma(a+b)n!}\frac{\Gamma(1-a-n+m)}{\Gamma(1-a-n)m!}\right) \\
&\times \int_0^{\infty} x^{-1+k}\left(\mathrm{e}^{-\lambda^{-k} x^k}\right)^{m+b} \mathrm{d}x \\
&- \frac{c^a k(-1+k)}{\lambda^k B(a,b)} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\Gamma(a+b+n)\,\Gamma(1-a-n+m)(1-c)^n}{\Gamma(a+b)\,\Gamma(1-a-n)\,n!\,m!} \\
&\times \int_0^{\infty} x^{-1+k}\left(\mathrm{e}^{-\lambda^{-k} x^k}\right)^{m+b} \log(x)\,\mathrm{d}x \\
&+ \frac{c^a k\,(m+b)}{\lambda^{2k} B(a,b)} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\Gamma(a+b+n)\,\Gamma(1-a-n+m)(1-c)^n}{\Gamma(a+b)\,\Gamma(1-a-n)\,n!\,m!} \\
&\times \int_0^{\infty} x^{-1+2k}\left(\mathrm{e}^{-\lambda^{-k} x^k}\right)^{m+b} \mathrm{d}x\,.
\end{aligned}$$

$$\begin{aligned}
H(f) = &-\frac{c^a}{(m+b)B(a,b)} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\Gamma(a+b+n)(1-c)^n}{\Gamma(a+b)n!}\frac{\Gamma(1-a-n+m)}{\Gamma(1-a-n)m!} \\
&\times \log\left(\frac{c^a k}{\lambda^k B(a,b)} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\Gamma(a+b+n)(1-c)^n}{\Gamma(a+b)n!}\frac{\Gamma(1-a-n+m)}{\Gamma(1-a-n)m!}\right) \\
&- \frac{c^a k(-1+k)}{\lambda^k B(a,b)} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\Gamma(a+b+n)\,\Gamma(1-a-n+m)(1-c)^n}{\Gamma(a+b)\,\Gamma(1-a-n)\,n!\,m!} \\
&\times \int_0^{\infty} (x)^{-1+k}\left(\mathrm{e}^{-\lambda^{-k} x^k}\right)^{m+b} \log(x)\,\mathrm{d}x \\
&+ \frac{c^a}{(m+b)B(a,b)} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\Gamma(a+b+n)\,\Gamma(1-a-n+m)(1-c)^n}{\Gamma(a+b)\,\Gamma(1-a-n)\,n!\,m!}\,. \qquad (2.8)
\end{aligned}$$

Note that, the integral on the right–hand side of (2.8) can be evaluated by numerical integration.

**2.4. Mean Residue Life Function.** The mean residue life function is defined as

$$\begin{aligned}
K(x) &= \frac{1}{S(x)} \int_x^{\infty} (y-x)\,f(y)\,\mathrm{d}y \\
&= \frac{1}{S(x)} \int_x^{\infty} y\,f(y)\,\mathrm{d}y - x \\
&= \frac{1}{S(x)} \left[E(Y) - \int_0^{x} y\,f(y)\,\mathrm{d}y\right] - x\,,
\end{aligned}$$

where $f(y)$, $S(x)$ and $E(Y)$ are as given in (1.10), (1.6) and (2.2), respectively and

$$\int_0^x y f(y)\mathrm{d}y = \frac{c^a k}{\lambda^k B(a,b)} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\Gamma(a+b+n)\Gamma(1-a-n+m)(1-c)^n}{\Gamma(a+b)\,\Gamma(1-a-n)n!m!} \times \int_0^x y^k \left(\mathrm{e}^{-\lambda^{-k} y^k}\right)^{m+b} \mathrm{d}y\,.$$

$$= \frac{c^a\, k\, \lambda^{-k}}{B(a,b)} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\Gamma(a+b+n)\,\Gamma(1-a-n+m)(1-c)^n}{\Gamma(a+b)\,\Gamma(1-a-n)\,n!\,m!} \times \int_0^x \mathrm{e}^{-(m+b)\,\lambda^{-k}\,y^k}\, y^k\, \mathrm{d}y$$

$$= \frac{c^a\, k\, \lambda^{-k}}{B(a,b)} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\Gamma(a+b+n)\,\Gamma(1-a-n+m)(1-c)^n}{\Gamma(a+b)\,\Gamma(1-a-n)\,n!\,m!} \times \int_0^x y^k\, G_{0,1}^{1,0}\left( (m+b)\,\lambda^{-k}\,y^{p/q} \,\middle|\, \begin{matrix} - \\ 0 \end{matrix} \right) \mathrm{d}y\,, \tag{2.9}$$

where $\mathrm{e}^{-g(x)} = G_{0,1}^{1,0}\left( g(x) \,\middle|\, \begin{matrix} - \\ 0 \end{matrix} \right)$, $k = p/q$, $p \geq 1$, $q \geq 1$ are natural co-prime numbers and

$$\int_0^x y^t\, G_{0,1}^{1,0}\left( (m+b)\,\beta\, y^{p/q} \,\middle|\, \begin{matrix} - \\ 0 \end{matrix} \right) \mathrm{d}y = \frac{q\, x^{p\,(t+1)}}{p(2\,\pi)^{(q-1)/2}}\, G_{p,p+q}^{q,p}\left( \frac{((m+b)\,\lambda^{-k})^q\, x^p}{q^q} \,\middle|\, \begin{matrix} \frac{-t}{p}, \frac{1-t}{p}, \ldots, \frac{p-t-1}{p}, - \\ 0, \frac{-t-1}{p}, \frac{t}{p}, \ldots, \frac{p-t-2}{p} \end{matrix} \right). \tag{2.10}$$

Equation (2.10) is obtained by making use of Equation (13) of [5].

**2.5. Mean Deviation.** The mean deviation about the mean is defined by

$$\delta(X) = \int_0^{\infty} |x - E(X)|\, f(x)\,\mathrm{d}x = \int_0^{E(X)} (E(X) - x)\; f(x)\,\mathrm{d}x + \int_{E(X)}^{\infty} (x - E(X))\; f(x)\,\mathrm{d}x\,.$$

where $E(X)$ can be evaluated by letting $r = 1$ in (2.2). The mean deviation can easily be evaluated by numerical integration.

## 3. Parameter Estimation

In this section, we will make use of the MBW, Transmuted–Weibull(TW) [1], Kumaraswamy modified Weibull (KwMW) [9], Extended Weibull (ExtW) [21], Exponential–Weibull (EW) [5], Gamma–Weibull (GW) [22], Generalized modified Weibull (GMW) [4], Modified Weibull (MW) [15], Generalized gamma (GG) [26], Two parameter Weibull (Weibull) and Two parameter gamma (Gamma) distributions to model two well–known real data sets, namely the 'Carbon fibres' [19] and the 'Cancer patients' [16] data sets. The parameters of the MBW distribution can be estimated from the loglikelihood of the samples in conjunction with the *NMaximize* command in the symbolic computational package *Mathematica*. Additionally, three goodness-of-fit measures are proposed to compare the density estimates.

**3.1. Maximum Likelihood Estimation.** In order to estimate the parameters of the proposed MBW density function as defined in Equation (1.7), the loglikelihood of the sample is maximized with respect to the parameters. Given the data $x_i$, $i = 1, \ldots, n$, the loglikelihood function is

$$\begin{aligned}\ell(\lambda,\, k,\, a,\, b,\, c) = n\, &\{a\, \log(c) + \log(k) -\, k\, \log(\lambda) - \log(\mathrm{B}(a,b))\} \\ &+ (k-1)\sum_{i=1}^{n} \log(x_i) + b \sum_{i=1}^{n} \log\left(\mathrm{e}^{-{x_i}^k/\lambda^k}\right) \\ &+ (a-1)\sum_{i=1}^{n} \log\left(1-\mathrm{e}^{-{x_i}^k/\lambda^k}\right) \\ &- (a+b)\sum_{i=1}^{n} \log\left\{1-(1-c)\left(1-\mathrm{e}^{-{x_i}^k/\lambda^k}\right)\right\} \end{aligned} \tag{3.1}$$

where $f(x)$ is as given in (1.7). The associated nonlinear loglikehood system $\frac{\partial \ell(\theta)}{\partial \theta} = 0$ for MLE estimator derivation reads as follows:

$$\begin{aligned}\frac{\partial \ell(\theta)}{\partial \lambda} &= -\frac{k\, n}{\lambda} + b\sum_{i=1}^{n} k\, \lambda^{-1-k}\, x_i^k + (a-1)\sum_{i=1}^{n} -\frac{\mathrm{e}^{-\lambda^{-k}\, x_i^k}\, k\, \lambda^{-1-k}\, x_i^k}{1-\mathrm{e}^{-\lambda^{-k}\, x_i^k}} \\ &\quad - (a+b)\sum_{i=1}^{n} \frac{(1-c)\, \mathrm{e}^{-\lambda^{-k}\, x_i^k}\, k\, \lambda^{-1-k}\, x_i^k}{1-(1-c)\left(1-\mathrm{e}^{-\lambda^{-k}\, x_i^k}\right)} = 0 \\ \frac{\partial \ell(\theta)}{\partial k} &= n\left\{\frac{1}{k} - \log(\lambda)\right\} + \sum_{i=1}^{n} \log(x_i) \\ &\quad + b\sum_{i=1}^{n}\left\{\lambda^{-k}\, \log(\lambda)\, x_i^k - \lambda^{-k}\log(x_i)\, x_i^k\right\} \\ &\quad - (a-1)\sum_{i=1}^{n} \frac{\mathrm{e}^{-\lambda^{-k} x_i^k}\, \{\lambda^{-k}\log(\lambda)\, x_i^k - \lambda^{-k}\log(x_i)\, x_i^k\}}{1-\mathrm{e}^{-\lambda^{-k}\, x_i^k}} \\ &\quad - (a+b)\sum_{i=1}^{n} \frac{(1-c)\mathrm{e}^{-\lambda^{-k} x_i^k}\, \{\lambda^{-k}\log(\lambda)\, x_i^k - \lambda^{-k}\log(x_i)\, x_i^k\}}{1-(1-c)\left(1-\mathrm{e}^{-\lambda^{-k} x_i^k}\right)} = 0 \\ \frac{\partial \ell(\theta)}{\partial a} &= n\{\log(c) - \psi^{(0)}(a) + \psi^{(0)}(a+b)\} + \sum_{i=1}^{n} \log\left(1-\mathrm{e}^{-\lambda^{-k} x_i^k}\right) \\ &\quad - \sum_{i=1}^{n} \log\left\{1-(1-c)\left(1-\mathrm{e}^{-\lambda^{-k}\, x_i^k}\right)\right\} = 0 \\ \frac{\partial \ell(\theta)}{\partial b} &= n\{-\psi^{(0)}(b) + \psi^{(0)}(a+b)\} \\ &\quad + \sum_{i=1}^{n} \log\left(\mathrm{e}^{-\lambda^{-k}\, x_i^k}\right) - \sum_{i=1}^{n} \log\left\{1-(1-c)\left(1-\mathrm{e}^{-\lambda^{-k} x_i^k}\right)\right\} = 0 \\ \frac{\partial \ell(\theta)}{\partial c} &= \frac{a\, n}{c} - (a+b)\sum_{i=1}^{n} \frac{1-\mathrm{e}^{-\lambda^{-k}\, x_i^k}}{1-(1-c)\left(1-\mathrm{e}^{-\lambda^{-k}\, x_i^k}\right)} = 0\,. \end{aligned} \tag{3.2}$$

Where $\psi^{(0)}(\cdot)$ is the polygamma function. The above equations cannot be solved analytically and statistical software can be used to solve them numerically.

**3.2. Goodness-of-Fit Statistics.** To verify the goodness-of-fit of certain statistical models, some goodness-of-fit statistics shall be used. They are computed using the symbolic computation package *Mathematica*. The following goodness-of-fit statistics are considered: the Anderson-Darling, Cramér-von Mises and Akaike Information Criterion (AIC) statistics for comparison purposes. The Anderson-Darling and Cramér-von Mises statistics are widely utilized to determine how closely a specific distribution whose associated cumulative distribution function denoted by cdf(·) fits the empirical distribution associated with a given data set. Upper tail percentiles of the asymptotic distributions of Anderson-Darling and Cramér-von Mises statistics were tabulated in [19]. The distribution having the better fit will be the one whose goodness-of-fit statistic is the smallest.

## 4. Empirical illustrations

In this section, we present two applications where the MBW model is compared with other related models, namely Transmuted–Weibull(TW) [1], Kumaraswamy modified Weibull (KwMW) [9] Extended Weibull (ExtW) [21], Exponential–Weibull (EW) [5], Gamma–Weibull (GW) [22], Generalized modified Weibull (GMW) [4], Modified Weibull (MW) [15], Generalized gamma (GG) [26], Two parameter Weibull (Weibull) and Two parameter gamma (Gamma) distributions. We make use of two data sets: first, the Carbon fibres data set [19] and, secondly, the Cancer patients data set [16].

- The classical gamma (Gamma) distribution with density function
$$f(x) = \frac{x^{\xi-1}\,\mathrm{e}^{-x/\phi}}{\phi^{\xi}\,\Gamma(\xi)}, \quad x>0,\, \phi,\, \xi>0\,.$$
- The classical Weibull (Weibull) distribution with density function
$$f(x) = \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1}\mathrm{e}^{-(x/\lambda)^{k}}, \quad x>0,\, k,\, \lambda>0\,.$$
- The generalize gamma (GG) distribution [26] with density function
$$f(x) = \frac{k\,\lambda^{-\xi}\,x^{\xi-1}\,\mathrm{e}^{-\lambda^{-k}\,x^{k}}}{\Gamma(\xi/k)}, \quad x>0,\, \xi,\, k,\, \lambda>0\,.$$
- The modified Weibull (MW) distribution [15] with density function
$$f(x) = \alpha\,x^{\gamma-1}(\gamma+\lambda x)\,\mathrm{e}^{\left(\lambda x-\,\alpha x^{\gamma}\,\mathrm{e}^{\lambda x}\right)}, \quad x>0,\, \gamma,\, \alpha>0,\, \lambda\geq 0.$$
- The generalized modified Weibull (GMW) distribution [4] with density function
$$f(x) = \varphi\,\alpha\,x^{\gamma-1}\,(\gamma+\lambda x)\,\mathrm{e}^{\left(\lambda x-\alpha x^{\gamma}\,\mathrm{e}^{\lambda x}\right)}\left\{1-\mathrm{e}^{\left(-\alpha x^{\gamma}\,\mathrm{e}^{\lambda x}\right)}\right\}^{\varphi-1},$$
$$x>0,\, \gamma,\, \alpha,\, \varphi>0,\, \lambda\geq 0.$$
- The gamma–Weibull distribution [22] with density function
$$f(x) = \frac{k\,\lambda^{-k-\xi}\,x^{\,\xi+k-1}\mathrm{e}^{-\lambda^{-k}\,x^{k}}}{\Gamma(1+\xi/k)}, \quad x>0,\, \xi+k>0,\, \lambda>0\,.$$
- The exponential–Weibull (EW) distribution [5] with density function
$$f(x) = \left(\lambda+\beta\,k\,x^{k-1}\right)\,\mathrm{e}^{-\lambda\,x-\beta\,x^{k}}, \quad x>0,\, \lambda,\, \beta,\, k>0\,.$$
- The Transmuted–Weibull(TW) [1] with density function
$$f(x) = \frac{\eta\mathrm{e}^{-\left(\frac{x}{\sigma}\right)^{\eta}}\left(\frac{x}{\sigma}\right)^{\eta-1}\left\{2\lambda\,\mathrm{e}^{-\left(\frac{x}{\sigma}\right)^{\eta}}-\lambda+1\right\}}{\sigma}, \quad x>0,\, \sigma,\, \eta,\, \lambda>0\,.$$

- The extended Weibull (ExtW) distribution [21] with density function

$$f(x) = a\,(c + b\,x)\,x^{-2+b}\,\mathrm{e}^{-c/x - a x^{b}\mathrm{e}^{-c/x}}, \quad x > 0,\ a,\ b > 0,\ c \geq 0\,.$$

- The Kumaraswamy modified Weibull (KwMW) distribution [9] with density function

$$f(x) = a\,b\,\alpha\,x^{\gamma-1}\,(\gamma + \lambda\,x)\,\mathrm{e}^{\left(\lambda\,x - \alpha\,x^{\gamma}\,\mathrm{e}^{\lambda\,x}\right)}\left\{1 - \mathrm{e}^{\left(-\alpha\,x^{\gamma}\,\mathrm{e}^{\lambda\,x}\right)}\right\}^{a-1}$$

$$\times\left[1 - \left\{1 - \mathrm{e}^{\left(-\alpha\,x^{\gamma}\,\mathrm{e}^{\lambda\,x}\right)}\right\}^{a}\right]^{b-1}, \quad x > 0,\ a,\ b,\ \alpha,\ \gamma > 0,\ \lambda \geq 0\,.$$

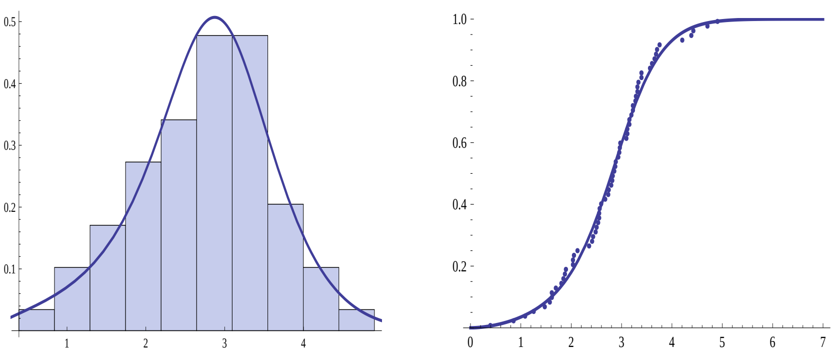


**Figure 5.** Left panel: The MBW density estimate superimposed on the histogram for Carbon fibres data . Right panel: The MBW cdf estimates and empirical cdf.

Note that: The empirical cdf can be plotted using the following code in mathematica.

ListPlot[Table[{data[[i]], i/n-1/(2n)},{i, 1,n}]].

**4.1. The Carbon Fibres Data Set.** The first data set represents the uncensored real data set on the breaking stress of carbon fibres (in Gba) as reported in [5]. The data are (n = 66): 3.70, 2.74, 2.73, 2.50, 3.60, 3.11, 3.27, 2.87, 1.47, 3.11, 3.56, 4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.90, 1.57, 2.67, 2.93, 3.22, 3.39, 2.81, 4.20, 3.33, 2.55, 3.31, 3.31, 2.85, 1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.70, 2.03, 1.89, 2.88, 2.82, 2.05, 3.65, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.35, 2.55, 2.59, 2.03, 1.61, 2.12, 3.15, 1.08, 2.56, 1.80, 2.53.

**4.2. The Cancer Patients Data Set.** The second data set represents the remission times (in months) of a random sample of 128 bladder cancer patients as reported in [16]. The data are 0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90 , 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69.

The pdf and cdf estimates of the MBW distribution are plotted in Figures 5 and 6 for the Carbon fibres and Cancer patients data, respectively. The estimated hazard

**Table 1.** Estimates of the Parameters, Goodness-of-Fit Statistics and Loglikelihood for the Carbon Fibres Data

| Distributions | Estimates | | | | |
|---|---|---|---|---|---|
| Gamma $(\xi\,,\ \phi)$ | 7.48803 | 0.368528 | | | |
| Weibull $(k\,,\ \lambda)$ | 3.4412 | 47.0505 | | | |
| GG $(k\,,\ \lambda\,,\ \xi)$ | 4.0735 | 3.34592 | 3.09225 | | |
| MW$(\alpha\,,\gamma\,,\lambda)$ | 0.021813 | 2.709212 | 0.248518 | | |
| GMW$(\varphi,\ \alpha\,,\gamma\,,\lambda\ )$ | 5.49894 | 0.436399 | 0.148117 | 0.516284 | |
| GW $(k\,,\ \xi\,,\ \lambda)$ | 3.4412 | $1.6\times10^{-7}$ | 3.06226 | | |
| EW $(k\,,\ \lambda\,,\ \beta)$ | 3.73666 | 0.0170948 | 0.01401 | | |
| TW $(\eta\,,\sigma\,,\lambda\,)$ | 3.441197 | 3.745584 | 1 | | |
| ExtW $(\ a\,,\ b\,,\ c)$ | 16.1979 | $1\times10^{-7}$ | 8.05671 | | |
| KwMW$(\alpha\,,\gamma\,,\lambda\,,a\,,b)$ | 0.14981 | 1.7994 | 0.49987 | 0.64975 | 0.17111 |
| MBW $(\lambda,\ k,\ a\,,b\,,c)$ | 1.65934 | 2.23218 | 0.78685 | 0.55408 | 0.07248 |
| Distributions | | $A_0^*$ | $W_0^*$ | AIC | $\ell(\hat{\Theta})$ |
| Gamma $(\xi\,,\ \phi)$ | | 1.32674 | 0.248153 | 186.335 | -91.1675 |
| Weibull $(k\,,\ \lambda)$ | | 0.491678 | 0.0843011 | **176.135** | -86.0676 |
| GG $(k\,,\ \lambda\,,\ \xi)$ | | 0.487573 | 0.0811144 | 177.835 | -85.9175 |
| MW$(\alpha\,,\gamma\,,\lambda)$ | | 0.485662 | 0.0793299 | 177.727 | -85.8636 |
| GMW$(\varphi,\ \alpha\,,\gamma\,,\lambda\ )$ | | 0.385439 | 0.0627953 | 178.746 | -85.3731 |
| GW $(k\,,\ \xi\,,\ \lambda)$ | | 0.491678 | 0.0843011 | 178.135 | -86.0676 |
| EW $(k\,,\ \lambda\,,\ \beta)$ | | 0.403649 | 0.06479 | 177.044 | -85.5218 |
| TW $(\eta\,,\sigma\,,\lambda\,)$ | | 0.491678 | 0.0843011 | 178.135 | -86.0676 |
| ExtW $(\ a\,,\ b\,,\ c)$ | | 2.26745 | 0.416152 | 207.47 | -100.735 |
| KwMW$(\alpha\,,\gamma\,,\lambda\,,a\,,b)$ | | 1.29338 | 0.213215 | 185.980 | -87.9902 |
| MBW $(\lambda,\ k,\ a\,,b\,,c)$ | | **0.24516** | **0.034375** | 179.226 | **-84.613** |

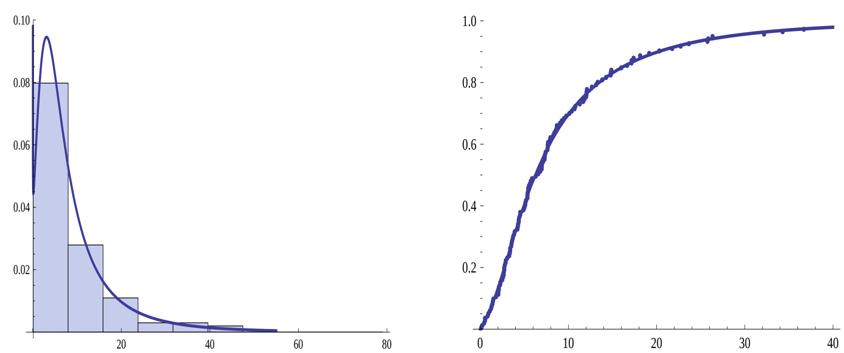


**Figure 6.** Left panel: The MBW density estimate superimposed on the histogram for Cancer patients data . Right panel: The MBW cdf estimates and empirical cdf.

rate function of MBW distribution are plotted in Figure 7. It can be seen that both shapes of hazard rate function, for carbon fibers and cancer patients data sets are like bathtub shaped hazard rate function. The estimates of the parameters and the values of AIC, Anderson-Darling and Cramér-von Mises goodness–of–fit statistics are given in Tables 1 and 2 for the Carbon fibres and Cancer patients data, respectively. It is seen that the proposed MBW model provides the best fit for both data sets when considering Anderson-Darling and Cramér-von Mises goodness–of–fit statistics and is a competitive model when considering AIC.

**Table 2.** Estimates of the Parameters, Goodness-of-Fit Statistics and Loglikelihood for the Cancer Patients Data

| Distributions | Estimates | | | | |
|---|---|---|---|---|---|
| Gamma $(\xi, \phi)$ | 1.17251 | 7.98766 | | | |
| Weibull $(k, \lambda)$ | 1.04783 | 10.651 | | | |
| GG $(k, \lambda, \xi)$ | 0.520095 | 0.595104 | 1.94927 | | |
| MW$(\alpha, \gamma, \lambda)$ | 0.093887 | 1.047834 | $3.6 \times 10^{-11}$ | | |
| GMW$(\varphi, \alpha, \gamma, \lambda)$ | 2.796005 | 0.453691 | 0.654409 | $5.8 \times 10^{-13}$ | |
| GW $(k, \xi, \lambda)$ | 0.520095 | 1.42917 | 0.595104 | | |
| EW $(k, \lambda, \beta)$ | 1.04783 | $1 \times 10^{-7}$ | 0.093887 | | |
| TW $(\eta, \sigma, \lambda)$ | 1.133310 | 14.61979 | 0.744922 | | |
| ExtW $(a, b, c)$ | 1.9621 | $1 \times 10^{-21}$ | 3.74383 | | |
| KwMW$(\alpha, \gamma, \lambda, a, b)$ | 0.639622 | 0.381865 | 0.029602 | 0.375 | 0.322843 |
| MBW $(\lambda, k, a, b, c)$ | 0.32113 | 0.52381 | 1.29997 | 0.41823 | 0.053809 |
| **Distributions** | | $A_0^*$ | $W_0^*$ | AIC | $\ell(\hat{\Theta})$ |
| Gamma $(\xi, \phi)$ | | 0.77625 | 0.136063 | 830.736 | -413.368 |
| Weibull $(k, \lambda)$ | | 0.963452 | 0.154303 | 832.174 | -414.087 |
| GG $(k, \lambda, \xi)$ | | 0.300873 | 0.04526 | **827.708** | -410.854 |
| MW$(\alpha, \gamma, \lambda)$ | | 0.963452 | 0.154303 | 834.174 | -414.087 |
| GMW$(\varphi, \alpha, \gamma, \lambda)$ | | 0.271984 | 0.04050 | 829.36 | -410.68 |
| GW $(k, \xi, \lambda)$ | | 0.300873 | 0.045261 | **827.708** | -410.854 |
| EW $(k, \lambda, \beta)$ | | 0.963452 | 0.154303 | 834.174 | -414.087 |
| TW $(\eta, \sigma, \lambda)$ | | 0.563397 | 0.0882597 | 829.916 | -411.958 |
| ExtW $(a, b, c)$ | | 13.3317 | 2.49818 | 1034.9 | -514.498 |
| KwMW$(\alpha, \gamma, \lambda, a, b)$ | | 18.8864 | 3.68568 | 979.652 | -484.826 |
| MBW $(\lambda, k, a, b, c)$ | | **0.076133** | **0.0119393** | 828.612 | **-409.306** |

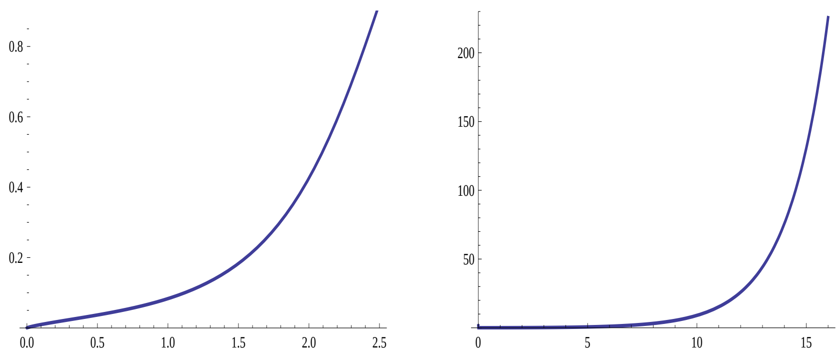


**Figure 7.** Left panel: The estimated MBW hazard rate function for carbon data . Right panel: The estimated MBW hazard rate function for cancer patients data .

## 5. Discussion

There has been a growing interest among statisticians and applied researchers in constructing flexible lifetime models in order to improve the modeling of survival data. As a result, significant progress has been made towards the generalization of some well–known lifetime models, which have been successfully applied to problems arising in several areas of research. In particular, several authors have proposed new distributions which are based on the traditional Weibull model. In this paper, we introduce a five–parameter distribution which is obtained by applying the modified beta technique to the Weibull model. Interestingly, our proposed model has bathtub and up side down bathtub shaped hazard rate function.We studied some of its statistical properties. We also provided computable representations of the positive and negative moments, the factorial moments, the moment generating function, the mean residue life function, the mean deviation and the associated Shannon's entropy. The proposed distribution was applied to two data sets and shown to provide a better fit than other related models. The distributional results developed in this article should find numerous applications in the physical and biological sciences, reliability theory, hydrology, medicine, meteorology, engineering and survival analysis.


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